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PIECEWISE-CONSTANT COLLOCATION FOR FIRST-KIND BOUNDARY INTEGRAL EQUATIONS

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Abstract

We examine the piecewise-constant collocation method, with collocation points the mid-points of subintervals, for first-kind integral equations with logarithmic kernels on polygonal boundaries. Previously this method had been shown to converge subject to certain restrictions on the angles at the corners of the polygon. Here, by considering a slightly modified collocation method, we are able to remove any restrictions on these angles, and to generalise slightly the meshes which may be used. Moreover, the modification leads to new results on the convergence of preconditioned two-(or multi-) grid methods for solving the resultant linear systems.

1. Introduction

Many boundary-integral methods in potential theory (e.g. [14]) require the numerical solution of the equation

$$-\frac{1}{\pi}\int_\Gamma \log|\mathbf{x}-\xi|u(\xi)\,d\Gamma(\xi) = g(\mathbf{x}), \qquad \mathbf{x}\in\Gamma, \tag{1.1}$$

or

$$\mathscr{V}u = g,$$

where $g:\Gamma\to\mathbf{R}$ is given, $u:\Gamma\to\mathbf{R}$ is to be found, and Γ is the boundary of a polygonal domain in $\mathbf{R}^2$. We assume throughout that the transfinite diameter of Γ is not equal to 1, so that (1.1) is uniquely solvable for g sufficiently smooth. We shall discuss the numerical solution of (1.1) by the following very simple collocation scheme. Subdivide Γ into n segments (where each corner of Γ is a break point), and choose as *collocation points*

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the mid-points of the segments. Then the approximate solution u_j $(h = 1/n)$ is defined to be constant on each segment, and to satisfy

$$-\frac{1}{\pi}\int_{\Gamma}\log|\mathbf{x}^* - \xi|u_h(\xi)\,d\Gamma(\xi) = g(\mathbf{x}^*), \tag{1.2}$$

for all collocation points $\mathbf{x}^*$.

This method has been in practical use for many years, and seems to have originally been proposed by Symm [19]. However, in contrast to the less practical Galerkin methods which are well analysed (e.g. [13], [5], [9], [15], [20]), its numerical analysis is still incomplete. For smooth Γ and a uniform grid, the convergence of u_h to u was first demonstrated by De Hoog [10]. The results of [10] were considerably generalised by Arnold, Wendland and Saranen ([3], [17]), who studied spline collocation schemes of arbitrary order applied to a variety of equations, with method (1.2) for (1.1) contained as a special case, but their analysis was restricted to smooth Γ. Subsequently, Costabel and Stephan [8] were able to extend the ideas in [3] to the practically important case of polygonal Γ, but only for the special numerical method where u is approximated by a *continuous piecewise linear* function, and (1.1) is collocated at the break points. However, the techniques of [3], [17] for dealing with piecewise-constant (or, more generally, even-order) spline approximation are more deeply rooted in Fourier analysis than those for odd-order splines, and hence appear less easily adaptable to the case of polygonal Γ. (The Fourier analysis technique applies naturally only when $\mathscr{V}$ is a smooth perturbation of a pure convolution operator; this is the case when Γ is smooth, but not when it is polygonal.)

A new approach to the analysis of the polygonal case is given in [22]. In this approach, Γ is parametrised by $\gamma : [-\pi, \pi] \to \Gamma$, and any $\nu : \Gamma \to \mathbf{R}$, is identified with $\nu(s) = \nu(\gamma(s))$, $s \in [-\pi, \pi]$. Then (1.1) is written

$$-\frac{1}{\pi}\int_{-\pi}^{\pi}\log|\gamma(s) - \gamma(\sigma)||\gamma'(\sigma)|u(\sigma)\,d\sigma = g(s), \qquad s \in [-\pi, \pi],$$

which we abbreviate as

$$Kw = g, \tag{1.3}$$

where $w = |\gamma'|u$. Let A denote the operator

$$A\nu(s) = -\frac{1}{\pi}\int_{-\pi}^{\pi}\log|2\sin(s-\sigma)/2|\nu(\sigma)\,d\sigma + \frac{1}{2\pi}\int_{-\pi}^{\pi}\nu(\sigma)\,d\sigma. \tag{1.4}$$

The first term is merely the operator K when Γ is the unit circle parametrised by $\gamma(s) = (\cos s, \sin s)$; the second term is a compact perturbation. Then A is invertible (on appropriate function spaces), and (1.3) may be written

$$(I + A^{-1}(K - A))w = f := A^{-1}g. \tag{1.5}$$

Thus w satisfies a (nonstandard) second-kind equation. The principal idea in [22] was to observe that (1.2) can similarly be converted into a (nonstandard) projection method for this second-kind equation, and then to invoke well-known arguments for the analysis of such methods. This works beautifully when Γ is smooth, for then $A^{-1}(K-A)$ is compact. But when Γ is polygonal, $A^{-1}(K-A)$ turns out to contain a noncompact (Mellin) convolution component, which makes the analysis of the numerical method more difficult. Consequently [22] obtained convergence results for (1.2) when Γ is polygonal, only under certain unnatural restrictions on the angles subtended by the boundary at each corner of Γ, and on the mesh which could be used.

The present paper has two purposes. The first is to propose a slightly modified collocation method for (1.1), and to prove its convergence in the presence of corners of any angle. In the process we also generalise the kind of meshes which can be used somewhat, although further work is required to prove results for the graded meshes needed to obtain optimal convergence rates. The modified collocation method proposed is similar in spirit to that proposed in [6] for collocation for standard second-kind boundary integral equations on polygons. There the modification was a device for proving general stability results, but was not usually needed in practice [1], [6]. We shall see in Section 3 that a similar situation pertains to the proposed modification of (1.2).

