SPECHT MODULES FOR FINITE REFLECTION GROUPS

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1. Introduction. Over fields of characteristic zero, there are well known constructions of the irreducible representations, due to A. Young, and of irreducible modules, called Specht modules, due to W. Specht, for the symmetric groups S_n which are based on elegant combinatorial concepts connected with Young tableaux etc. (see, e.g. [**13**]). James [**12**] extended these ideas to construct irreducible representations and modules over an arbitrary field. Al-Aamily, Morris and Peel [**1**] showed how this construction could be extended to cover the Weyl groups of type B_n. In [**14**] Morris described a possible extension of James' work for Weyl groups in general. Later, the present author and Morris [**8**] gave an alternative generalisation of James' work which is an extended improvement and extension of the original approach suggested by Morris. We now give a possible extension of James' work for finite reflection groups in general.

2. Some general results on finite reflection groups. In this section we establish the notation and state some results on finite reflection groups which are required later. Standard references for this material are N. Bourbaki [**3**], R. W. Carter [**4**], J. E. Humphreys [**10**] [**11**], Grove and Benson [**7**].

Let V be l-dimensional Euclidean space over the real field $\mathbb{R}$ equipped with a positive definite inner product (,). For $\alpha \in V$, $\alpha \neq 0$, let τ_α be the *reflection* in the hyperplane orthogonal to α, that is, τ_α is the linear transformation on V defined by

$$\tau_\alpha(v) = v - 2\frac{(\alpha, v)}{(\alpha, \alpha)}\alpha$$

for all $v \in V$. Let Φ be a root system in V and π be a simple system in Φ with corresponding positive system Φ^+ and negative system Φ^-. Then the finite reflection group

$$\mathcal{W} = \mathcal{W}(\Phi) = \langle \tau_\alpha \mid \tau_\alpha^2 = e, (\tau_\alpha \tau_\beta)^{m_{\alpha\beta}} = e, \alpha, \beta \in \pi \text{ and } \alpha \neq \beta \rangle$$

where e is the identity element of $\mathcal{W}$ and $m_{\alpha\beta}$ is the order of $\tau_\alpha\tau_\beta$. Let $l(w)$ denote the *length* of w and the *sign* of w, $s(w)$, is defined by $s(w) = (-1)^{l(w)}$, $w \in \mathcal{W}$.

We note the following facts which are required later.

2.1. There are $|\mathcal{W}|$ simple systems (positive systems) in Φ given by $w\pi(w\Phi^+)$, $w \in \mathcal{W}$. The group $\mathcal{W}$ acts transitively on the set of simple systems.

2.2. To each root system Φ, there corresponds a graph Γ called the *Coxeter graph* (or *Dynkin diagram*) of $\mathcal{W}$, whose nodes are in one-to-one correspondence with the elements of π. A finite reflection group is *irreducible* if its Coxeter graph is connected. Finite irreducible reflection groups have been classified and correspond to root systems of type $A_l (l \geq 1)$, $B_l (l \geq 2)$, $C_l (l \geq 3)$, $D_l (l \geq 4)$, E_6, E_7, E_8, F_4, G_2, H_3, H_4, $I_2(p)$ ($p = 5$ or $p \geq 7$). For example $\mathcal{W}(A_l) \cong S_{l+1}$, the symmetric group on the set $\{1, 2, \ldots, l+1\}$. As our aim in

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this paper is to generalise ideas from the symmetric groups, the root system and simple system are given in this case. If $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_{l+1}\}$ is the standard basis for $\mathbb{R}^{l+1}$, then

$$\pi = \{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_l = \epsilon_l - \epsilon_{l+1}\}$$

$$\Phi = \{\epsilon_i - \epsilon_j \mid 1 \le i, j \le l+1\}$$

$$\Phi^+ = \{\epsilon_i - \epsilon_j \mid 1 \le i < j \le l+1\}$$

2.3. A *subsystem* Ψ of Φ is a subset of Φ which is itself a root system in the space which it spans. A subsystem Ψ is said to be *additively closed* if α, $\beta \in \Psi$, $\alpha + \beta \in \Phi$ implies that $\alpha + \beta \in \Psi$. From now on we assume that Ψ is an additively closed subsystem of Φ. The *finite reflection subgroup* $\mathcal{W}(\Psi)$ of $\mathcal{W}$ corresponding to the subsystem Ψ is the subgroup of $\mathcal{W}$ generated by the τ_α, $\alpha \in \Psi$. If Ψ and Υ are subsystems of Φ which span subspaces U and W of V respectively, then Ψ and Υ are *isomorphic* if there exists a vector space isomorphism $\theta: U \to W$ such that $\theta(\Psi) = \Upsilon$ and

$$\frac{(\theta(\alpha), \theta(\beta))}{(\theta(\beta), \theta(\beta))} = \frac{(\alpha, \beta)}{(\beta, \beta)} \qquad \text{for all } \alpha, \beta \in \Psi.$$

