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UNITARY PROPAGATION OF OPERATOR DATA

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Abstract We study the problem of finding conditions which guarantee that given operator data propagate under the control of unitary operators. This is a kind of moment problem and the way we solve it is based on unitary dilations. Comparing it with other results in this area, our approach has the advantages of

(i) allowing the data to be unbounded operators,

(ii) considering arbitrary dilations, not necessarily regular ones.

The relationship between the moment problem and harmonizable multivariate discrete processes is indicated.

Keywords: systems of contractions with unitary power dilations; vector and operator moment problems; harmonizable multivariate discrete processes

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1. Introduction

A family $\boldsymbol{T} = \{T_\xi\}_{\xi \in X}$ of commuting linear operators on a complex inner product space $\mathcal{E}$ (we always use $\langle \cdot, - \rangle$ to denote the inner product, regardless of the space) indexed by a non-empty set X is said to have a *unitary dilation* if there exists a complex Hilbert space $\mathcal{K} \supset \mathcal{E}$ (isometric embedding) and a family $\boldsymbol{U} = \{U_\xi\}_{\xi \in X}$ of commuting unitary operators on $\mathcal{K}$ such that

$$\langle \boldsymbol{T}^{\boldsymbol{x}} f, g \rangle = \langle \boldsymbol{U}^{\boldsymbol{x}} f, g \rangle, \quad \boldsymbol{x} \in \mathbb{Z}[X]_+, \ f, g \in \mathcal{E}, \tag{1.1}$$

where $\mathbb{Z}[X]_+$ is the set of all non-negative integer-valued functions on X with finite support,

$$\boldsymbol{T}^{\boldsymbol{x}} = \prod_{\xi \in X} T_\xi^{\boldsymbol{x}(\xi)} \quad \text{and} \quad \boldsymbol{U}^{\boldsymbol{x}} = \prod_{\xi \in X} U_\xi^{\boldsymbol{x}(\xi)}.$$

Such a $\boldsymbol{U}$ is called a *unitary dilation* of $\boldsymbol{T}$. The condition (1.1) is obviously equivalent to

$$\boldsymbol{T}^{\boldsymbol{x}} = P\boldsymbol{U}^{\boldsymbol{x}}|_{\mathcal{E}}, \quad \boldsymbol{x} \in \mathbb{Z}[X]_+,$$

where P is the orthogonal projection of $\mathcal{K}$ onto $\bar{\mathcal{E}}$, the closure of $\mathcal{E}$ in $\mathcal{K}$.

The problem is to determine whether, for a given family $\{A_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathbb{Z}[X]_+}$ of linear operators from a complex inner product space $\mathcal{D}$ to a complex Hilbert space $\mathcal{H}$, there exists a family $\boldsymbol{T} = \{T_\xi\}_{\xi\in X}$ of commuting contractions on $\mathcal{H}$ having a unitary dilation and such that

$$A_{\boldsymbol{x}} = \boldsymbol{T}^{\boldsymbol{x}} A_{\boldsymbol{0}}, \quad \boldsymbol{x} \in \mathbb{Z}[X]_+. \tag{1.2}$$

Our main result, Theorem 4.1, solves this problem (which can be regarded as a moment problem) for a family $\{A_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathbb{Z}[X]_+}$ of arbitrary cardinality, allowing the solution $\boldsymbol{T}$ to have a unitary dilation with no further restrictions.* Calling this a moment problem is a matter of language; in no way does it limit the range of applicability. In particular, applications in stochastic processes deserve more attention. An important special case of (1.2) takes the form

$$h_{\boldsymbol{x}} = \boldsymbol{T}^{\boldsymbol{x}} h_{\boldsymbol{0}}, \quad \boldsymbol{x} \in \mathbb{Z}[X]_+, \tag{1.3}$$

where $\{h_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathbb{Z}[X]_+}$ is a family of vectors in $\mathcal{H}$. Denote by $\mathbb{T}^X$ the Cartesian product of card X-copies of the unit circle $\mathbb{T} \stackrel{\text{def}}{=} \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and equip it with the topology of pointwise convergence. Let $\boldsymbol{U} = \{U_\xi\}_{\xi\in X}$ be a unitary dilation of $\boldsymbol{T}$ consisting of operators acting on a Hilbert space $\mathcal{K}$ and let E be the joint spectral measure† of $\boldsymbol{U}$ defined on Borel subsets of $\mathbb{T}^X$, i.e.

$$U_\xi = \int_{\mathbb{T}^X} \lambda_\xi E(\mathrm{d}\boldsymbol{\lambda}), \quad \xi \in X,\ \boldsymbol{\lambda} = \{\lambda_\zeta\}_{\zeta\in X}. \tag{1.4}$$

Denote by P the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$ and define the vector Borel measure ϕ on $\mathbb{T}^X$ by $\phi(\cdot) = PE(\cdot)h_{\boldsymbol{0}}$. Then, by (1.3) and (1.4), we have

$$h_{\boldsymbol{x}} = \int_{\mathbb{T}^X} \boldsymbol{\lambda}^{\boldsymbol{x}} \phi(\mathrm{d}\boldsymbol{\lambda}), \quad \boldsymbol{x} \in \mathbb{Z}[X]_+,$$

with

$$\boldsymbol{\lambda}^{\boldsymbol{x}} \stackrel{\text{def}}{=} \prod_{\xi\in X} \lambda_\xi^{\boldsymbol{x}(\xi)}.$$

Hence, $\{h_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathbb{Z}[X]_+}$ is a harmonizable multivariate discrete process‡ (or, in another terminology, a random field); its stationary dilation $\{f_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathbb{Z}[X]_+}$ is of the form $f_{\boldsymbol{x}} \stackrel{\text{def}}{=} \boldsymbol{U}^{\boldsymbol{x}} h_{\boldsymbol{0}}$. As multivariate processes can be generalized to the operator case [**9**], Theorem 4.1 offers some sufficient conditions for so-called weak operator harmonizability as well; all this is given in terms of covariance kernels. It should be pointed out that the operators $A_{\boldsymbol{x}}$ in the moment problem (1.2) are allowed to be *unbounded*, which may be of some prospective interest.