However, the modification in [6] found itself a more practical role in the acceleration of convergence of multigrid procedures for solving the collocation equations. (See [2], which dealt with a discrete collocation method.) The second purpose of our paper, then, is to prove similar results for the multigrid solution of the modified version of (1.2). Here again the correspondence between (1.3) and (1.5) and the correspondence between their respective discretisations is of prime importance. The multigrid method for (1.2) is in fact derived as a method for the discrete version of (1.5). The smoothing operator in the algorithm is the discretisation of $K-A$ preconditioned by the inverse of the discretisation of A. Since the discretisation of A is a circulant matrix it is very efficiently inverted by FFT, and the preconditioning step does not increase the operation count significantly.

In Section 2, various required results on the discretisation of A and other operators on the unit circle are derived. In Section 3 the properties of (1.1), (1.2), when Γ is a polygon are discussed. Then Section 4 discusses the modified collocation method, and Section 5 describes the related multigrid algorithm.

Before leaving this introduction, we describe some notation needed later. We assume Γ has corners $\mathbf{x}_0, \mathbf{x}_2, \dots, \mathbf{x}_{2r} = \mathbf{x}_0$, and that the exterior angle at each $\mathbf{x}_{2l}$ is $(1+\chi_l)\pi$. Let $\mathbf{x}_{2l-1}$ be the mid-point of the straight line joining $\mathbf{x}_{2l-2}$ to $\mathbf{x}_{2l}$. We use Γ_j to denote the portion of Γ joining $\mathbf{x}_{j-1}$ to $\mathbf{x}_j$. Then we shall choose fixed integers m_j, $j = 1, \ldots, 2r$ and, for any $N \in \mathbf{N}$, divide Γ_j up into $m_j N$ equal segments. This yields a mesh on Γ with $N = (m_1 + \cdots + m_{2r})N$ segments. It is uniform on each Γ_j, but (in contrast to [22]), not necessarily uniform over all of Γ. We now construct a special parametrisation γ of Γ as follows. For $j = 1, \ldots, 2r$ set

$$M_j = m_1 + \cdots + m_j,$$

and $M = M_{2r}$. Next define points $S_j \in [-\pi, \pi]$ by

$$\begin{aligned} S_0 &= -\pi, \\ S_j &= -\pi + (M_j/M)2\pi, \qquad j = 1, \ldots, 2r. \end{aligned}$$

Finally define $\gamma : [-\pi, \pi] \to \Gamma$ by

$$\gamma(s) = \mathbf{x}_{j-1} + \left(\frac{s - S_{j-1}}{S_j - S_{j-1}}\right)(\mathbf{x}_j - \mathbf{x}_{j-1}), \qquad s \in [S_{j-1}, S_j],$$

for $j = 1, \ldots, 2r$.

Let $h = 2\pi/n$, and introduce the uniform mesh on $[-\pi, \pi]$:

$$s_j = -\pi + jh, \qquad j = 0, \ldots, n$$

and the mid-points of subintervals

$$t_j = (s_{j-1} + s_j)/2, \qquad j = 1, \ldots, n.$$

Then γ maps this mesh on $[-\pi, \pi]$ to the mesh on Γ defined above. Let X_j denote the characteristic function of $[s_{j-1}, s_j]$ and set $V_h = \operatorname{span}\{X_1, \ldots, X_n\}$. For $\nu : [-\pi, \pi] \to \mathbf{R}$, define $Q_h\nu \in V_h$ by requiring $Q_h\nu$ to interpolate ν at t_j, $j = 1, \ldots, n$.

Then (1.2) is equivalent to seeking $u_h \in V_h$ such that $w_h := |\gamma'|u_h$ satisfies

$$Kw_h(t_j) = g(t_j), \qquad j = 1, \ldots, n, \tag{1.6}$$

or equivalently,

$$Q_h K w_h = Q_h g. \tag{1.7}$$

For later reference, the (j, k)th element of the coefficient matrix of the system (1.6) is

$$KX_k(t_j) = -\frac{1}{\pi}\int_{s_{k-1}}^{s_k} \log|\gamma(t_j) - \gamma(\sigma)|\, d\sigma. \tag{1.8}$$

2. The circle

With $*$ denoting convolution, the operator A defined in (1.4) may be written

$$A\nu = \Lambda * \nu, \tag{2.1}$$

where $\Lambda(s) = (-1/\pi)\log|2e^{-1/2}\sin(s/2)|$.

Let $\mathbf{Z}$ denote the set of all integers. For $\nu \in C^\infty$ (i.e. $D^r\nu$ is continuous and 2π-periodic for all $r \geq 0$), define the Fourier transform of ν by

$$\hat{\nu}(m) = \frac{1}{\sqrt{(2\pi)}}\int_{-\pi}^{\pi} \nu(s)\exp(-ims)\,ds, \qquad m \in \mathbf{Z}.$$

Then

$$\nu(s) = \frac{1}{\sqrt{(2\pi)}}\sum_{m\in\mathbf{Z}} \hat{\nu}(m)\exp(ims), \qquad s \in [-\pi, \pi].$$

For $t \in \mathbf{R}$, define the norm $|\nu|_t$ on C^∞ by

$$|\nu|_t^2 = \sum_{m\neq 0} |m|^{2t}|\hat{\nu}(m)|^2 + |\hat{\nu}(0)|^2,$$

and define the usual Sobolev space H^t to be the completion of C^∞ under this norm. If L is a bounded linear operator on H^t, then $|L|_t$ denotes its norm.

