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SOME COMBINATORIAL THEOREMS ON MONOTONICITY

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1. **Introduction.** P. Erdös and G. Szekeres [1] proved that from any $\binom{2n-2}{n-1}+1$ points in the plane one can always choose $n+1$ of them which are the vertices of a convex polygon, thus answering a question due to Miss Esther Klein (who later became Mrs. G. Szekeres). Indeed, in [1] they proved that from any $\binom{p+q-2}{p-1}+1$ points, (but not necessarily from fewer) no two with equal abscissae, one can always select $p+1$ of them, say (x_i, y_i), $i=0, 1, \ldots, p$, satisfying

$$\frac{y_1-y_0}{x_1-x_0} \le \frac{y_2-y_1}{x_2-x_1} \le \cdots \le \frac{y_p-y_{p-1}}{x_p-x_{p-1}}, \quad x_0 < x_1 < \cdots < x_p,$$

or some $q+1$ of them satisfying

$$\frac{y_1-y_0}{x_1-x_0} > \frac{y_2-y_1}{x_2-x_1} > \cdots > \frac{y_q-y_{q-1}}{x_q-x_{q-1}}, \quad x_0 < x_1 < \cdots < x_q.$$

In §3 we prove the more general theorem which results when the points are replaced by an arbitrary ordered set and the "slope function" is replaced by an arbitrary real-valued function on the pairs of points. In §2 we describe the problem in terms of real-valued functions on vertices of a digraph D and find a generalization of Gallai's theorem relating monotone paths in D to its chromatic number. As a particular case we obtain another theorem of Erdös and Szekeres concerning monotone subsequences of sequences of real numbers.

2. **A generalization of Gallai's theorem.** A digraph D is an ordered pair (V, E) where $V=V(D)=\{v_1, \ldots, v_n\}$ is the finite vertex-set and $E=E(D)$ (the directed edge-set) is a subset of $\{(v_i, v_j) \mid i\neq j\}$. A sequence $v_0, v_1, \ldots, v_k$ of distinct vertices is called a *path* of length k terminating at v_k if

$$(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$$

are edges of D. If (v_k, v_0) is also an edge then we have a *cycle*.

A mapping $\varphi: V \to X$ from V to an arbitrary set X is an *m-coloring* of D if $|X|=m$ and $\varphi(v)\neq\varphi(w)$ whenever (v, w) is an edge of D. The chromatic number $\chi(D)$ is the least m such that there exists an m-coloring of D.

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T. Gallai [2] proved that if digraph D has no path of length p then $\chi(D) \leq p$.

Let R denote the reals and f any function $f: V \rightarrow R$. The path $v_0, v_1, \ldots, v_k$ is called f-nondecreasing if

$$f(v_0) \leq f(v_1) \leq \cdots \leq f(v_k)$$

and f-decreasing if

$$f(v_0) > f(v_1) > \cdots > f(v_k).$$

We generalize Gallai's theorem to

THEOREM 1. *If there exists a function $f: V \rightarrow R$ so that digraph D contains no f-nondecreasing path of length p and no f-decreasing path of length q then $\chi(D) \leq pq$. (When f is a constant function this reduces to Gallai's theorem.)*

Proof. Split D into subdigraphs (V, E_1) and (V, E_2) where

$$E_1 = \{(v, w) \in E \mid f(v) \leq f(w)\}, \qquad E_2 = \{(v, w) \in E \mid f(v) > f(w)\}.$$

(Of course $E_1 \cap E_2 = \varnothing$ and $E_1 \cup E_2 = E$.) These subgraphs contain no paths of length p or q respectively, so by Gallai's theorem there is a p-coloring of (V, E_1) and a q-coloring of (V, E_2), say

$$\varphi_1: V \rightarrow X_1, \quad |X_1| = p; \qquad \varphi_2: V \rightarrow X_2, \quad |X_2| = q.$$

The mapping

$$\varphi: V \rightarrow X_1 \times X_2, \qquad \varphi(v) = (\varphi_1(v), \varphi_2(v))$$

is clearly a pq-coloring of D. Hence $\chi(D) \leq pq$.