* The moment problem (1.2) has been solved by Sebestyén for card $X = 1$ [**20, 21**] (see also [**16**] for a recent approach). Its variant with so-called regular unitary dilations has been solved by Găvruţă and Păunescu for card $X = 2$ [**6**] and by Popovici and Sebestyén for arbitrary X [**15**].

† E is the product of spectral measures of unitary operators $\{U_\xi\}_{\xi\in X}$ defined on Borel subsets of the compact Hausdorff space $\mathbb{T}^X$ (for more details see [**23**, Proposition 4] and [**3**]).

‡ The notion of a harmonizable process is attributed to Rozanov [**18**]. This circle of ideas has been developed by many authors [**1, 5, 7–14, 17, 19, 22, 28, 29**].

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2. Preliminaries

The n-dimensional complex Euclidean space $\mathbb{C}^n$ is equipped with the standard inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \bar{w}_j$ for $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ in $\mathbb{C}^n$. Throughout what follows, $\mathcal{D}$ and $\mathcal{E}$ are complex inner product spaces and $\mathcal{H}$ is a complex Hilbert space. We denote by $\boldsymbol{L}(\mathcal{D}, \mathcal{E})$ (respectively, $\boldsymbol{B}(\mathcal{D}, \mathcal{E})$) the set of all linear (respectively, bounded linear) operators from $\mathcal{D}$ to $\mathcal{E}$. We shall abbreviate $\boldsymbol{L}(\mathcal{D}, \mathcal{D})$ to $\boldsymbol{L}(\mathcal{D})$ and $\boldsymbol{B}(\mathcal{D}, \mathcal{D})$ to $\boldsymbol{B}(\mathcal{D})$. The identity operator on $\mathcal{D}$ is denoted by $I_{\mathcal{D}}$. If $f, g \in \mathcal{E}$, then $f \otimes g \in \boldsymbol{B}(\mathcal{E})$ is defined by

$$(f \otimes g)(h) = \langle h, g\rangle f, \quad h \in \mathcal{E}.$$

Given a non-empty set Y, we write $\mathcal{E}[Y]$ for the set of all maps $f : Y \to \mathcal{E}$ with finite support $\{\xi \in Y : f(\xi) \neq 0\}$.

We denote by $\mathbb{Z}$ the additive group of all integers. Let X be a non-empty set. We write $\mathbb{Z}[X]$ for the additive group of all functions $\boldsymbol{x} : X \to \mathbb{Z}$ with finite support $\{\xi \in X : \boldsymbol{x}(\xi) \neq 0\}$ equipped with pointwise defined group operation. Define the sets

$$\begin{aligned} \mathbb{Z}[X]_+ &= \{\boldsymbol{x} \in \mathbb{Z}[X]; \boldsymbol{x}(\xi) \geqslant 0 \text{ for all } \xi \in X\}, \\ \mathbb{Z}[X]_\pm &= \mathbb{Z}[X]_+ \cup (-\mathbb{Z}[X]_+), \\ \mathbb{Z}[X]_\pm^{\mathrm{C}} &= \mathbb{Z}[X] \setminus \mathbb{Z}[X]_\pm. \end{aligned}$$

If Y is a subset of X, then we can think of $\mathbb{Z}[Y]$ as a subset of $\mathbb{Z}[X]$. Given $\boldsymbol{x} \in \mathbb{Z}[X]$, we denote by $\boldsymbol{x}_{\mathrm{pos}}, \boldsymbol{x}_{\mathrm{neg}} \in \mathbb{Z}[X]_+$ the positive and the negative parts of $\boldsymbol{x}$, i.e.

$$\left.\begin{aligned} \boldsymbol{x}_{\mathrm{pos}}(\xi) &\overset{\mathrm{def}}{=} (\boldsymbol{x}(\xi))_{\mathrm{pos}}, \\ \boldsymbol{x}_{\mathrm{neg}}(\xi) &\overset{\mathrm{def}}{=} (\boldsymbol{x}(\xi))_{\mathrm{neg}}, \end{aligned}\right\} \quad \xi \in X,$$

where $n_{\mathrm{pos}} \overset{\mathrm{def}}{=} \max\{n, 0\}$ and $n_{\mathrm{neg}} \overset{\mathrm{def}}{=} -\min\{n, 0\}$ for $n \in \mathbb{Z}$.

The ensuing lemma characterizes families of operators having unitary dilations; it is an adaptation of the results from [**25**] to the present context.