By applying the convolution theorem to (2.1), (computing $\widehat{\Lambda}$ by contour integration,) we obtain the well-known representation

$$A\nu(s) = \frac{1}{\sqrt{(2\pi)}}\left[\sum_{m\neq 0} |m|^{-1}\hat{\nu}(m)\exp(ims) + \hat{\nu}(0)\right], \qquad s \in [-\pi, \pi]. \tag{2.2}$$

As is easily shown, $|A\nu|_{t+1} = |\nu|_t$, for all $\nu \in H^t$, and so A is an isometry from H^t to H^{t+1}. Its inverse $A^{-1} : H^{t+1} \to H^t$ is given by

$$A^{-1}\nu(s) = \frac{1}{\sqrt{(2\pi)}}\left[\sum_{m\neq 0} |m|\hat{\nu}(m)\exp(ims) + \hat{\nu}(0)\right],$$

and hence

$$A^{-1} = -DH + J = -HD + J, \tag{2.3}$$

where H is the Hilbert transform:

$$\begin{aligned} H\nu(s) &= -\frac{1}{2\pi}p.v.\int_{-\pi}^{\pi} \cot\left(\frac{s-\sigma}{2}\right)\nu(\sigma)\,d\sigma \\ &= \frac{1}{\sqrt{(2\pi)}}\left[\sum_{m\neq 0} i\,\mathrm{sign}(m)\hat{\nu}(m)\exp(ims)\right], \end{aligned}$$

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D is the 2π-periodic (distributional) differentiation operator, and J is the linear functional

$$J\nu = \frac{1}{2\pi}\int_{-\pi}^{\pi} \nu(\sigma)\, d\sigma.$$

Clearly

$$|H|_0 = 1. \tag{2.4}$$

Consider now the integral equation

$$Aw = g,$$

and the piecewise constant collocation method defined by seeking $w_h \in V_h$ (defined in Section 1) which solves

$$Q_h A w_h = Q_h g = Q_h A w.$$

For any operator L on H^0, let L_h denote its restriction to V_h. Then

$$Q_h A_h w_h = Q_h A w. \tag{2.5}$$

This method is analysed in [22]. The technique used there is to first show $Q_h A_h : V_h \to V_h$ invertible, and then to show that the operator

$$B_h := (Q_h A_h)^{-1} Q_h A, \tag{2.6}$$

is uniformly bounded on H^0, for all n. Then $B_h \to I$ pointwise on H^0, as $N \to \infty$. If now $g \in H^1$, then $w \in H^0$, and by (2.5) we have

$$|w - w_h|_0 = |(I - B_h)w|_0 \to 0, \quad \text{as } N \to \infty.$$

In this paper we shall need some generalisations of the results of [22]. Related calculations can be found in [16]. First observe that operators A and AH above are both of the form

$$L\nu(s) = \frac{1}{\sqrt{(2\pi)}}\left[\sum_{m\in\mathbf{Z}} \zeta_m \hat{\nu}(m) \exp(ims)\right], \tag{2.7a}$$

where

$$|\zeta_0| \le 1, \qquad \text{and} \qquad |\zeta_m| \le |m|^{-1}, \quad \text{for all } m \neq 0. \tag{2.7b}$$

Any such L is a bounded linear operator from H^t to H^{t+1}.

Now let $\omega = \exp(ih)$, and define the $n \times n$ Fourier matrix $F = (f_{jk})$ by

$$f_{jk} = n^{-\frac{1}{2}}\, \overline{\omega}^{(j-1)(k-1)}, \qquad j, k = 1, \dots, n.$$

Observe that $F^* = F^{-1}$. We shall be interested in the basis $\{e_h^j, j = 1, \dots, n\}$ of the complex extension $\widetilde{V}_j = \{\nu_1 + i\nu_2 : \nu_1, \nu_2 \in V_h\}$ of V_h, which is obtained from $\{X_j\}$ by the transformation:

$$[e_h^1, e_h^2, \dots, e_h^n]^T = F^*[X_1, X_2, \dots, X_n]^T. \tag{2.8}$$

Since $F^*F = I$, we have $(e_h^j, e_h^k) = h\delta_{jk}$ (with the usual inner product on H^0), i.e. $\{e_h^j\}$ is an orthogonal basis for $\widetilde{V}_h$.

THEOREM 1. *Let L be of the form* (2.7). *Then $Q_h L_h : \widetilde{V}_h \to \widetilde{V}_h$ has eigenvalues*

$$\lambda_j = \sqrt{n} L e_h^j(t_1), \qquad j = 1, \dots, n,$$

and corresponding eigenfunctions e_h^j, $j = 1, \dots, n$. When all its eigenvalues are nonzero, the matrix representing $Q_h L_h$ (with respect to $\{X_j\}$) may be inverted in $O(n \log n)$ operations.