It follows that

$$\mathcal{W}(\Psi) \cong \mathcal{W}(\Upsilon) = \theta \mathcal{W}(\Psi)\theta^{-1}.$$

The subsystems Ψ and Υ are *conjugate under* $\mathcal{W}$ if $\Upsilon = w\Psi$ for some $w \in \mathcal{W}$; in this case $\mathcal{W}(w\Psi) = w\mathcal{W}(\Psi)w^{-1}$ since $\tau_{w(\alpha)} = w\tau_\alpha w^{-1}$ for $\alpha \in \Psi$. Note that isomorphic subsystems are not necessarily conjugate.

2.4. The graphs which are Dynkin diagrams of subsystems of Φ may be obtained up to conjugacy by a standard algorithm due independently to E. B. Dynkin, A. Borel and J. de Siebenthal (see e.g. [**4**]).

2.5. If $w \in \mathcal{W}$ and U is the subspace of V composed of all vectors fixed by w, then w is a product of reflections corresponding to roots in the orthogonal complement $U^\perp$ of U. [**4**]

2.6. The simple system J of Ψ can always be chosen such that $J \subset \Phi^+$. [**15**]

2.7. The set $D_\Psi = \{w \in \mathcal{W} \mid w(j) \in \Phi^+ \text{ for all } j \in J\}$ is a *distinguished set of coset representatives* of $\mathcal{W}(\Psi)$ in $\mathcal{W}$, that is, each element $w \in \mathcal{W}$ has unique expression of the form $d_\Psi w_\Psi$, where $d_\Psi \in D_\Psi$ and $w_\Psi \in \mathcal{W}(\Psi)$ and furthermore d_Ψ is the element of minimal length in the coset $d_\Psi \mathcal{W}(\Psi)$. [**15**]

3. Specht modules for finite reflection groups. Let Φ be a root system with simple system π and Coxeter graph Γ and let Ψ be a subsystem of Φ with simple system $J \subset \Phi^+$ and Coxeter graph Δ. If $\Psi = \bigcup_{i=1}^{r} \Psi_i$, where the Ψ_i are the indecomposable components of Ψ, then let J_i be a simple system in Ψ_i $(i = 1, 2, \ldots, r)$ and $J = \bigcup_{i=1}^{r} J_i$. Let $\Psi^\perp$ be the largest subsystem in Φ orthogonal to Ψ and let $J^\perp \subset \Phi^+$ be the simple system of $\Psi^\perp$.

Let Ψ' be a subsystem of Φ which is contained in $\Phi \backslash \Psi$, with simple system $J' \subset \Phi^+$ and Coxeter graph Δ'. If $\Psi' = \bigcup_{i=1}^{s} \Psi'_i$, where the Ψ'_i are the indecomposable components

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of Ψ' then let J_i' be a simple system in $\Psi_i'(i=1,2,\ldots,s)$ and $J'=\bigcup_{i=1}^{s} J_i'$. Let $\Psi'^{\perp}$ be the largest subsystem in Φ orthogonal to Ψ' and let $J'^{\perp}\subset\Phi^+$ be the simple system of $\Psi'^{\perp}$.

Let $\bar{J}$ stand for the ordered set $\{J_1,J_2,\ldots,J_r;J_1',J_2',\ldots,J_s'\}$, where in addition the elements in each J_i and J_i' are ordered. Let

$$\mathcal{T}_{J,J'}=\{w\bar{J}\mid w\in\mathcal{W}\}$$

Now we consider under what conditions the elements in the set $\mathcal{T}_{J,J'}$ are distinct.