Lemma 2.1. *If $\boldsymbol{T} = \{T_\xi\}_{\xi \in X} \subset \boldsymbol{L}(\mathcal{E})$ is a family of commuting operators, then the following conditions are equivalent:*

(a) *$\boldsymbol{T}$ has a unitary dilation;*

(b) *for all integers $m, n \geqslant 1$, for all maps $\lambda_1, \dots, \lambda_m \in \mathbb{C}^n[\mathbb{Z}[X]_+]$ such that*

$$\sum_{\substack{\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}[X]_+, \\ \boldsymbol{x} - \boldsymbol{y} = \boldsymbol{u}}} \langle \lambda_k(\boldsymbol{x}), \lambda_l(\boldsymbol{y})\rangle = 0, \quad \boldsymbol{u} \in \mathbb{Z}[X]_\pm^{\mathrm{C}}, \quad k, l = 1, \dots, m, \tag{2.1}$$

and for all vectors $v_1, \dots, v_m \in \mathcal{E}$, the inequality

$$\sum_{k,l=1}^m \sum_{\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}[X]_+} \langle \boldsymbol{T}^{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{pos}}} v_k, \boldsymbol{T}^{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{neg}}} v_l\rangle \langle \lambda_k(\boldsymbol{x}), \lambda_l(\boldsymbol{y})\rangle \geqslant 0 \tag{2.2}$$

holds;

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(c) *for any integer $n \geqslant 1$ and for all maps $f_1, \dots, f_n \in \mathcal{E}[\mathbb{Z}[X]_+]$ such that*

$$\sum_{j=1}^{n} \sum_{\substack{\boldsymbol{x},\boldsymbol{y}\in\mathbb{Z}[X]_+,\\ \boldsymbol{x}-\boldsymbol{y}=\boldsymbol{u}}} f_j(\boldsymbol{x}) \otimes f_j(\boldsymbol{y}) = 0, \quad \boldsymbol{u} \in \mathbb{Z}[X]_\pm^{\mathrm{C}}, \tag{2.3}$$

the inequality

$$\sum_{j=1}^{n} \sum_{\boldsymbol{x},\boldsymbol{y}\in\mathbb{Z}[X]_+} \langle \boldsymbol{T}^{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{pos}}} f_j(\boldsymbol{x}), \boldsymbol{T}^{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{neg}}} f_j(\boldsymbol{y})\rangle \geqslant 0$$

holds.

Idea of the proof. Apply [**25**, Theorem 1] to the group $\mathfrak{G} = \mathbb{Z}[X]$ with involution $\boldsymbol{x}^* \overset{\mathrm{def}}{=} -\boldsymbol{x}$, the set $\mathfrak{X} = \mathbb{Z}[X]_\pm$, the inner product space $\mathcal{D} = \mathcal{E}$ and the function $\omega : \mathbb{Z}[X]_\pm \times \mathcal{E} \times \mathcal{E} \to \mathbb{C}$ given by

$$\omega(\boldsymbol{x}, f, g) = \langle \boldsymbol{T}^{\boldsymbol{x}_{\mathrm{pos}}} f, \boldsymbol{T}^{\boldsymbol{x}_{\mathrm{neg}}} g\rangle, \quad \boldsymbol{x} \in \mathbb{Z}[X]_\pm,\ f, g \in \mathcal{E},$$

and argue as in the proofs of [**25**, Theorem 3] and [**24**, Theorem 29]. We leave the details to the reader. □

Recall that a family $\boldsymbol{T} = \{T_\xi\}_{\xi\in X} \subset \boldsymbol{B}(\mathcal{H})$ of commuting contractions has a *regular* unitary dilation [**27**, Chapter I, § 9] if and only if

$$\sum_{\boldsymbol{x},\boldsymbol{y}\in\mathbb{Z}[X]_+} \langle \boldsymbol{T}^{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{pos}}} f(\boldsymbol{x}), \boldsymbol{T}^{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{neg}}} f(\boldsymbol{y})\rangle \geqslant 0, \quad f \in \mathcal{H}[\mathbb{Z}[X]_+].$$

Clearly, the above condition implies Lemma 2.1 (c) with $\mathcal{E} = \mathcal{H}$.

3. Contractive solutions

For $\xi \in X$, we define $\boldsymbol{e}_\xi \in \mathbb{Z}[X]_+$ by

$$\boldsymbol{e}_\xi(\zeta) = \begin{cases} 1 & \text{if } \zeta = \xi, \\ 0 & \text{if } \zeta \neq \xi. \end{cases}$$

Given a family $\{A_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathbb{Z}[X]_+} \subset \boldsymbol{L}(\mathcal{D}, \mathcal{H})$, we set

$$\mathcal{R}_A = \operatorname{lin} \bigcup_{\boldsymbol{x}\in\mathbb{Z}[X]_+} A_{\boldsymbol{x}}(\mathcal{D}), \qquad \mathcal{H}_A = \bar{\mathcal{R}}_A.$$

The next lemma extends [**16**, Theorem 3.1] to the multidimensional setting. Despite this setting, it becomes slightly simpler and its proof is essentially shorter than that in [**16**].