A digraph D is transitive tournament if its n vertices can be labelled $v_1, v_2, \ldots, v_n$ in such a way that $(v_i, v_j) \in E$ if and only if $i<j$. Then $\chi(D)=n$. Since any sequence of real numbers $a_1, a_2, \ldots, a_n$ can be considered as the values of a function on the vertices of a transitive tournament, we have immediately

COROLLARY (*Erdös and Szekeres* [1]). *Every real sequence of length more than pq contains a nondecreasing subsequence of length more than p or a decreasing subsequence of length more than q.*

3. **A generalization of the Erdös-Szekeres theorem.** Let $D_1=(V_1, E_1)$ and $D_2=(V_2, E_2)$ be digraphs. We call a mapping $\varphi: V_1 \rightarrow V_2$ compatible provided [3]

$$(\varphi(v), \varphi(w)) \in E_2 \quad \text{whenever } (v, w) \in E_1.$$

A digraph without cycles is *acyclic*.

Let $D_{p,q}$ denote the digraph with vertex set

$$V_{pq} = \{(i, j) \mid i = 0, 1, \ldots, p-1; \quad j = 0, 1, \ldots, q-1\}$$

and edge set

$$E_{p,q} = \{((a, b,)(c, d)) \mid a < c \quad \text{or } b < d\}.$$

LEMMA 1. *Let $D=(V, E)$ be an acyclic digraph and $f\colon V \to R$ be a function so that D has no f-nondecreasing path of length p and no f-decreasing path of length q. Then there is a compatible mapping $\varphi\colon V \to V_{p,q}$.*

Proof. We construct such a mapping. For any $v \in V$, let $\alpha(v)$ resp. $\beta(v)$ denote the maximum length of f-nondecreasing resp. f-decreasing paths terminating at v. Consider the mapping

$$\varphi\colon V \to V_{p,q}$$

defined by

$$\varphi(v) = (\alpha(v), \beta(v)) \in V_{p,q} \quad \text{for } v \in V.$$

If $(v, w) \in E$, then, because D is acyclic, $\alpha(v) < \alpha(w)$ or $\beta(v) < \beta(w)$, i.e. φ is a compatible mapping.

Let $v=(x_1, y_1)$, $w=(x_2, y_2)$ be vertices in $V_{p,q}$. We write $v \le w$ if $x_1 \le x_2$ and $y_1 \le y_2$. A subset $A \subset V_{p,q}$ is called *thin* if no two distinct vertices v, w in A satisfy $v \le w$. Obviously, the vertices in a thin subset can be ordered

$$(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$$

in such a way that

$$x_1 < x_2 < \cdots < x_k; \qquad y_1 > y_2 > \cdots > y_k.$$

Hence the number of thin subsets of cardinality k is $\binom{p}{k}\binom{q}{k}$, and $V_{p,q}$ has

$$\sum_{k=0}^{\infty} \binom{p}{k}\binom{q}{k} = \binom{p+q}{p}$$

thin subsets.

We are about to consider mappings from the edge-set of a digraph to the reals; these are conveniently described in terms of the dual digraph. The dual of digraph D is the digraph D^*, whose vertices are the edges of D and two vertices of D^*, say (a, b), (c, d), (these are edges of D) are joined by a directed edge from the former to the latter whenever $b=c$.

THEOREM 2. *If the acyclic digraph D has chromatic number $\chi(D) > \binom{p+q}{p}$ then there is no compatible mapping*

$$V(D^*) \to V_{p,q}.$$

In other words, for any mapping

$$\varphi\colon E(D) \to V_{p,q}$$

(from the edges of D to the vertices of $D_{p,q}$) there exist two "adjacent" edges (a, b), (b, c) of D such that $\varphi((a, b)) \ge \varphi((b, c))$, i.e.

$$\varphi((a, b)) = (x_1, y_1), \qquad \varphi((b, c)) = (x_2, y_2)$$

with $x_1 \ge x_2$, $y_1 \ge y_2$.