Lemma 3.1. *$|\mathcal{T}_{J,J'}|=|\mathcal{W}|$ if and only if $\mathcal{W}(J^{\perp})\cap\mathcal{W}(J'^{\perp})=\langle e\rangle$.*

Proof. See Lemma 3.1 [**8**].

Now we can give our principal definition.

Definition 3.2. Let Ψ and Ψ' be subsystems of Ψ with simple systems J and J' respectively such that $\Psi'\subseteq\Phi\backslash\Psi$ and $J\subset\Phi^+$, $J'\subset\Phi^+$. The pair $\{J,J'\}$ is called a *useful system* in Φ if $\mathcal{W}(J)\cap\mathcal{W}(J')=\langle e\rangle$ and $\mathcal{W}(J^{\perp})\cap\mathcal{W}(J'^{\perp})=\langle e\rangle$.

Remark 1. If $\{J,J'\}$ is a useful system in Φ, then $\{wJ,wJ'\}$ is also a useful system in Φ, for $w\in\mathcal{W}$. Thus, from now on $\mathcal{T}_{J,J'}$ will be denoted by $\mathcal{T}_{\Delta}$.

Remark 2. If $\{J,J'\}$ is a useful system in Φ then $\Psi\cap\Psi'=\varnothing$ and $\Psi^{\perp}\cap\Psi'^{\perp}=\varnothing$. However the converse is not true in general.

Definition 3.3. Let $\{J,J'\}$ be a useful system in Φ. Then the elements of $\mathcal{T}_{\Delta}$ are called Δ-*tableaux*, the J_i and J_i' are called the *rows* and the *columns* of $\{J,J'\}$ respectively.

Definition 3.4. Two Δ-tableaux $\bar{J}$ and $\bar{K}$ are *row-equivalent*, written $\bar{J}\sim\bar{K}$, if there exists $w\in\mathcal{W}(J)$ such that $\bar{K}=w\bar{J}$. The equivalence class which contains the Δ-tableau $\bar{J}$ is $\{\bar{J}\}$ and is called a Δ-*tabloid*.

Let τ_{Δ} be the set of all Δ-tabloids. It is clear that the number of distinct elements in τ_{Δ} is $[\mathcal{W}:\mathcal{W}(J)]$ and by (2.7) we have

$$\tau_{\Delta}=\{\{d\bar{J}\}\mid d\in D_{\Psi}\}$$

We note that if $\bar{J}=\{J;J'\}$ then $dJ\subset\Phi^+$ but dJ' need not be a subset of Φ^+.

We now give an example to illustrate the construction of a Δ-tabloid. In this example and later examples we use the following notation. If $\pi=\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ is a simple system in Φ and $\alpha\in\Phi$, then $\alpha=\sum_{i=1}^{n}a_i\alpha_i$, where $a_i\in\mathbb{Z}$. From now on α is denoted by $a_1a_2\ldots a_n$ and $\tau_{\alpha_1},\tau_{\alpha_2},\ldots,\tau_{\alpha_n}$ are denoted by $\tau_1,\tau_2,\ldots,\tau_n$ respectively.

Example 3.5. Let $\Phi=\mathbf{D_4}$ with simple system

$$\pi=\{\alpha_1=\epsilon_1-\epsilon_2,\alpha_2=\epsilon_2-\epsilon_3,\alpha_3=\epsilon_3-\epsilon_4,\alpha_4=\epsilon_3+\epsilon_4\}.$$

Let $\Psi_1=\mathbf{A_3}$ be the subsystem of $\mathbf{D_4}$ with $J=\{1000,0100,0010\}$. Let $\Psi'=\mathbf{2A_1}$ be the subsystem of Φ which is contained in $\Phi\backslash\Psi$, with simple system $J'=\{1101,0111\}$. Since