Lemma 3.1. *Let $\{A_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathbb{Z}[X]_+} \subset \boldsymbol{L}(\mathcal{D},\mathcal{H})$ be a family of operators. Then, for every $\xi \in X$, the following conditions are equivalent:*

(i) *there exists a contraction $T_\xi \in \boldsymbol{B}(\mathcal{H})$ such that*

$$T_\xi A_{\boldsymbol{x}} = A_{\boldsymbol{e}_\xi+\boldsymbol{x}}, \quad \boldsymbol{x} \in \mathbb{Z}[X]_+; \tag{3.1}$$

(ii) *for all maps $h_1, h_2 \in \mathcal{D}[\mathbb{Z}[X]_+]$,*

$$\sum_{k,l=1}^{2} \sum_{\boldsymbol{x},\boldsymbol{y}\in\mathbb{Z}[X]_+} \langle A_{(k-l)_{\mathrm{pos}}\boldsymbol{e}_\xi+\boldsymbol{x}} h_k(\boldsymbol{x}), A_{(k-l)_{\mathrm{neg}}\boldsymbol{e}_\xi+\boldsymbol{y}} h_l(\boldsymbol{y})\rangle \geqslant 0; \tag{3.2}$$

(iii) *for any map $h \in \mathcal{D}[\mathbb{Z}[X]_+]$,*

$$\left\| \sum_{\boldsymbol{x}\in\mathbb{Z}[X]_+} A_{\boldsymbol{e}_\xi+\boldsymbol{x}} h(\boldsymbol{x}) \right\| \leqslant \left\| \sum_{\boldsymbol{x}\in\mathbb{Z}[X]_+} A_{\boldsymbol{x}} h(\boldsymbol{x}) \right\|. \tag{3.3}$$

Moreover, if either condition (ii) or (iii) holds for every $\xi \in X$, then there exits a family $\{T_\xi\}_{\xi\in X} \subset \boldsymbol{B}(\mathcal{H})$ of commuting contractions satisfying (3.1).

Proof. Note first that the inequality (3.2) can be rewritten in the form

$$\|g_1\|^2 + \|g_2\|^2 + 2\,\mathrm{Re}\langle g_1, \hat{g}_2\rangle \geqslant 0, \tag{3.4}$$

where $g_j \overset{\mathrm{def}}{=} \sum_{\boldsymbol{x}\in\mathbb{Z}[X]_+} A_{\boldsymbol{x}} h_j(\boldsymbol{x})$ and $\hat{g}_2 \overset{\mathrm{def}}{=} \sum_{\boldsymbol{x}\in\mathbb{Z}[X]_+} A_{\boldsymbol{e}_\xi+\boldsymbol{x}} h_2(\boldsymbol{x})$.

(i) $\Rightarrow$ (ii). Since $\hat{g}_2 = T_\xi(g_2)$ and T_ξ is a contraction, we get

$$\begin{aligned} -2\,\mathrm{Re}\langle g_1, \hat{g}_2\rangle &= -2\,\mathrm{Re}\langle g_1, T_\xi(g_2)\rangle \\ &\leqslant 2\|g_1\|\|T_\xi(g_2)\| \\ &\leqslant 2\|g_1\|\|g_2\| \leqslant \|g_1\|^2 + \|g_2\|^2, \end{aligned}$$

which implies (3.4).

(ii) $\Rightarrow$ (iii). Take $h \in \mathcal{D}[\mathbb{Z}[X]_+]$ and define $h_1, h_2 \in \mathcal{D}[\mathbb{Z}[X]_+]$ by $h_2 = h$ and

$$h_1(\boldsymbol{x}) = \begin{cases} -h(\boldsymbol{x}-\boldsymbol{e}_\xi) & \text{if } \boldsymbol{x}-\boldsymbol{e}_\xi \in \mathbb{Z}[X]_+, \\ 0 & \text{otherwise.} \end{cases}$$

Then $g_1 = -\hat{g}_2$. Together with (3.4), this yields $\|\hat{g}_2\|^2 \leqslant \|g_2\|^2$, which implies (3.3).

(iii) $\Rightarrow$ (i). It follows from (3.3) that there exists a unique contraction $\tilde{T}_\xi \in \boldsymbol{B}(\mathcal{H}_A)$ such that

$$\tilde{T}_\xi A_{\boldsymbol{x}} f = A_{\boldsymbol{e}_\xi+\boldsymbol{x}} f \quad \text{for all } \boldsymbol{x} \in \mathbb{Z}[X]_+,\ f \in \mathcal{D}.$$

Define the operator $T_\xi \in \boldsymbol{B}(\mathcal{H})$ by $T_\xi = \tilde{T}_\xi \oplus I_{\mathcal{H}\ominus\mathcal{H}_A}$. Then T_ξ is a contraction which satisfies (3.1). Moreover, if this is done for every $\xi \in X$, then the operators T_ξ, $\xi \in X$, commute. This completes the proof. □

As in [**16**, Theorem 3.1], one can deduce from Lemma 3.1 (i) that the inequality (3.2) holds for all finite sequence $h_1, \dots, h_m \in \mathcal{D}[\mathbb{Z}[X]_+]$ (however, this requires a usage of Sz.-Nagy's dilation theorem [**26**]).

Regarding Lemma 3.1 (as well as Corollary 3.2 and Theorem 4.1), one can check that if $\mathcal{H}_{\boldsymbol{A}} = \mathcal{H}$, then there exists at most one operator $T_\xi \in \boldsymbol{B}(\mathcal{H})$ satisfying (3.1). On the other hand, if there exists exactly one operator $T_\xi \in \boldsymbol{B}(\mathcal{H})$ satisfying (3.1), then $\mathcal{H}_{\boldsymbol{A}} = \mathcal{H}$.

Applying Lemma 3.1 and Ando's dilation theorem [**2**], we get the following corollary.