PROOF. Let $\mathscr{A} = (\alpha_{j,k})$ be the matrix of $Q_h L_h$ with respect to the basis $\{X_j\}$. Then

$$\begin{aligned}
\alpha_{j,k} &= L X_k(t_j) \\
&= \frac{1}{\sqrt{(2\pi)}} \sum_{m \in \mathbf{Z}} \zeta_m \widehat{X}_k(m) \exp(imt_j) \\
&= \frac{1}{n} \sum_{m \in \mathbf{Z}} \zeta_m \rho_m \exp(im(t_j - t_k)) \\
&= \frac{1}{n} \sum_{m \in \mathbf{Z}} \zeta_m \rho_m \omega^{m(j-k)},
\end{aligned}$$

where

$$\begin{aligned}
\rho_m &= (n/(m\pi)) \sin((m\pi)/n), \qquad & m \neq 0 \\
&= 1, & m = 0.
\end{aligned} \tag{2.9}$$

Hence, for $j = 1, \dots, n-1$, we have

$$\alpha_{j+1,k+1} = \alpha_{j,k}, \qquad k = 1, \dots, n-1,$$

and

$$\alpha_{j+1,1} = \alpha_{j,n}.$$

That is, $\mathscr{A}$ is a circulant matrix and its eigenspaces may be determined explicitly. (See, for example, [7].)

Let $\gamma = (\gamma_1, \dots, \gamma_n)^{\mathsf{T}}$ denote (the transpose of) the first row of $\mathscr{A}$. This is enough to determine $\mathscr{A}$ uniquely, and we write $\mathscr{A} = \operatorname{circ} \gamma$. Then a straightforward calculation shows

$$\mathscr{A} F^* = F^* \operatorname{diag}(\lambda_1, \dots, \lambda_n), \tag{2.10}$$

where $\lambda = (\lambda_1, \dots, \lambda_n)^{\mathsf{T}}$ is given by

$$\lambda = \sqrt{n} F^* \gamma. \tag{2.11}$$

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Thus each λ_j is an eigenvalue of $\mathscr{A}$, and also of $Q_h L_h$. The corresponding eigenvector of $\mathscr{A}$ is the jth column (or equivalently, the jth row) of F^*. By (2.8), then, the corresponding eigenvector of $Q_h L_h$ is e_h^j. Also, by (2.11) and (2.8),

$$\lambda_j = \sum_{k=1}^{n} L X_k(t_1)\omega^{(j-1)(k-1)} = \sqrt{n} L e_h^j(t_1),$$

as required. If now all the λ_j are nonzero, $\mathscr{A}$ has the inverse given, using (2.10), by

$$\mathscr{A}^{-1} = F^* \operatorname{diag}(\lambda_1^{-1}, \dots, \lambda_n^{-1})F.$$

Then $\mathscr{A}^{-1}$ is also a circulant, $\mathscr{A}^{-1} = \operatorname{circ}\delta$, say. For any $\mathbf{x} \in \mathbf{C}^n\backslash\{\mathbf{0}\}$, let $\mathbf{x}^{-1}$ denote the vector $(x_1^{-1}, \dots, x_n^{-1})^{\mathsf{T}}$. Then analogously to (2.11), $\lambda^{-1} = \sqrt{n}F^*\delta$ so that

$$f\delta = (1/\sqrt{n})F\lambda^{-1} = (1/n)F(F^*\gamma)^{-1}.$$

So $\mathscr{A}^{-1}$ may be completely determined by finding δ, which in turn requires two discrete Fourier transforms and one inversion. Using FFT, this amounts to $O(N\log N)$ operations (recall $n = MN$, with M fixed).

COROLLARY 2.

(i) *The eigenvalues of $Q_h A_h$ are $\lambda_1 = 1$, and for $j = 2, \dots, n$,*

$$\lambda_j = \frac{n}{\pi}\sin\frac{(j-1)\pi}{n}\sum_{k=0}^{\infty}(-1)^k\left\{\frac{1}{(kn+(j-1))^2} + \frac{1}{((k+1)n-(j-1))^2}\right\}.$$

(ii) *The eigenvalues of $Q_h A H_h$ are $\mu_1 = 0$, and for $j = 2, \dots, n$,*

$$\mu_j = i\frac{n}{\pi}\sin\frac{(j-1)\pi}{n}\sum_{k=0}^{\infty}(-1)^k\left\{\frac{1}{(kn+(j-1))^2} - \frac{1}{((k+1)n-(j-1))^2}\right\}.$$

In both cases the corresponding eigenfunctions are e_h^j.

PROOF. Some calculation shows

$$\hat{e}_h^j(m) = \begin{cases} \sqrt{(2\pi/n)}\rho_m \exp(-imt_1), & m = j-1 \pmod{n} \\ 0, & m \neq j-1 \pmod{n} \end{cases} \tag{2.12}$$

with ρ_m given by (2.9). Then, applying Theorem 1 with $L = A$ yields

$$\lambda_1 = A1(x_1) = 1,$$

and, for $j = 2, \dots, n$,

$$\lambda_j = \sum_{m \equiv j-1 (\mathrm{mod}\, n)} |m|^{-1} \rho_m$$
$$= \frac{n}{\pi} \sin \frac{(j-1)\pi}{n} \sum_{k \in \mathbf{Z}} (-1)^k |j-1+kn|^{-1} (j-1+kn)^{-1},$$

which yields part (i).

On the other hand, noting that

$$AH\nu(s) = \frac{1}{\sqrt{(2\pi)}} \sum_{m \neq 0} im^{-1} \hat{\nu}(m) \exp(ims),$$

applying Theorem 1 with $L = AH$ yields $\mu_1 = AH1(x_1) = 0$, while, for $j = 2, \dots, n$,

$$\mu_j = \sum_{m \equiv j-1 (\mathrm{mod}\, n)} im^{-1} \rho_m$$
$$= i\frac{n}{\pi} \sin \frac{(j-1)\pi}{n} \sum_{k \in \mathbf{Z}} (-1)^k (j-1+kn)^{-2},$$

which yields (ii).