$\mathscr{W}(J) \cap \mathscr{W}(J') = \langle e \rangle$ and $\mathscr{W}(J^{\perp}) \cap \mathscr{W}(J'^{\perp}) = \langle e \rangle$, then $\{J, J'\}$ is a useful system in Φ. Then τ_{Δ} contains the Δ – tabloids

$$
\begin{aligned}
\{\bar{J}\} &= \{1000, 0100, 0010; 1101, 0111\} \\
\{\tau_4 \bar{J}\} &= \{1000, 0101, 0010; 1100, 0110\} \\
\{\tau_2 \tau_4 \bar{J}\} &= \{1100, 0001, 0110; 1000, 0010\} \\
\{\tau_1 \tau_2 \tau_4 \bar{J}\} &= \{0100, 0001, 1110; -1000, 0010\} \\
\{\tau_3 \tau_2 \tau_4 \bar{J}\} &= \{1110, 0001, 0100; 1000, -0010\} \\
\{\tau_1 \tau_3 \tau_2 \tau_4 \bar{J}\} &= \{0110, 0001, 1100; -1000, -0010\} \\
\{\tau_2 \tau_1 \tau_3 \tau_2 \tau_4 \bar{J}\} &= \{0010, 0101, 1000; -1100, -0110\} \\
\{\tau_4 \tau_2 \tau_1 \tau_3 \tau_2 \tau_4 \bar{J}\} &= \{0010, 0100, 1000; -1101, -0111\}.
\end{aligned}
$$

The group $\mathscr{W}$ acts on τ_{Δ} according to

$$\sigma\{\overline{wJ}\} = \{\overline{\sigma wJ}\} \qquad \text{for all } \sigma \in \mathscr{W}.$$

This action is well defined, for if $\{\overline{w_1 J}\} = \{\overline{w_2 J}\}$, then there exists $\rho \in \mathscr{W}(w_1 J)$ such that $\overline{\rho w_1 J} = \overline{w_2 J}$. Hence since $\sigma \rho \sigma^{-1} \in \mathscr{W}(\sigma w_1 J)$ and $\overline{\sigma w_2 J} = \overline{\sigma \rho w_1 J} = (\sigma \rho \sigma^{-1})(\overline{\sigma w_1 J})$, we have $\{\overline{\sigma w_1 J}\} = \{\overline{\sigma w_2 J}\}$.

Now if K is arbitrary field, let M^{Δ} be the K-space whose basis elements are the Δ-tabloids. Extend the action of $\mathscr{W}$ on τ_{Δ} linearly on M^{Δ}, then M^{Δ} becomes a $K\mathscr{W}$-module. Then we have the following lemma.

Lemma 3.6. *The $K\mathscr{W}$-module M^{Δ} is a cyclic $K\mathscr{W}$-module generated by any one tabloid and* $\dim_K M^{\Delta} = [\mathscr{W} : \mathscr{W}(J)]$.

Now we proceed to consider the possibility of constructing a $K\mathscr{W}$-module which corresponds to the Specht module in the case of symmetric groups. In order to do this we need to define a Δ-polytabloid.

Definition 3.7. Let $\{J, J'\}$ be a useful system in Φ. Let

$$\kappa_{J'} = \sum_{\sigma \in \mathscr{W}(J')} s(\sigma)\sigma \quad \text{and} \quad e_{J,J'} = \kappa_{J'}\{\bar{J}\},$$

where s is the sign function defined in Section 2. Then $e_{J,J'}$ is called the generalized Δ-*polytabloid* associated with J.

If $w \in \mathscr{W}(\Phi)$, then

$$
\begin{aligned}
w\kappa_{J'} &= \sum_{\sigma \in \mathscr{W}(J')} s(\sigma) w\sigma \\
&= \sum_{\sigma \in \mathscr{W}(J')} s(\sigma)(w\sigma w^{-1})w \\
&= \left\{ \sum_{\sigma \in \mathscr{W}(wJ')} s(\sigma)\sigma \right\} w.
\end{aligned}
$$

Hence, for all $w \in \mathscr{W}(\Phi)$, we have

$$we_{J,J'} = \kappa_{wJ'}\{\overline{wJ}\} = e_{wJ,wJ'}. \tag{3.1}$$

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Let $S^{J,J'}$ be the subspace of M^Δ generated by $e_{wJ,wJ'}$ where $w \in \mathcal{W}$. Then, by (3.1), $S^{J,J'}$ is a $K\mathcal{W}$-submodule of M^Δ, which is called a *generalized Specht module.* Then we have the following theorem.

THEOREM 3.8. *The $K\mathcal{W}$-module $S^{J,J'}$ is a cyclic submodule generated by any Δ-polytabloid.*

The following proposition notes some isomorphisms between Specht modules.