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Proof. Let

$$\varphi: V(D^*) \to V_{p,q}$$

be any mapping. For each $v \in V(D)$ consider the set

$$P_v = \{\varphi((u, v)) \mid (u, v) \in E(D)\}.$$

Let X_v denote the "maximal" points of P_v:

$$X_v = \{Z \in P_v \mid Z' \geq Z \text{ for no } Z'(\neq Z) \in P_v\}.$$

Of course the sets X_v are thin subsets of $V_{p,q}$. Because there are only $\binom{p+q}{p}$ such thin subsets, and (by hypothesis) this number is less than $\chi(D)$, the assignment

$$v \to X_v$$

cannot be a coloring. Hence there must be an edge (u, w) of D for which $X_u = X_w$. The point $\varphi((u, w))$ is in P_w (by definition), so there is a point $Z \in X_w$ with $Z \geq \varphi((u, w))$. But $X_w = X_u \subset P_u$, so Z is a point in P_u. From the definition of P_u, there is an edge (t, u) of D for which $Z = \varphi((t, u))$. Thus

$$\varphi((t, u)) \geq \varphi((u, w)),$$

so φ is not compatible.

Combining Theorem 2 and Lemma 1 (applied to D^*), we have

THEOREM 3. *Let F: $E(D) \to R$ be a function from the edge-set of the acyclic digraph D to the reals. If $\chi(D) > \binom{p+q}{p}$ then D contains either a path $v_0, v_1, \ldots, v_{p+1} \in V(D)$ with*

$$f((v_0, v_1)) \leq f((v_1, v_2)) \leq \ldots \leq f((v_p, v_{p+1}))$$

or a path $v_0, v_1, \ldots, v_{q+1} \in V(D)$ with

$$f((v_0, v_1)) > f((v_1, v_2)) > \cdots > f((v_q, v_{q+1})).$$

Consider the transitive tournament T_n with vertex set $V(T_n) = \{v_1, \ldots, v_n\}$ and edge-set

$$E(T_n) = \{(v_i, v_j) \mid 1 \leq i < j \leq n\}.$$

Let $F(p, q)$ be the smallest integer such that if $n > F(p, q)$, then, for any function

$$f: E(T_n) \to R,$$

T_n contains either a path $v_0, v_1, \ldots, v_p \in V(T_n)$ with

$$f((v_0, v_1)) \leq f((v_1, v_2)) \leq \cdots \leq f((v_{p-1}, v_p)), \tag{1}$$

or a path $v_0, v_1, \ldots, v_q \in V(T_n)$ with

$$f((v_0, v_1)) > f((v_1, v_2)) > \cdots > f((v_{q-1}, v_q)). \tag{2}$$

As an immediate consequence of the last theorem we have the following generalization of the Erdös–Szekeres result [1]:

$$F(p, q) \leq \binom{p+q-2}{p-1}.$$

Indeed, we have

THEOREM 4. $F(p, q) = \binom{p+q-2}{p-1}$.

Proof. It remains to show $F(p, q) \geq \binom{p+q-2}{p-1}$. First observe that $F(p, 2)=p$ and $F(2, q)=q$, so it suffices to show that

$$F(p, q) \geq F(p-1, q)+F(p, q-1). \tag{3}$$

For convenience, we will say that the function

$$f\colon E(T_n) \to R$$

has property $[p, q]$ provided T_n contains no path $v_0, \ldots, v_p$, of length p, satisfying (1) and no path $v_0, \ldots, v_q$, of length q, satisfying (2).

Let $l=F(p-1, q)$, $m=F(p, q-1)$, $n=l+m$, and T_l, T_m, T_n be the transitive tournaments:

$$\begin{aligned} V(T_l) &= \{v_1, \ldots, v_l\}, & E(T_l) &= \{(v_i, v_j) \mid 1 \leq i < j \leq l\}; \\ V(T_m) &= \{v_{l+1}, \ldots, v_{l+m}\}, & E(T_m) &= \{(v_i, v_j) \mid l+1 \leq i < j \leq l+m\}; \\ V(T_n) &= \{v_1, \ldots, v_n\}, & E(T_n) &= \{(v_i, v_j) \mid 1 \leq i < j \leq n\}. \end{aligned}$$

From the definitions of l, m, there exist functions

$$g\colon E(T_l) \to R,\ h\colon E(T_m) \to R$$

such that g has property $[p-1, q]$, and h has property $[p, q-1]$.

Choose a number M greater than all the values of g and h, and define

$$f\colon E(T_n) \to R$$

by

$$f((v_i, v_j)) = \begin{cases} g((v_i, v_j)) & \text{if } 1 \leq i < j \leq l, \\ h((v_i, v_j)) & \text{if } l+1 \leq i < j \leq n = l+m, \\ M & \text{if } 1 \leq i \leq l < j \leq n. \end{cases}$$

It is easy to see that f has property $[p, q]$, and hence $F(p, q) \geq n = l+m$, establishing inequality (3).