REMARK. The eigenvalues $\{\lambda_j\}$ are well known (see, for example, [7], [10], [22]).

Observe now that $\lambda_j > 0$ for all $j = 1, \dots, n$ and recall the operator B_h defined by (2.6).

COROLLARY 3. $|B_h H_h|_0 \leq 1$, *for all* $h = 2\pi/n$, $n \in \mathbf{N}$.

PROOF. By Corollary 2,

$$B_h H_h e_h^j = (Q_h A_h)^{-1} Q_h A H_h e_h^j = (\mu_j/\lambda_j) e_h^j.$$

So if $\nu = \sum_{j=1}^n \nu_j e_h^j \in V_h$, then since $\{e_h^j\}$ is an orthogonal basis,

$$|B_h H_h \nu|_0^2 = \sum_{j=1}^n |u_j/\lambda_j|^2 |\nu_j|^2 (e_h^j, e_h^j)$$
$$\leq \max_j |u_j/\lambda_j|^2 \sum_{j=1}^n |\nu_j|^2 (e_h^j, e_h^j)$$
$$= \max_j |u_j/\lambda_j|^2 |\nu|_0^2.$$

The result follows since $|u_j/\lambda_j| \leq 1$ for $j = 1, \dots, n$.

Theorem 4. *For all* $h = 2\pi/n$, $n \in \mathbf{N}$, *we have*

$$|B_h|_0 \le B,$$

with $1.34 < B < 1.35$.

This result has been proved in [22] by (essentially) a careful Fourier analysis employing (2.2), and a result similar to Corollary 2 (i). We omit its proof here.

3. The polygon

Return to the collocation method of Section 1, and recall that each Γ_j, was divided into $m_j N$ subintervals. Introduce the ratios

$$d_l = \frac{m_{2l}}{m_{2l+1}} \frac{|\Gamma_{2l+1}|}{|\Gamma_{2l}|}, \quad l = 1, \dots, r,$$

(where $m_{2r+1} = m_1$, and $\Gamma_{2r+1} = \Gamma_1$). Recall also the operators A and H introduced in Section 2. The following decomposition of K generalises that obtained in [21].

Theorem 5. $K = A(I - HR + E)$ *where* E *is compact on* H^0, *and* $R = \sum_{l=1}^r R_l$, *where, for* $l = 1, \dots, r-1$,

$$R_l\nu(s) = \begin{cases} \int_{S_{2l}}^{S_{2l+1}} r_l^+\left(\frac{S_{2l}-s}{\sigma-S_{2l}}\right)\frac{\nu(\sigma)d\sigma}{\sigma-S_{2l}}, & s \in [S_{2l-1}, S_{2l}] \\ \int_{S_{2l-1}}^{S_{2l}} r_l^-\left(\frac{s-S_{2l}}{S_{2l}-\sigma}\right)\frac{\nu(\sigma)d\sigma}{(S_{2l}-\sigma)}, & s \in [S_{2l}, S_{2l+1}] \\ 0, & s \notin [S_{2l-1}, S_{2l+1}], \end{cases}$$

and

$$R_r\nu(s) = \begin{cases} \int_{S_{2r}}^{2\pi+S_1} r_r^+\left(\frac{S_{2r}-s}{\sigma-S_{2r}}\right)\frac{\nu(\sigma)d\sigma}{\sigma-S_{2r}}, & s \in [S_{2r-1}, S_{2r}] \\ \int_{S_{2r-1}-2\pi}^{S_0} r_r^-\left(\frac{s-S_0}{S_0-\sigma}\right)\frac{\nu(\sigma)d\sigma}{(S_0-\sigma)}, & s \in [S_0, S_1] \\ 0, & s \notin [S_0, S_1] \cup [S_{2r-1}, S_{2r}], \end{cases}$$

with

$$r_l^\pm = \pm\frac{1}{\pi}\left\{\frac{\sigma + d_l^{\pm 1}\cos\chi_l\pi}{\sigma^2 + (d_l^{\pm 1})^2 + 2\sigma d_l^{\pm 1}\cos\chi_l\pi} - \frac{1}{1+\sigma}\right\}.$$

Furthermore, for each $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$|R|_0 < \max\left\{\left(1 - \cos\frac{\chi_l\pi}{2}\right) : l = 1, \dots, r\right\} + \varepsilon$$

provided

$$\max\{|d_l - 1| : l = 1, \dots, r\} < \delta.$$

PROOF. By (2.3),

$$K = A(I + A^{-1}(K - A)) = A(I - HD(K - A) + E),$$

with E compact. A minor variation of the argument in [21, Section 5] then shows that

$$D(K - A) = \sum_{l=1}^{r} R_l + E,$$

with E again compact, and R_l as given above, and the first part of the theorem follows.