PROPOSITION 3.9. *Let $\{J,J'\}$ be a useful system in Φ. Then we have the following isomorphisms.*

(i) *If $w \in \mathcal{W}$, then $S^{J,J'} \cong S^{wJ,wJ'}$.*
(ii) *If $w \in \mathcal{W}(J)$, then $S^{J,J'} \cong S^{J,wJ'}$.*
(iii) *If $w \in \mathcal{W}(J')$, then $S^{J,J'} \cong S^{wJ,J'}$.*

Proposition 3.9 says that a generalized Specht module is dependent only on the Dynkin diagrams Δ and Δ' of J and J' respectively, thus, from now on it will be denoted by $S^{\Delta,\Delta'}$.

A Specht module is spanned by the $e_{wJ,wJ'}$ for all $w \in \mathcal{W}$; the next lemma shows that we need only consider a certain subset of $\mathcal{W}$.

LEMMA 3.10. *Let $\{J,J'\}$ be a useful system in Φ. Then $S^{\Delta,\Delta'}$ is generated by $e_{dJ,dJ'}$, where $d \in D_{\Psi'}$.*

Proof. See Lemma 3.10 [**8**].

LEMMA 3.11. *Let $\{J,J'\}$ be a useful system in Φ and let $d \in D_\Psi$. If $\{\overline{dJ}\}$ appears in $e_{J,J'}$ then it appears only once.*

Proof. See Lemma 3.11 [**8**].

COROLLARY 3.12. *If $\{J,J'\}$ is a useful system in Φ, then $e_{J,J'} \neq 0$.*

The following lemma shows that the extra condition $\mathcal{W}(J) \cap \mathcal{W}(J') = \langle e \rangle$ in our definition of a useful system is necessary. Unfortunately this condition which is a group theoretical one is not easily checked and it would be useful if it could be replaced by a criterion in terms of the root system only.

LEMMA 3.13. *If there exists $w \in \mathcal{W}(J) \cap \mathcal{W}(J')$ such that w has order 2, and $s(w) = -1$ then $e_{J,J'} = 0$.*

Proof. See Lemma 3.13 [**8**].

EXAMPLE 3.14. Let $\Phi = \mathbf{B_3}$ and $\pi = \{\alpha_1 = \epsilon_1 - \epsilon_2,\ \alpha_2 = \epsilon_2 - \epsilon_3,\ \alpha_3 = \epsilon_3\}$. Let $\Psi = \mathbf{3A_1}$ be the subsystem of Φ with simple system $J = \{\alpha_1 = \epsilon_1 - \epsilon_2,\ \tilde{\alpha} = \epsilon_1 + \epsilon_2,\ \alpha_3 = \epsilon_3\}$ and let $\Psi' = \mathbf{3A_1}$ be the subsystem of Φ with $J' = \{\alpha_2 = \epsilon_2 - \epsilon_3,\ \alpha_1 + \alpha_2 + \alpha_3 = \epsilon_1,\ \alpha_2 + 2\alpha_3 = \epsilon_2 + \epsilon_3\}$. Then $\Psi \cap \Psi' = \varnothing$. But

$$\mathcal{W}(J) = \{e,\ \tau_1,\ \tau_3,\ \tau_1\tau_3,\ \tau_2\tau_3\tau_1\tau_2\tau_3\tau_1\tau_2,\ \tau_2\tau_3\tau_1\tau_2\tau_3\tau_1\tau_2\tau_1,\ \tau_3\tau_2\tau_3\tau_1\tau_2\tau_3\tau_1\tau_2,\ \tau_3\tau_2\tau_3\tau_1\tau_2\tau_3\tau_1\tau_2\tau_1\},$$

$$\mathcal{W}(J') = \{e,\ \tau_2,\ \tau_1\tau_2\tau_3\tau_2\tau_1,\ \tau_3\tau_2\tau_3,\ \tau_3\tau_2\tau_3\tau_1\tau_2\tau_3\tau_1\tau_2\tau_1,\ \tau_3\tau_2\tau_3\tau_2,\ \tau_3\tau_2\tau_3\tau_1\tau_2\tau_3\tau_2\tau_1,\ \tau_1\tau_2\tau_3\tau_1\tau_2\tau_1\}.$$

It follows that $w = \tau_3\tau_2\tau_3\tau_1\tau_2\tau_3\tau_1\tau_2\tau_1 \in \mathcal{W}(J) \cap \mathcal{W}(J')$ and $e_{J,J'} = 0$.