Corollary 3.2. *If $\{A_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathbb{Z}_+^2} \subset \boldsymbol{L}(\mathcal{D},\mathcal{H})$, then the following conditions are equivalent (we identify $\mathbb{Z}_+^2$ with $\mathbb{Z}[X]_+$, where $X = \{1,2\}$):*

(i) *there exists a pair $\boldsymbol{T} = (T_1, T_2) \in \boldsymbol{B}(\mathcal{H})^2$ of commuting contractions having a unitary dilation and such that $A_{\boldsymbol{x}} = \boldsymbol{T}^{\boldsymbol{x}} A_{\boldsymbol{0}}$ for all $\boldsymbol{x} \in \mathbb{Z}_+^2$;*

(ii) *for every $j = 1, 2$ and for all $h_1, h_2 \in \mathcal{D}[\mathbb{Z}_+^2]$,*

$$\sum_{k,l=1}^{2} \sum_{\boldsymbol{x},\boldsymbol{y}\in\mathbb{Z}_+^2} \langle A_{(k-l)_{\mathrm{pos}}\boldsymbol{e}_j+\boldsymbol{x}} h_k(\boldsymbol{x}), A_{(k-l)_{\mathrm{neg}}\boldsymbol{e}_j+\boldsymbol{y}} h_l(\boldsymbol{y})\rangle \geqslant 0.$$

4. Dilatability of solutions

We are now ready to prove the main result of the paper.

Theorem 4.1. *Suppose that we are given $\{A_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathbb{Z}[X]_+} \subset \boldsymbol{L}(\mathcal{D},\mathcal{H})$. Then the following conditions are equivalent.*

(i) *There exists a family $\boldsymbol{T} = \{T_\xi\}_{\xi\in X} \subset \boldsymbol{B}(\mathcal{H})$ of commuting contractions having a unitary dilation and such that (1.2) holds.*

(ii) *For all integers $m, n \geqslant 1$, for all maps $\lambda_1, \dots, \lambda_m \in \mathbb{C}^n[\mathbb{Z}[X]_+]$ satisfying (2.1) and for all maps $h_1, \dots, h_m \in \mathcal{D}[\mathbb{Z}[X]_+]$, the following inequality holds:*

$$\sum_{k,l=1}^{m} \sum_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{s},\boldsymbol{t}\in\mathbb{Z}[X]_+} \langle A_{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{pos}}+\boldsymbol{s}} h_k(\boldsymbol{s}), A_{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{neg}}+\boldsymbol{t}} h_l(\boldsymbol{t})\rangle \langle \lambda_k(\boldsymbol{x}), \lambda_l(\boldsymbol{y})\rangle \geqslant 0.$$

(iii) *For any integer $n \geqslant 1$ and for all maps $h_1, \dots, h_n \in \mathcal{D}[\mathbb{Z}[X]_+^2]$ such that*

$$\sum_{j=1}^{n} \sum_{\substack{\boldsymbol{x},\boldsymbol{y}\in\mathbb{Z}[X]_+,\\ \boldsymbol{x}-\boldsymbol{y}=\boldsymbol{u}}} \sum_{\boldsymbol{s},\boldsymbol{t}\in\mathbb{Z}[X]_+} (A_{\boldsymbol{s}} h_j(\boldsymbol{x},\boldsymbol{s})) \otimes (A_{\boldsymbol{t}} h_j(\boldsymbol{y},\boldsymbol{t})) = 0, \quad \boldsymbol{u} \in \mathbb{Z}[X]_\pm^{\mathrm{C}}, \tag{4.1}$$

the following inequality holds:

$$\sum_{j=1}^{n} \sum_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{s},\boldsymbol{t}\in\mathbb{Z}[X]_+} \langle A_{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{pos}}+\boldsymbol{s}} h_j(\boldsymbol{x},\boldsymbol{s}), A_{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{neg}}+\boldsymbol{t}} h_j(\boldsymbol{y},\boldsymbol{t})\rangle \geqslant 0.$$

Proof. (i) $\Rightarrow$ (ii). Take $h_1, \dots, h_m \in \mathcal{D}[\mathbb{Z}[X]_+]$ and $\lambda_1, \dots, \lambda_m \in \mathbb{C}^n[\mathbb{Z}[X]_+]$ satisfying (2.1). Define

$$v_k = \sum_{\boldsymbol{s} \in \mathbb{Z}[X]_+} A_{\boldsymbol{s}} h_k(\boldsymbol{s}), \quad k = 1, \dots, m.$$

Applying the implication (a) $\Rightarrow$ (b) of Lemma 2.1 and using (1.2), we get

$$\begin{aligned} 0 &\leqslant \sum_{k,l=1}^{m} \sum_{\boldsymbol{x},\boldsymbol{y} \in \mathbb{Z}[X]_+} \langle \boldsymbol{T}^{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{pos}}} v_k, \boldsymbol{T}^{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{neg}}} v_l \rangle \langle \lambda_k(\boldsymbol{x}), \lambda_l(\boldsymbol{y}) \rangle \\ &= \sum_{k,l=1}^{m} \sum_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{s},\boldsymbol{t} \in \mathbb{Z}[X]_+} \langle \boldsymbol{T}^{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{pos}}} A_{\boldsymbol{s}} h_k(\boldsymbol{s}), \boldsymbol{T}^{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{neg}}} A_{\boldsymbol{t}} h_l(\boldsymbol{t}) \rangle \langle \lambda_k(\boldsymbol{x}), \lambda_l(\boldsymbol{y}) \rangle \\ &= \sum_{k,l=1}^{m} \sum_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{s},\boldsymbol{t} \in \mathbb{Z}[X]_+} \langle A_{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{pos}}+\boldsymbol{s}} h_k(\boldsymbol{s}), A_{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{neg}}+\boldsymbol{t}} h_l(\boldsymbol{t}) \rangle \langle \lambda_k(\boldsymbol{x}), \lambda_l(\boldsymbol{y}) \rangle. \end{aligned}$$