To prove the second part, it is sufficient to show that, for any $\varepsilon > 0$ and, for any particular $l \in \{1, \dots, r-1\}$, there exists $\delta > 0$ such that

$$|R_l \nu|_0 \le (1 - \cos\frac{\chi_l \pi}{2} + \varepsilon)|\nu|_{[S_{2l-1}, S_{2l+1}]}|_0, \tag{3.1}$$

when $|d_l - 1| < \delta$. To prove (3.1)we assume, without loss of generality, that

$$S_{2l-1} = a^- < 0 = S_{2l} < a^+ = S_{2l+1}. \tag{3.2}$$

Then

$$R_l \nu(s) = \begin{cases} \int_0^{a^+} r_l^+ \left(-\frac{s}{\sigma}\right) \frac{\nu(\sigma)\, d\sigma}{\sigma}, & s \in [a^-, 0] \\ -\int_{a^-}^{0} r_l^- \left(-\frac{s}{\sigma}\right) \frac{\nu(\sigma)\, d\sigma}{\sigma}, & s \in [0, a^+] \\ 0, & s \notin [a^-, a^+]. \end{cases} \tag{3.3}$$

Define the Mellin transform of $\nu : [0, \infty) \to \mathbf{R}$ by

$$\hat{\nu}(w) = \int_0^\infty s^{iw} \nu(s) \frac{ds}{s}.$$

Then it is well known (eg in [9], [21]) that

$$|R_l \nu|_0 \le \max_{\lambda \in \mathbf{R}} \{|\hat{r}_l^+(\lambda - i/2)|, |\hat{r}_l^-(\lambda - i/2)|\} \times |\nu|_{[a^-, a^+]}|_0. \tag{3.4}$$

From integral transform tables,

$$\begin{aligned} \hat{r}_l^\pm(\lambda - i/2) &= \pm \frac{1}{\cosh \pi\lambda} \left[d_l^{\pm(i\lambda - \frac{1}{2})} \cos\left(i\lambda - \frac{1}{2}\right) \chi_l \pi - 1 \right] \\ &= h_l^\pm(d_l, \lambda), \text{ say.} \end{aligned}$$

Let us consider

$$h_l^+(d_l, \lambda) = \frac{1}{\cosh \pi\lambda} \left[d_l^{(i\lambda - \frac{1}{2})} \cos\left(i\lambda - \frac{1}{2}\right) \chi_l \pi - 1 \right].$$

The first derivative of this function with respect to d_l is bounded on $(d_l, \lambda) \in [1/2, 3/2] \times \mathbf{R}$. Hence there exists a constant C_0 such that, for all $(d_l, \lambda) \in [1/2, 3/2] \times \mathbf{R}$,

$$|h_l^+(d_l, \lambda) - h_l^+(1, \lambda)| \le C_0 |d_l - 1|. \tag{3.5}$$

But (see [21]),

$$\max_{\lambda \in \mathbf{R}} |h_l^+(1, \lambda)| = 1 - \cos \frac{\chi_l \pi}{2}. \tag{3.6}$$

Now let $\varepsilon > 0$ and choose $0 < \delta < \max\{\frac{1}{2}, \varepsilon/C_0\}$. Then (3.5), (3.6) show

$$\max_{\lambda \in \mathbf{R}} |h_l^+(d_l, \lambda)| < 1 - \cos \frac{\chi_l \pi}{2} + \varepsilon,$$

provided $|D_l - 1| < \delta$. An identical bound can be proved for h_l^-. Using these in (3.4) proves (3.1), as required.

From the last part of the theorem, it is clear that by choosing m_{2l}/m_{2l+1} sufficiently close to $|\Gamma_{2l}|/|\Gamma_{2l+1}|$, for $l = 1, \dots, r$ we can guarantee

$$|R|_0 < 1. \tag{3.7}$$

Let us assume for the remainder of the paper that such a choice of $\{m_j\}$ has been made. Using Theorem 5, (1.3) may then be written

$$(I - (HR - E))w = f := A^{-1} g. \tag{3.8}$$

Similarly (1.7) may be written

$$Q_h A(I - HR + E)w_h = Q_h A f,$$

or, equivalently,

$$(I - B_h(HR - E))w_h = B_h f, \tag{3.9}$$

with B_h given by (2.6).

By (2.4) and (3.7), HR is a contraction on H^0. Standard uniqueness results for (1.1) imply $I - (HR - E)$ is one-one on H^0. So the Fredholm alternative ensures $I - (HR - E)$ is invertible on H^0, and (3.8) has a unique solution in H^0 whenever $g \in H^1$. By (2.4) and Theorem 4,

$$|B_h HR|_0 \le B|R|_0$$

uniformly in N, with B as in Theorem 4. Suppose now

$$B|R|_0 < 1. \tag{3.10}$$

Since $B_h \to I$ pointwise on H^0, as $N \to \infty$, standard arguments then show that $I - B_h(HR - E)$ has a uniformly bounded inverse on H^0, and then (3.8), (3.9) give

$$\begin{aligned} |w - w_h| &= |(I - B_h(HR - E))^{-1}(I - B_h)w|_0 \\ &\le C|(I - B_h)w|_0 \to 0, \quad \text{as } N \to \infty. \end{aligned}$$

However, $|R|_0 \to 1$ when any $\chi_l \to \pm 1$. Since $1.34 < B < 1.35$, the requirement (3.10) leads to unnatural restrictions on the angles at each of the corners of Γ. This is similar to the situation for second-kind boundary integral equations on polygons [1]. We circumvent it by a modification similar to that used for second-kind equations in [6].

4. Modified collocation

For any fixed nonnegative integer $i^* \le (\min_j m_j)N$, let I^h denote the region

$$[S_0, S_0 + i^* h] \cup \left(\bigcup_{l=1}^{r-1} [S_{2l} - i^* h, S_{2l} + i^* h] \right) \cup [S_{2r} - i^* h, S_{2r}].$$

Then define the *modified operator* K^h by

$$K^h \nu(s) = -\frac{1}{\pi} \left\{ \int_{I^h} \log|2e^{-\frac{1}{2}} \sin((s-\sigma)/2)| \nu(\sigma)\, d\sigma \right.$$
$$\left. + \int_{I \setminus I^h} \log|\gamma(s) - \gamma(s)| \nu(\sigma)\, d\sigma \right\}.$$

When $i^* = 0$, $K^h = K$. When $i^* > 0$, the kernel of K, $\log|\gamma(s) - \gamma(\sigma)|$, is replaced by the kernel of A, $\log|2e^{-\frac{1}{2}} \sin(s-\sigma)/2|$, when $\sigma \in I^h$, and $s \in [-\pi, \pi]$.

