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ON THE ISOPERIMETRIC PROBLEM FOR THE LAPLACIAN WITH ROBIN AND WENTZELL BOUNDARY CONDITIONS

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We consider the eigenvalues of the Laplacian $-\Delta u = \lambda u$ in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, equipped with Robin boundary conditions

$$\frac{\partial u}{\partial \nu} + \alpha u = 0$$

or generalized Wentzell boundary conditions

$$\Delta u + \beta \frac{\partial u}{\partial \nu} + \gamma u = 0,$$

on $\partial\Omega$, where ν is the outer unit normal to $\partial\Omega$, and for us α, β, γ are nonzero constants. In this context, isoperimetric problems, also known as shape optimization problems, involve minimizing (or maximizing) a given eigenvalue with respect to the domain Ω (assumed to have a fixed volume); see, for example, the survey article of Payne [11].

The classical Faber–Krahn inequality states that the first eigenvalue of the Dirichlet Laplacian is smallest when Ω is a ball. This was recently extended to the Robin case when $\alpha > 0$ by Daners [2]. Our first result is that in this case the ball is the *unique* domain with this property, at least amongst all domains of class C^2. The method of proof uses a functional of the level sets of the first eigenfunction, together with some tools from geometric measure theory, to estimate the first eigenvalue from below. This is combined with a rearrangement of the ball's eigenfunction onto the domain Ω and the usual isoperimetric inequality.

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For the second Robin eigenvalue, we prove that the unique minimizing domain when $\alpha > 0$ is the disjoint union of two equal balls (which is again the same as in the Dirichlet case), and set the proof up so it works for the Robin p-Laplacian. For the higher eigenvalues, we show that it is in general impossible for a minimizer to exist independently of $\alpha > 0$. This is done by considering perturbations of the corresponding Dirichlet and Neumann problems.

When $\alpha < 0$, we prove that every eigenvalue behaves like $-\alpha^2$ as $\alpha \to -\infty$, provided only that Ω is bounded with C^1 boundary. This extends a result of Lou and Zhu [10] for the first eigenvalue. Our proof involves constructing an explicit test function to use in the minimax formula for the nth eigenvalue.

For the Wentzell problem, we prove (or, in most cases, re-prove) general operator properties, including those in the less-studied case $\beta < 0$, where the problem is ill-posed in some sense. (This contrasts with the well-behaved case $\beta > 0$.) In particular, when $\beta < 0$, we give a new proof of the compactness of the resolvent and the structure of the spectrum, at least if $\partial\Omega$ is smooth. This is based on the operator matrix approach of Engel [5] for $\beta > 0$, which de-couples the problem into a Dirichlet-type operator acting in the interior of the domain, and a Dirichlet-to-Neumann action on the boundary of the domain.

We prove Faber–Krahn-type inequalities in the general case $\beta, \gamma \neq 0$. These are based on their Robin counterparts, via the elementary trick of identifying every Wentzell eigenvalue as that of a suitable Robin problem, together with a type of fixed point argument. In the 'best' case $\beta, \gamma > 0$ we exploit this further to establish a type of equivalence property between the Wentzell and Robin minimizers for all eigenvalues. In particular, this yields a minimizer of the second Wentzell eigenvalue. Finally, we also prove a Cheeger-type inequality for the first eigenvalue in this case (see [1, 6]).

The new material in this thesis has been published in [3, 4, 7–9]. The thesis itself is available online at http://hdl.handle.net/2123/5972.

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