(ii) $\Rightarrow$ (i). Take $\xi \in X$ and $h_1, h_2 \in \mathcal{D}[\mathbb{Z}[X]_+]$. Define $\lambda_1, \lambda_2 \in \mathbb{C}[\mathbb{Z}[X]_+]$ by

$$\lambda_k(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} = k\boldsymbol{e}_\xi, \\ 0 & \boldsymbol{x} \neq k\boldsymbol{e}_\xi, \end{cases} \qquad k = 1, 2.$$

Note that (2.1) holds for $m = 2$. Hence, by (ii), we have

$$\begin{aligned} 0 &\leqslant \sum_{k,l=1}^{2} \sum_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{s},\boldsymbol{t} \in \mathbb{Z}[X]_+} \langle A_{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{pos}}+\boldsymbol{s}} h_k(\boldsymbol{s}), A_{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{neg}}+\boldsymbol{t}} h_l(\boldsymbol{t}) \rangle \langle \lambda_k(\boldsymbol{x}), \lambda_l(\boldsymbol{y}) \rangle \\ &= \sum_{k,l=1}^{2} \sum_{\boldsymbol{s},\boldsymbol{t} \in \mathbb{Z}[X]_+} \langle A_{(k-l)_{\mathrm{pos}}\boldsymbol{e}_\xi+\boldsymbol{s}} h_k(\boldsymbol{s}), A_{(k-l)_{\mathrm{neg}}\boldsymbol{e}_\xi+\boldsymbol{t}} h_l(\boldsymbol{t}) \rangle. \end{aligned}$$

It follows from Lemma 3.1 that there exists a family

$$\tilde{\boldsymbol{T}} = \{\tilde{T}_\xi\}_{\xi \in X} \subset \boldsymbol{B}(\mathcal{H}_{\boldsymbol{A}})$$

of commuting contractions such that $\tilde{T}_\xi A_{\boldsymbol{x}} = A_{\boldsymbol{e}_\xi+\boldsymbol{x}}$ for all $\xi \in X$ and $\boldsymbol{x} \in \mathbb{Z}[X]_+$. Arguing as in the proof of the implication (i) $\Rightarrow$ (ii), we show that the family $\tilde{\boldsymbol{T}}$ satisfies Lemma 2.1 (b) with $\mathcal{E} = \mathcal{H}_{\boldsymbol{A}}$ (verify (2.2) first for all $v_1, \dots, v_m \in \mathcal{R}_{\boldsymbol{A}}$ and then, using the continuity of $\tilde{T}_\xi$, for all $v_1, \dots, v_m \in \mathcal{H}_{\boldsymbol{A}}$). As a consequence, $\tilde{\boldsymbol{T}}$ has a unitary dilation. Define the family $\boldsymbol{T} = \{T_\xi\}_{\xi \in X} \subset \boldsymbol{B}(\mathcal{H})$ by $T_\xi = \tilde{T}_\xi \oplus I_{\mathcal{H} \ominus \mathcal{H}_{\boldsymbol{A}}}$ for $\xi \in X$. It is now easily seen that $\boldsymbol{T}$ has all the required properties listed in (i).

(i) $\Rightarrow$ (iii). Take $h_1, \dots, h_n \in \mathcal{D}[\mathbb{Z}[X]_+^2]$ satisfying (4.1). Define $f_1, \dots, f_n \in \mathcal{H}[\mathbb{Z}[X]_+]$ by

$$f_j(\boldsymbol{x}) = \sum_{\boldsymbol{s} \in \mathbb{Z}[X]_+} A_{\boldsymbol{s}} h_j(\boldsymbol{x}, \boldsymbol{s}), \quad \boldsymbol{x} \in \mathbb{Z}[X]_+, \ j = 1, \dots, n.$$

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Then clearly $f_1, \ldots, f_n$ satisfy (2.3). Applying the implication (a) $\Rightarrow$ (c) of Lemma 2.1 and using (1.2) we get

$$\begin{aligned} 0 &\leqslant \sum_{j=1}^{n} \sum_{\boldsymbol{x},\boldsymbol{y} \in \mathbb{Z}[X]_+} \langle \boldsymbol{T}^{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{pos}}} f_j(\boldsymbol{x}), \boldsymbol{T}^{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{neg}}} f_j(\boldsymbol{y})\rangle \\ &= \sum_{j=1}^{n} \sum_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{s},\boldsymbol{t} \in \mathbb{Z}[X]_+} \langle A_{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{pos}}+\boldsymbol{s}} h_j(\boldsymbol{x},\boldsymbol{s}), A_{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{neg}}+\boldsymbol{t}} h_j(\boldsymbol{y},\boldsymbol{t})\rangle. \end{aligned}$$