We then define the *modified collocation solution* w_h to (1.3) by

$$Q_h K^h w_h = Q_h g. \tag{4.1}$$

When $i^* = 0$, the method coincides with (1.7). When $i^* > 0$, the modification is equivalent to changing the (j, k)th matrix element (1.8) to

$$-\frac{1}{\pi} \int_{s_{k-1}}^{s_k} \log|2e^{-\frac{1}{2}} \sin((s_j - \sigma)/2)|\, d\sigma,$$

when k is such that $[s_{k-1}, s_k] \subseteq I^h$.

For the analysis of (4.1) we define the truncation operator:

$$T^h \nu(s) = 0, \qquad s \in I^h$$
$$T^h \nu(s) = \nu(s), \qquad s \in [-\pi, \pi] \setminus I^h.$$

Then we can write

$$K^h = A + (K - A) T^h. \tag{4.2}$$

Moreover, $|T^h|_0 = 1$, and $T^h \to I$ pointwise on H^0, as $N \to \infty$. The identity,

$$T^h B_h - I = (T^h - I) + T^h(B_h - I),$$

and the fact that $B_h \to I$ pointwise on H^0, as $N \to \infty$ then imply that $T^h B_h \to I$ pointwise on H^0, as $N \to \infty$. Now, for $w \in H^0$, define

$$d_M(w, h) = \max\{|w - B_h w|_0, |w - T^h w|_0\},$$

Then, by the above remarks,

$$d_M(w, h) \to 0 \quad \text{as } N \to \infty,$$

for any $w \in H^0$. Also, introduce P_h, the orthogonal projection of H^0 onto V_h. Then $|P_h|_0 = 1$, and $P_h \to I$ pointwise on H^0 as $N \to \infty$. Throughout this section, C will denote a generic constant independent of N.

The following technical result is at the heart of the theory of (4.1).

THEROEM 6. *For each $\varepsilon > 0$, there exists i^* independent of N, such that*

$$|(I - P_h)RT^h|_0 < \varepsilon, \tag{4.3}$$

for all N sufficiently large.

PROOF. Recall the mesh on $[-\pi, \pi]$ defined in Section 1, with mesh diameter $h = 2\pi/n$, where $n = (m_1 + \cdots + m_{2r})N = MN$. Clearly the theorem will be proved if we can show (4.3) with R replaced by R_l for arbitrary $l \in \{1, \ldots, r\}$. Without loss of generality we assume $l \le r - 1$, so that we can adopt the simplification (3.2), (3.3). Set

$$m^- = m_{2l}N, \qquad m^+ = m_{2l+1}N.$$

Then the mesh on $[S_{2l-1}, S_{2l+1}] = [a^-, a^+]$ coincides with the points

$$s_j = jh, \qquad j = -m^-, \ldots, 0, \ldots, m^+.$$

For any $\nu : [-\pi, \pi] \to \mathbf{R}$, let ν_j denote its restriction to $[s_{j-1}, s_j]$. Observe that there exists a constant C such that

$$\sup_{\sigma \in [0, \infty)} |\sigma^j D^j r_l^{\pm}(\sigma)| \le C, \qquad j = 0, 1. \tag{4.4}$$

Then for $\nu \in H^0$,

$$|(I - P_h)(R_l T^h \nu)|_0^2 = \left(\sum_{j=-m^-+1}^{0} + \sum_{j=1}^{m^+} \right) |(I - P_h)(R_l T^h \nu)_j|_0^2. \tag{4.5}$$

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Let us investigate the second sum on the right-hand side of (4.5). It may be written

$$|(I-P_h)(R_l T^h \nu)_1|_0^2 + \sum_{j=2}^{m^+} |(I-P_h)(R_l T^h \nu)_j|_0^2 = b_1 + b_2, \tag{4.6}$$

say. But, for $j = 2, \ldots, m^+$, using well known polynomial approximation results in H^0, we have

$$|(I-P_h)(R_l T^h \nu)_j|_0^2 \le Ch^2 |(D(R_l T^h \nu)_j|_0^2, \tag{4.7}$$

and

$$D(R_l T^h \nu)_j(s) = -\int_{a^-}^{-i^* h} \frac{1}{\sigma}(Dr_l^-)\left(-\frac{s}{\sigma}\right) \frac{\nu(\sigma)\, d\sigma}{\sigma}, \qquad s \in [0, a^+].$$

So if $|\nu|_0 \le 1$, we have

$$\begin{aligned} |D(R_l T^h \nu)_j(s)| &\le \int_{a^-}^{-i^* h} \frac{1}{\sigma^2} \left|Dr_l^- \left(-\frac{s}{\sigma}\right)\right| |\nu(\sigma)|\, d\sigma \\ &\le \left\{ \int_{a^-}^{-i^* h} \frac{1}{\sigma^4} \left|Dr_l^- \left(-\frac{s}{\sigma}\right)\right|^2 d\sigma \right\}^{\frac{1}{2}}, \qquad s \in [0, a^+]. \end{aligned} \tag{4.8}$$

