INTEGRALS INVOLVING E-FUNCTIONS AND ASSOCIATED LEGENDRE FUNCTIONS

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§ 1. *Introductory.* The formulae to be proved are as follows.

If $p \geqq q+1$, $R(l)>0$, $R(\alpha_r - l + m + n) > -1$, $R(\alpha_r - l + m - n) > 0$, $r = 1, 2, \ldots, p$,

$$\int_1^\infty E\{p\,;\ \alpha_r : q\,;\ \rho_s : z/(\lambda-1)\}(\lambda-1)^{l-1}(\lambda^2-1)^{-\frac{1}{2}m} P_n^{-m}(\lambda)\,d\lambda$$

$$= \frac{\pi}{\Gamma(m+n+1)\Gamma(m-n)}$$

$$\times \Bigg[\frac{2^n z^{l-m-n-1}}{\cos n\pi \sin(l-m-n)\pi} E\begin{pmatrix} m+n+1,\ n+1,\ \alpha_1 - l + m + n + 1,\ \ldots,\ \alpha_p - l + m + n + 1 & : \tfrac{1}{2}e^{\pm i\pi} z \\ 2n+2,\ m+n-l+2,\ \rho_1 - l + m + n + 1,\ \ldots,\ \rho_q - l + m + n + 1 & \end{pmatrix}$$

$$+ \frac{2^{-n-1} z^{l-m+n}}{\cos n\pi \sin(l-m+n)\pi} E\begin{pmatrix} m-n,\ -n,\ \alpha_1 - l + m - n,\ \ldots,\ \alpha_p - l + m - n : & \tfrac{1}{2}e^{\pm i\pi} z \\ -2n,\ m-n-l+1,\ \rho_1 - l + m - n,\ \ldots,\ \rho_q - l + m - n & \end{pmatrix}$$

$$- \frac{2^{l-m} \sin(l-m)\pi}{\sin(l-m+n)\pi \sin(l-m-n)\pi} E\begin{pmatrix} l,\ l-m,\ \alpha_1,\ \ldots,\ \alpha_p & : \tfrac{1}{2}e^{\pm i\pi} z \\ l-m+n+1,\ l-m-n,\ \rho_1,\ \ldots,\ \rho_q & \end{pmatrix} \Bigg] \quad \ldots\ldots(1)$$

If $p \geqq q+1$, $R(l+m)>0$, $R(\alpha_r - l - m + n) > -1$, $R(\alpha_r - l - m - n) > 0$, $r = 1, 2, \ldots, p$,

$$\int_1^\infty E\{p\,;\ \alpha_r : q\,;\ \rho_s : z/(\lambda-1)\}(\lambda-1)^{l-1}(\lambda^2-1)^{\frac{1}{2}m} P_n^{-m}(\lambda)\,d\lambda$$

$$= -\frac{2^{-n-1}\sin(m+n)\pi}{\cos n\pi \sin(l+m+n)\pi} z^{l+m+n} E\begin{pmatrix} -n,\ -n-m,\ \alpha_1 - l - m - n,\ \ldots,\ \alpha_p - l - m - n & : \tfrac{1}{2}e^{\pm i\pi} z \\ -2n,\ 1-l-m-n,\ \rho_1 - l - m - n,\ \ldots,\ \rho_q - l - m - n & \end{pmatrix}$$

$$+ \frac{2^n \sin(m-n)\pi}{\cos n\pi \sin(l+m-n)\pi} z^{l+m-n-1}$$

$$\times E\begin{pmatrix} n+1,\ n-m+1,\ \alpha_1 - l - m + n + 1,\ \ldots,\ \alpha_p - l - m + n + 1 : & \tfrac{1}{2}e^{\pm i\pi} z \\ 2n+2,\ 2-l-m+n,\ \rho_1 - l - m + n + 1,\ \ldots,\ \rho_q - l - m + n + 1 & \end{pmatrix}$$

$$+ \frac{2^{l+m}\sin l\pi \sin n\pi}{\sin(l+m+n)\pi \sin(l+m-n)\pi} E\begin{pmatrix} l+m,\ l,\ \alpha_1,\ \ldots,\ \alpha_p & : \tfrac{1}{2}e^{\pm i\pi} z \\ l+m+n+1,\ l+m-n,\ \rho_1\ \ldots,\ \rho_q & \end{pmatrix}. \quad \ldots\ldots(2)$$

If $p \geqq q+1$, $R(l)>0$, $R(l+m)>0$, $R(\alpha_r - l - m + n) > -1$,

$$\int_1^\infty E\{p\,;\ \alpha_r :\ q\,;\ \rho_s :\ z/(\lambda-1)\}(\lambda-1)^{l-1}(\lambda^2-1)^{\frac{1}{2}m} Q_n^{-m}(\lambda)\,d\lambda$$

$$= -\frac{\pi 2^n z^{l+m-n-1}}{\sin(l+m-n)\pi} E\begin{pmatrix} n+1,\ n-m+1,\ \alpha_1 - l - m + n + 1,\ \ldots,\ \alpha_p - l - m + n + 1 : & \tfrac{1}{2}e^{\pm i\pi} z \\ 2n+2,\ 2-l-m+n,\ \rho_1 - l - m + n + 1,\ \ldots,\ \rho_q - l - m + n + 1 & \end{pmatrix}$$

$$+ \frac{\pi 2^{l+m-1}}{\sin(l+m-n)\pi} E\begin{pmatrix} l,\ \ l+m,\ \ \alpha_1,\ \ldots,\ \alpha_p : & \tfrac{1}{2}e^{\pm i\pi} z \\ l+m+n+1,\ l+m-n,\ \rho_1,\ \ldots,\ \rho_q & \end{pmatrix}. \quad \ldots\ldots(3)$$

The method of proof is outlined in § 2. Many special cases can be derived from these formulae. An example is given in § 3.