(iii) $\Rightarrow$ (i). Take $\xi \in X$ and $h_1, h_2 \in \mathcal{D}[\mathbb{Z}[X]_+]$. Define $f \in \mathcal{D}[\mathbb{Z}[X]_+^2]$ by

$$f(\boldsymbol{x},\boldsymbol{s}) = \begin{cases} h_1(\boldsymbol{s}) & \text{for } \boldsymbol{x} = \boldsymbol{e}_\xi, \\ h_2(\boldsymbol{s}) & \text{for } \boldsymbol{x} = 2\boldsymbol{e}_\xi, \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\sum_{\substack{\boldsymbol{x},\boldsymbol{y} \in \mathbb{Z}[X]_+,\\ \boldsymbol{x}-\boldsymbol{y}=\boldsymbol{u}}} \sum_{\boldsymbol{s},\boldsymbol{t} \in \mathbb{Z}[X]_+} (A_{\boldsymbol{s}} f(\boldsymbol{x},\boldsymbol{s})) \otimes (A_{\boldsymbol{t}} f(\boldsymbol{y},\boldsymbol{t})) = 0 \quad \text{for all } \boldsymbol{u} \in \mathbb{Z}[X]_\pm^{\mathrm{C}}.$$

Hence, by (iii), we get

$$\begin{aligned} 0 &\leqslant \sum_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{s},\boldsymbol{t} \in \mathbb{Z}[X]_+} \langle A_{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{pos}}+\boldsymbol{s}} f(\boldsymbol{x},\boldsymbol{s}), A_{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{neg}}+\boldsymbol{t}} f(\boldsymbol{y},\boldsymbol{t})\rangle \\ &= \sum_{k,l=1}^{2} \sum_{\boldsymbol{s},\boldsymbol{t} \in \mathbb{Z}[X]_+} \langle A_{(k-l)_{\mathrm{pos}} e_\xi+\boldsymbol{s}} h_k(\boldsymbol{s}), A_{(k-l)_{\mathrm{neg}} e_\xi+\boldsymbol{t}} h_l(\boldsymbol{t})\rangle. \end{aligned}$$

According to Lemma 3.1 there exists a family $\tilde{\boldsymbol{T}} = \{\tilde{T}_\xi\}_{\xi \in X} \subset \boldsymbol{B}(\mathcal{H}_{\boldsymbol{A}})$ of commuting contractions such that $\tilde{T}_\xi A_{\boldsymbol{x}} = A_{\boldsymbol{e}_\xi+\boldsymbol{x}}$ for all $\xi \in X$ and $\boldsymbol{x} \in \mathbb{Z}[X]_+$. Arguing as in the proof of the implication (i) $\Rightarrow$ (iii), we show that the family $\{\tilde{T}_\xi|_{\mathcal{R}_{\boldsymbol{A}}}\}_{\xi \in X} \subset \boldsymbol{B}(\mathcal{R}_{\boldsymbol{A}})$ satisfies Lemma 2.1 (c) with $\mathcal{E} = \mathcal{R}_{\boldsymbol{A}}$. Applying Lemma 2.1 and then exploiting the continuity of $\tilde{T}_\xi$, we deduce that the family $\tilde{\boldsymbol{T}}$ has a unitary dilation. As a consequence, the family $\boldsymbol{T} = \{T_\xi\}_{\xi \in X} \subset \boldsymbol{B}(\mathcal{H})$ defined by $T_\xi = \tilde{T}_\xi \oplus I_{\mathcal{H} \ominus \mathcal{H}_{\boldsymbol{A}}}$, $\xi \in X$, has all the required properties listed in (i). □

Corollary 4.2. *Suppose that* $\{A_{\sigma,\boldsymbol{x}}\}_{\boldsymbol{x} \in \mathbb{Z}[X]_+} \subset \boldsymbol{L}(\mathcal{D},\mathcal{H})$, $\sigma \in \Sigma$, *is a net of families of operators and* $\{A_{\boldsymbol{x}}\}_{\boldsymbol{x} \in \mathbb{Z}[X]_+} \subset \boldsymbol{L}(\mathcal{D},\mathcal{H})$ *is a family such that*

$$\lim_{\sigma \in \Sigma} A_{\sigma,\boldsymbol{x}} h = A_{\boldsymbol{x}} h, \quad h \in \mathcal{D},\ \boldsymbol{x} \in \mathbb{Z}[X]_+.$$

If for every $\sigma \in \Sigma$ *there exists a family* $\boldsymbol{T}_\sigma = \{T_{\sigma,\xi}\}_{\xi \in X} \subset \boldsymbol{B}(\mathcal{H})$ *of commuting contractions having a unitary dilation and such that* $A_{\sigma,\boldsymbol{x}} = \boldsymbol{T}_\sigma^{\boldsymbol{x}} A_{\sigma,\boldsymbol{0}}$ *for all* $\boldsymbol{x} \in \mathbb{Z}[X]_+$, *then there exists a family* $\boldsymbol{T} = \{T_\xi\}_{\xi \in X} \subset \boldsymbol{B}(\mathcal{H})$ *of commuting contractions having a unitary dilation and such that* $A_{\boldsymbol{x}} = \boldsymbol{T}^{\boldsymbol{x}} A_{\boldsymbol{0}}$ *for all* $\boldsymbol{x} \in \mathbb{Z}[X]_+$.