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LEMMA 3.15. *Let* $\{J, J_1'\}$ *and* $\{J, J_2'\}$ *be useful systems in* Φ. *If* $\Psi_1' \subseteq \Psi_2'$, *then* $S^{J,J_2'}$ *is a* $K\mathscr{W}$*-submodule of* $S^{J,J_1'}$, *where* J_1' *and* J_2' *are simple systems for* Ψ_1' *and* Ψ_2' *respectively.*

Now we consider under what conditions $S^{\Delta,\Delta'}$ is irreducible.

LEMMA 3.16. *Let* $\{J, J'\}$ *be a useful system in* Φ *and let* $d \in D_\Psi$. *Then the following conditions are equivalent:*

(i) $\{\overline{dJ}\}$ *appears with non-zero coefficient in* $e_{J,J'}$.
(ii) *There exists* $\sigma \in \mathscr{W}(J')$ *such that* $\sigma\{\bar{J}\} = \{\overline{dJ}\}$.
(iii) *There exists* $\rho \in \mathscr{W}(J)$ *and* $\sigma \in \mathscr{W}(J')$ *such that* $d = \sigma\rho$.

Proof. See Lemma 3.16 [**8**].

LEMMA 3.17. *Let* $\{J, J'\}$ *be a useful system in* Φ *and let* $d \in D_\Psi$. *If* $\{\overline{dJ}\}$ *appears in* $e_{J,J'}$ *then* $d\Psi \cap \Psi' = \varnothing$.

Proof. See Lemma 3.17 [**8**].

LEMMA 3.18. *Let* $\{J, J'\}$ *be a useful system in* Φ *and let* $d \in D_\Psi$. *Let* $d\,\Psi \cap \Psi' \neq \varnothing$. *Then* $\kappa_{J'}\{\overline{dJ}\} = 0$.

The converse of Lemma 3.17 is not true in general, which leads to the following definition.

DEFINITION 3.19. A useful system $\{J, J'\}$ in Φ is called a *good system* if $d\,\Psi \cap \Psi' = \varnothing$ for $d \in D_\Psi$ then $\{\overline{dJ}\}$ appears with non-zero coefficient in $e_{J,J'}$.

LEMMA 3.20. *Let* $\{J, J'\}$ *be a good system in* Φ *and let* $d \in D_\Psi$.

(i) *If* $\{\overline{dJ}\}$ *does not appear in* $e_{J,J'}$ *then* $\kappa_{J'}\{\overline{dJ}\} = 0$.
(ii) *If* $\{\overline{dJ}\}$ *appears in* $e_{J,J'}$ *then there exists* $\sigma \in \mathscr{W}(J')$ *such that*

$$\kappa_{J'}\{\overline{dJ}\} = s(\sigma)e_{J,J'}$$

Proof. See Lemma 3.20 [**8**].

COROLLARY 3.21. *Let* $\{J, J'\}$ *be a good system in* Φ. *If* $m \in M^\Delta$ *then* $\kappa_{J'}m$ *is a multiple of* $e_{J,J'}$.

We now define a bilinear form $\langle\,,\rangle$ on M^Δ by setting

$$\langle\{\bar{J}_1\}, \{\bar{J}_2\}\rangle = \begin{cases} 1 & \text{if } \{\bar{J}_1\} = \{\bar{J}_2\}, \\ 0 & \text{otherwise.} \end{cases}$$

This is a symmetric, non-singular, $\mathscr{W}$-invariant bilinear form on M^Δ.

Now we can prove James' submodule theorem in this general setting.

THEOREM 3.22. *Let* $\{J, J'\}$ *be a good system in* Φ. *Let* U *be a submodule of* M^Δ. *Then either* $S^{\Delta,\Delta'} \subseteq U$ *or* $U \subseteq S^{\Delta,\Delta'^\perp}$ *where* $S^{\Delta,\Delta'^\perp}$ *is the complement of* $S^{\Delta,\Delta'}$ *in* M^Δ.

Proof. See Theorem 3.22 [**8**].

We can now prove our principal result.