§ 2. *Proofs of the formulae.* The formulae can be deduced from the following three formulae [*Q.J.M.* XI, 1940, pp. 98, 99].

If $R(z)>0$, $R(l)>0$,

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$$\int_1^\infty E\{\ :\ : z/(\lambda-1)\}(\lambda-1)^{l-1}(\lambda^2-1)^{-\frac{1}{2}m} P_n^{-m}(\lambda)\,d\lambda$$

$$=\frac{2^{-m}z^l}{\Gamma(m+n+1)\Gamma(m-n)} E\begin{pmatrix} m+n+1, m-n, l: 2/z \\ m+1 \end{pmatrix} \quad \text{.................(4)}$$

If $R(z)>0$, $R(l+m)>0$,

$$\int_1^\infty E\{\ :\ : z/(\lambda-1)\}(\lambda-1)^{l-1}(\lambda^2-1)^{\frac{1}{2}m} P_n^{-m}(\lambda)\,d\lambda$$

$$= -\frac{\sin n\pi}{\pi} z^{l+m} E\begin{pmatrix} -n, n+1, l+m: 2/z \\ m+1 \end{pmatrix} \quad \text{..............................(5)}$$

If $R(z)>0$, $R(l+m)>0$, $R(l)>0$,

$$\int_1^\infty E\{\ :\ : z/(\lambda-1)\}(\lambda-1)^{l-1}(\lambda^2-1)^{\frac{1}{2}m} Q_n^{-m}(\lambda)\,d\lambda$$

$$=\frac{z^l}{2\sin m\pi}\begin{Bmatrix} \sin n\pi\, z^m E(-n, n+1, l+m: 1+m: 2/z) \\ -2^m \sin(m+n)\pi\, E(n-m+1, -n-m, l: 1-m: 2/z) \end{Bmatrix} \quad \text{........(6)}$$

On applying the formula, where $p \geqq q+1$,

$$E(p\,;\ \alpha_r:\ q:\ \rho_s:\ z) = \pi^{p-q-1}\sum_{r=1}^{p}\left\{\prod_{s=1}^{p}{}' \sin(\alpha_s-\alpha_r)\pi\right\}^{-1}\prod_{t=1}^{q}\sin(\rho_t-\alpha_r)\pi$$

$$\times z^{\alpha_r} E\begin{Bmatrix} \alpha_r, \alpha_r-\rho_1+1, \ldots, \alpha_r-\rho_q+1:\ (-1)^{p-q-1}/z \\ \alpha_r-\alpha_1+1, \ldots * \ldots, \alpha_r-\alpha_p+1 \end{Bmatrix} \quad \text{..............................(7)}$$

to the R.H.S.'s of (4), (5) and (6), formulae (1), (2), (3), with $p=q=0$ are obtained. The formulae can then be generalised in the usual way.

§ 3. *Integral involving a Product of two Associated Legendre Functions.* In (3) take $z=2$, $p=2$, $q=1$, $\alpha_1=q-p$, $\alpha_2=q+p+1$, $\rho_1=q+1$, apply the formula

$$E\{q-p, q+p+1: q+1: 2/(\lambda-1)\} = 2^q\Gamma(q-p)\Gamma(q+p+1)(\lambda^2-1)^{-\frac{1}{2}q}P_p^{-q}(\lambda), \quad \text{............(8)}$$

and so obtain the following result.

If $R(l)>0$, $R(l+m)>0$, $R(q+p-m+n-l)>-2$, $R(q-p-m+n-l)>-1$, $R(m-q)>-1$

$$\int_1^\infty (\lambda-1)^{l-1}(\lambda^2-1)^{-\frac{1}{2}q}P_p^{-q}(\lambda)(\lambda^2-1)^{\frac{1}{2}m}Q_n^{-m}(\lambda)\,d\lambda$$

$$=\frac{\pi 2^{l+m-q-1}}{\Gamma(q-p)\,\Gamma(q+p+1)\sin(l+m-n)\pi}$$

$$\times\left[\begin{matrix} E\begin{pmatrix} l, l+m, q-p, q+p+1: e^{\pm i\pi} \\ l+m+n+1, l+m-n, q+1 \end{pmatrix} \\ -E\begin{pmatrix} n+1, n-m+1, q-p-l-m+n+1, q+p-l-m+n+2: e^{\pm i\pi} \\ 2n+2, 2-l-m+n, q-l-m+n+2 \end{pmatrix} \end{matrix}\right] \quad \text{.......(9)}$$

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