Proof. By Theorem 4.1, the family $\{A_{\sigma,\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathbb{Z}[X]_+}$ satisfies Theorem 4.1 (ii). After passing to the limit with σ, we see that the limit family $\{A_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathbb{Z}[X]_+}$ satisfies the same condition, which, by Theorem 4.1, completes the proof. □

As shown below, to find a solution $\boldsymbol{T}$ to (1.2) which has a unitary dilation, it is sufficient to do this for every double-truncated system $\{A_{\boldsymbol{x}}|_{\mathcal{C}}\}_{\boldsymbol{x}\in\mathbb{Z}[Y]_+}$, where $\mathcal{C}$ is a finite-dimensional linear subspace of $\mathcal{D}$ and Y is a finite subset of X.

Corollary 4.3. *If $\{A_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathbb{Z}[X]_+}\subset \boldsymbol{L}(\mathcal{D},\mathcal{H})$, then the following conditions are equivalent:*

(i) *there exists a family $\boldsymbol{T}=\{T_\xi\}_{\xi\in X}\subset \boldsymbol{B}(\mathcal{H})$ of commuting contractions having a unitary dilation and such that (1.2) holds;*

(ii) *for every finite subset Y of X, there exists a family $\boldsymbol{T}_Y=\{T_{Y,\xi}\}_{\xi\in Y}\subset \boldsymbol{B}(\mathcal{H})$ of commuting contractions having a unitary dilation and such that $A_{\boldsymbol{x}}=\boldsymbol{T}_Y^{\boldsymbol{x}}A_{\boldsymbol{0}}$ for all $\boldsymbol{x}\in\mathbb{Z}[Y]_+$;*

(iii) *for every finite-dimensional linear subspace $\mathcal{C}$ of $\mathcal{D}$, there exists a family $\boldsymbol{T}_{\mathcal{C}}=\{T_{\mathcal{C},\xi}\}_{\xi\in X}\subset \boldsymbol{B}(\mathcal{H})$ of commuting contractions having a unitary dilation and such that $A_{\boldsymbol{x}}|_{\mathcal{C}}=\boldsymbol{T}_{\mathcal{C}}^{\boldsymbol{x}}A_{\boldsymbol{0}}|_{\mathcal{C}}$ for all $\boldsymbol{x}\in\mathbb{Z}[X]_+$.*

Proof. Since $\mathbb{Z}[X]=\bigcup\{\mathbb{Z}[Y] : Y \text{ is a finite subset of } X\}$ and

$$\mathbb{Z}[X]\setminus\mathbb{Z}[X]_\pm=\bigcup\{\mathbb{Z}[Y]\setminus\mathbb{Z}[Y]_\pm : Y \text{ is a finite subset of } X\},$$

we may apply either condition (ii) or (iii) of Theorem 4.1 to get the equivalences (i) ⇔ (ii) and (i) ⇔ (iii). □

5. Concluding remarks

As mentioned in § 1, our considerations also concern vector processes, which are the subject of [**15**]. To make this precise, let us recall the content of [**15**, Theorem C (c)]: a family $\{h_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathbb{Z}[X]_+}$ of vectors in $\mathcal{H}$ is of the form (1.3), where $\boldsymbol{T}=\{T_\xi\}_{\xi\in X}\subset\boldsymbol{B}(\mathcal{H})$ is a family of commuting contractions having a regular unitary dilation (this is a kind of restriction which we refer to on page 690) [**4**, **27**], if and only if

$$\sum_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{s},\boldsymbol{t}\in\mathbb{Z}[X]_+} c(\boldsymbol{x},\boldsymbol{s})\overline{c(\boldsymbol{y},\boldsymbol{t})}\langle h_{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{pos}}+\boldsymbol{s}}, h_{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{neg}}+\boldsymbol{t}}\rangle \geqslant 0,\quad c\in\mathbb{C}[\mathbb{Z}[X]_+^2].$$

A vector version of Theorem 4.1 (iii) is as follows (with $\mathcal{D}=\mathbb{C}$ and $A_{\boldsymbol{x}}=h_{\boldsymbol{x}}\otimes 1$).

(iii*) For any integer $n\geqslant 1$ and for all functions $c_1,\dots,c_n\in\mathbb{C}[\mathbb{Z}[X]_+^2]$ such that

$$\sum_{j=1}^{n}\sum_{\substack{\boldsymbol{x},\boldsymbol{y}\in\mathbb{Z}[X]_+,\\ \boldsymbol{x}-\boldsymbol{y}=\boldsymbol{u}}}\sum_{\boldsymbol{s},\boldsymbol{t}\in\mathbb{Z}[X]_+} c_j(\boldsymbol{x},\boldsymbol{s})\overline{c_j(\boldsymbol{y},\boldsymbol{t})}h_{\boldsymbol{s}}\otimes h_{\boldsymbol{t}}=0,\quad \boldsymbol{u}\in\mathbb{Z}[X]_\pm^{\mathrm{C}},$$

the following inequality holds:

$$\sum_{j=1}^{n} \sum_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{s},\boldsymbol{t} \in \mathbb{Z}[X]_+} c_j(\boldsymbol{x},\boldsymbol{s})\overline{c_j(\boldsymbol{y},\boldsymbol{t})}\langle h_{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{pos}}+\boldsymbol{s}}, h_{(\boldsymbol{x}-\boldsymbol{y})_{\mathrm{neg}}+\boldsymbol{t}}\rangle \geqslant 0.$$

Comparing this with [**15**, Theorem C (c)], it is evident that the latter implies our condition (iii*).

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