THEOREM 3.23. *Let* $\{J, J'\}$ *be a good system in* Φ. *The* $K\mathscr{W}$*-module* $D^{\Delta,\Delta'} = S^{\Delta,\Delta'}/S^{\Delta,\Delta'} \cap S^{\Delta,\Delta'^\perp}$ *is zero or irreducible.*

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Proof. If U is a submodule of $S^{\Delta,\Delta'}$ then U is a submodule of M^{Δ} and by Theorem 3.22 either $S^{\Delta,\Delta'} \subseteq U$ in which case $U = S^{\Delta,\Delta'}$ or $U \subseteq S^{\Delta,\Delta'^{\perp}}$ and $U \subseteq S^{\Delta,\Delta'} \cap S^{\Delta,\Delta'^{\perp}}$, which completes the proof.

In the case of $K = \mathbb{Q}$ or any field of characteristic zero $\langle\, ,\rangle$ is an inner product and $D^{\Delta,\Delta'} = S^{\Delta,\Delta'}$. Thus if for a subsystem Ψ of Φ a good system $\{J, J'\}$ can be found, then we have a construction for irreducible $K\mathscr{W}$-modules. Hence it is essential to show for each subsystem that a good system exists which satisfies Definition 3.19.

In the following example, we show how a good system may be constructed in all cases for the finite reflection group of type $\mathbf{G_2}$. In [**9**], we present an algorithm for constructing a good system for certain subsystems; indeed this algorithm will give a good system with additional properties which will lead to the construction of a K-basis for our Specht modules $S^{\Delta,\Delta'}$, which correspond to the basis consisting of standard tableaux in the case of symmetric groups.

EXAMPLE 3.24. Let $\Phi = \mathbf{G_2}$ with simple system $\pi = \{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = -2\epsilon_1 + \epsilon_2 + \epsilon_3\}$. Let $g_1 = e$, $g_2 = \tau_2$, $g_3 = \tau_1\tau_2$, $g_4 = (\tau_1\tau_2)^2$, $g_5 = (\tau_1\tau_2)^3$, $g_6 = \tau_1$ be representatives of conjugacy classes $C_1, C_2, C_3, C_4, C_5, C_6$ respectively of $\mathscr{W}(\mathbf{G_2})$. Then the character table of $\mathscr{W}(\mathbf{G_2})$ is

	C_1	C_2	C_3	C_4	C_5	C_6
χ_1	1	1	1	1	1	1
χ_2	1	−1	1	1	1	−1
χ_3	1	−1	−1	1	−1	1
χ_4	1	1	−1	1	−1	−1
χ_5	2	0	−1	−1	2	0
χ_6	2	0	1	−1	−2	0

The non-conjugate subsystems of $\mathbf{G_2}$ are:

(1) $\Psi_1 = \mathbf{A_2}$ with simple system $J_1 = \{01, 31\}$,

(2) $\Psi_2 = \mathbf{A_1} + \tilde{\mathbf{A}}_1$ with simple system $J_2 = \{10, 32\}$,

(3) $\Psi_3 = \mathbf{A_1}$ with simple system $J_3 = \{10\}$,

(4) $\Psi_4 = \tilde{\mathbf{A}}_1$ with simple system $J_4 = \{01\}$,

(5) $\Psi_5 = \varnothing$ with simple system $J_5 = \varnothing$,

(6) $\Psi_6 = \mathbf{G_2}$ with simple system $J_6 = \{10, 01\}$.

Let $\Psi_4 = \tilde{\mathbf{A}}_1$ be the subsystem of Φ with simple system $J_4 = \{01\}$. Let $\Psi_1' = \mathbf{A_1} + \tilde{\mathbf{A}}_1$ be the subsystem of Φ which is contained in $\Phi\backslash\Psi_4$, with simple system $J_1' = \{11, 31\}$. Since $\mathscr{W}(J_4) \cap \mathscr{W}(J_1') = \langle e\rangle$ and $\mathscr{W}(J_4^{\perp}) \cap \mathscr{W}(J_1'^{\perp}) = \langle e\rangle$, then $\{J_4, J_1'\}$ is a useful system in Φ. Then τ_{Δ_4} contains Δ_4-tabloids:

$$\{\overline{J_4}\} = \{01; 11, 31\}, \{\overline{\tau_1 J_4}\} = \{31; 21, 01\},$$

$$\{\overline{\tau_2\tau_1 J_4}\} = \{32; 21, -01\}, \{\overline{\tau_1\tau_2\tau_1 J_4}\} = \{32; 11, -31\},$$

$$\{\overline{\tau_2\tau_1\tau_2\tau_1 J_4}\} = \{31; 10, -32\}, \{\overline{\tau_1\tau_2\tau_1\tau_2\tau_1 J_4}\} = \{01; -10, -32\}.$$