4. **Ordered hypergraphs.** If digraph D is acyclic there is an ordering $v_1, v_2 \ldots$ of the vertices such that $(v_i, v_j) \in E(D)$ implies $i<j$. Let $\mathscr{P}(k, D)$ be the set of k-tuples of vertices which form paths. Theorem 1 implies that if $\chi(D) > (k-1)^2$ then

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given any function f: $V(D) \to R$ there corresponds a k-tuple $P \in \mathscr{P}(k, D)$ such that f is monotone (nondecreasing or decreasing) on P. This suggests the following definitions. An ordered hypergraph is a pair (V, H) where V is an ordered set and H a set of some subsets of V. An *ordered hypergraph* (V, H) is said to have *property* M if, given any function f: $V \to R$ there is always an $X \in H$ such that f is monotone on X. For convenience we use the notation $[V]^k$ to denote the set of all k-element subsets of V. An ordered hypergraph (V, H) is called an ordered k-graph if $H \subset [V]^k$. (An ordered 2-graph is simply an acyclic digraph.)

In terms of these notations, the corollary of Theorem 1 states: If $|V| > (k-1)^2$ and $H = [V]^k$ then the ordered hypergraph (V, H) has property M. It follows that, if $M(k)$ is the smallest m for which there exists an ordered k-graph (V, H), with $|H| = m$, having property M, then

$$M(k) \leq \binom{k^2-2k+2}{k}.$$

On the other side, we have

THEOREM 5. $M(k) \geq k!/2$.

Proof. Let (V, H) be an ordered k-graph with $|V| = n$ and $|H| = m$. Consider the one-to-one functions f: $V \to \{1, 2, \ldots, n\}$. We proceed to show that if $m < k!/2$ then one of these functions is monotone on no k-tuple of H, i.e. the k-graph does not have property M.

Given any k-tuple of H, there are precisely $2\binom{n}{k}(n-k)! = 2n!k!$ functions which are monotone on this k-tuple. Hence there are at most $m2n!/k! < n!$ functions each monotone on some k-tuple. Since there are just $n!$ functions altogether, one of the functions is monotone on no k-tuple of H.

Returning to a digraph $D = (V, E)$, let H be the set of all k-tuples of V which form paths. If $\chi(D) > (k-1)^2$ we know, from Theorem 1, that the hypergraph (V, H) has property M. From Theorem 5 we have

COROLLARY. *An acyclic digraph D with $\chi(D) > (k-1)^2$ contains at least $k!/2$ paths of length $k-1$.*

5. **Concluding remarks.** There are some points in our exposition which are perhaps worth mentioning. The functions used need not be real-valued; actually, any ordered set could serve as the range. The functions were used to order the domain. In fact, we did not need the vertex-sets of the hypergraphs to be ordered. What we did was to compare two distinct orderings of the same finite set. The fact that one of the orderings was a priori fixed was not essential. Moreover, we could compare more than two orderings, i.e. we could deal with more than one function, demanding that they be simultaneously monotone along certain paths. This remark applies also to the earlier sections. We close with a related open problem. Let V

be the vertex-set and $[V]^2$ be the edge-set of a complete graph. Let $g(n)$ denote the least integer such that if $|V| \geq g(n)$ then for any function $f: [V]^2 \to R$ there is a path $v_0, v_1, \ldots, v_n$ of length n on which f is monotone nondecreasing, i.e.

$$f((v_0, v_1)) \leq f((v_1, v_2)) \leq \cdots \leq f((v_{n-1}, v_n)).$$

(Note that the graph has undirected edges). The problem is, of course, to determine the function $g(n)$. R. Graham (oral communication) described a function which shows that $g(n) > ([4/3] - \varepsilon)n$; D. Kleitman (oral communication) established that $g(n) < cn^2$.

If we allow the "paths" to pass through a vertex more than once (but the edges are still distinct) then the corresponding function, call it $g_1(n)$, was determined by Graham and Kleitman: $g_1(n) = n+1$ for $n \neq 3, 5$; $g(3) = 3$, $g(5) = 5$.

We remark that some results related to this paper were also obtained by V. Rödl (Prague).

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