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For $d=e,\ \tau_2\tau_1,\ \tau_1\tau_2\tau_1,\ \tau_1\tau_2\tau_1\tau_2\tau_1$ we have $d\Psi_4\cap\Psi_1'=\emptyset$. Since

$$e_{J_4,J_1'}=\{\bar{J}_4\}-\{\overline{\tau_2\tau_1J_4}\}-\{\overline{\tau_1\tau_2\tau_1J_4}\}+\{\overline{\tau_1\tau_2\tau_1\tau_2\tau_1J_4}\}$$

then $\{J_4,J_1'\}$ is a good system in Φ.

Now let K be a field and char $K=0$. Let M^{Δ_4} be the K-space whose basis consists of the Δ_4-tabloids. Let $S^{\Delta_4,\Delta_1'}$ be the corresponding $K\mathcal{W}$-submodule of M^{Δ_4}, then by definition of the Specht module we have

$$S^{\Delta_4\Delta_1'}=Sp\{e_{J_4,J_1'},e_{\tau_1J_4,\tau_1J_1'}\}$$

where

$$e_{J_4,J_1'}=\{\bar{J}_4\}-\{\overline{\tau_2\tau_1J_4}\}-\{\overline{\tau_1\tau_2\tau_1J_4}\}+\{\overline{\tau_1\tau_2\tau_1\tau_2\tau_1J_4}\},$$

$$e_{\tau_1J_4,\tau_1J_1'}=\{\overline{\tau_1J_4}\}-\{\overline{\tau_2\tau_1J_4}\}-\{\overline{\tau_1\tau_2\tau_1J_4}\}+\{\overline{\tau_2\tau_1\tau_2\tau_1J_4}\}.$$

Let T_4^1 be the matrix representation of $\mathcal{W}$ afforded by $S^{\Delta_4,\Delta_1'}$ with character ψ_4^1 and let $\tau_1\tau_2$ be the representative of the conjugacy class C_3. Then

$$\tau_1\tau_2(e_{J_4,J_1'})=e_{\tau_1J_4,\tau_1J_1'}-e_{J_4,J_1'},$$

$$\tau_1\tau_2(e_{\tau_1J_4,\tau_1J_1'})=-e_{J_4,J_1'}.$$

Thus we have

$$T_4^1(\tau_1\tau_2)=\begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix}\quad\text{and}\quad \psi_4^1(\tau_1\tau_2)=-1.$$

By a similar calculation to the above it can be shown that $\psi_4^1=\chi_5$. By the same method to the above, we have the following table.

Ψ	Ψ'	J'	Ch
$\mathbf{A}_2$	$\mathbf{A}_1$	$\{10\}$	χ_4
$\mathbf{A}_1+\tilde{\mathbf{A}}_1$	$\tilde{\mathbf{A}}_1$	$\{01\}$	χ_5
$\mathbf{A}_1$	$\mathbf{A}_2$	$\{01,31\}$	χ_3
$\tilde{\mathbf{A}}_1$	$\mathbf{A}_1+\tilde{\mathbf{A}}_1$	$\{11,31\}$	χ_5
$\mathbf{G}_2$	$\emptyset$	$\emptyset$	χ_1
$\emptyset$	$\mathbf{G}_2$	$\{10,01\}$	χ_2

We note that the irreducible modules corresponding to the characters χ_6 have not been obtained. We now show how an additional irreducible character is obtained. Let $\Psi_2'=\mathbf{A}_1$ be the subsystem of Φ with simple system $J_2'=\{11\}$. Then $\{J_4,J_2'\}$ is a useful system in Φ. Since $\Psi_2'\subset\Psi_1'$, by Lemma 3.15 $S^{\Delta_1,\Delta_1'}$ is a $K\mathcal{W}$-submodule of $S^{\Delta_1,\Delta_2'}$. By a similar calculation, the corresponding character of $\mathcal{W}$ afforded by $S^{\Delta_1,\Delta_2'}/S^{\Delta_1,\Delta_1'}$ is χ_6. Thus we have obtained a complete set of irreducible modules for $\mathbf{G}_2$.

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