Thus (4.7), (4.8) show that for $j = 2, \ldots, m^+$,

$$\begin{aligned} |(I-P_h)(R_l T^h \nu)_j|_0^2 &\le Ch^2 \int_{s_{j-i}}^{s_j} \int_{a^-}^{-i^* h} \frac{1}{\sigma^4} \left|Dr_l^- \left(-\frac{s}{\sigma}\right)\right|^2 d\sigma\, ds \\ &= Ch^2 \int_{a^-}^{-i^* h} \frac{1}{\sigma^2} \left\{ \int_{s_{j-1}}^{s_j} \left(-\frac{s}{\sigma}\right)^2 \left|Dr_l^- \left(-\frac{s}{\sigma}\right)\right|^2 \frac{1}{s^2}\, ds \right\} d\sigma \\ &\le Ch^2 \left\{ \int_{a^-}^{-i^* h} \frac{1}{\sigma^2}\, d\sigma \right\} \left\{ \int_{s_{j-1}}^{s_j} \frac{1}{s^2}\, ds \right\}, \end{aligned}$$

where the last inequality uses (4.4). Hence

$$\begin{aligned} |(I-P_h)(R_l T^h \nu)_j|_0^2 &\le Ch^2 \left(\frac{1}{i^* h} + \frac{1}{a^-}\right) \left(\frac{1}{s_{j-1}} - \frac{1}{s_j}\right) \\ &\le C\frac{h}{i^*} \left(\frac{1}{s_{j-1}} - \frac{1}{s_j}\right) \\ &\le C\frac{h^2}{i^*} s_{j-1}^{-2} = C\frac{1}{i^*} \left(\frac{1}{j-1}\right)^2. \end{aligned}$$

Hence,

$$b_2 \le C\frac{1}{i^*} \sum_{j=2}^{\infty} \frac{1}{(j-1)^2} \le C\frac{1}{i^*}, \tag{4.9}$$

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for all N. Also, if $|\nu|_0 \le 1$,

$$\begin{aligned} b_1 &\le |(R_l T^h \nu)_1|_0^2 \\ &= \int_0^{s_1} \left| \int_{a^-}^{-i^* h} r_l^- \left(-\frac{s}{\sigma}\right) \frac{\nu(\sigma)\, d\sigma}{\sigma} \right|^2 ds \\ &\le \int_0^{s_1} \int_{a^-}^{-i^* h} \frac{1}{\sigma^2} \left| r_l^- \left(-\frac{s}{\sigma}\right)\right|^2 d\sigma\, ds \\ &= \int_{a^-}^{-i^* h} \frac{1}{\sigma^2} \int_0^{s_1} \left| r_l^- \left(-\frac{s}{\sigma}\right)\right|^2 ds\, d\sigma \\ &\le Ch \left[\frac{1}{i^* h} + \frac{1}{a^-}\right] \le C \frac{1}{i^*}, \end{aligned} \tag{4.10}$$

where we have used (4.4) again. Collecting (4.9), (4.10) and recalling (4.6) shows the second sum in (4.5) is arbitrarily small for appropriate i^* and all N. The first sum in (4.5) may be estimated analogously, and the lemma follows.

Next we prove the main result of this section.

THEOREM 7. *Let $g \in H^1$. Then there exists a fixed $i^* \ge 0$ such that, for all N sufficiently large,* (4.1) *defines $w_h \in V_h$ uniquely, and*

$$|w - w_h|_0 \le C d_M(w, h) \to 0, \quad \text{as } N \to \infty,$$

where w is the unique solution of (1.3).

REMARK. Recall d_M is defined just prior to the statement of Theorem 6.

PROOF. Using (4.2), (4.1) may be written

$$Q_h(A + (K - A)T^h) w_h = Q_h g.$$

By Theorem 5, this is equivalent to

$$Q_h A(I - (HR - E)T^h) w_h = Q_h A f$$

with $f = A^{-1} g$, i.e.

$$(I - B_h(HR - E)T^h) w_h = B_h f, \tag{4.11}$$

with B_h defined by (2.6). We examine the solvability of (4.11).

Recall that, as discussed at the end of Section 3, $(I-(HR-E))$ is invertible on H^0, and hence (by (2.4), (3.7)), so is $I + (I - HR)^{-1} E$. Recall also that for any $\mathscr{L} : H^0 \to H^0$, the identities

$$(I - \mathscr{L} T^h)^{-1} = I + \mathscr{L}(I - T^h \mathscr{L})^{-1} T^h \tag{4.12}$$

$$(I - T^h \mathscr{L})^{-1} = I + T^h (I - \mathscr{L} T^h)^{-1} \mathscr{L} \tag{4.13}$$

are valid (when the inverses involved exist). By (2.4), Corollary 3, and Theorem 4,

$$|B_hHRT^h|_0 \le |B_hHP_hRT^h|_0 + |B_kH(I-P_h)RT^h|_0$$
$$\le |R|_0 + B|(I-P_h)RT^h|_0.$$

So, by Theorem 6 and (3.7), there exists fixed i^* such that $|B_hHRT^h|_0$ is bounded below 1 as $N \to \infty$, and so

$$|(I - B_hHRT^h)^{-1}|_0 \le C, \quad \text{as } N \to \infty.$$

By (4.13), then

$$|(I - T^hB_hHR)^{-1}|_0 \le C, \quad \text{as } N \to \infty. \tag{4.14}$$

Thus,

$$\begin{aligned}
&|(I - T^hB_hHR)^{-1}T^hB_hE - (I-HR)^{-1}E|_0 \\
&\quad \le |(I - T^hB_hHR)^{-1}(T^hB_h - I)E|_0 \\
&\qquad + |(I - T^hB_hHR)^{-1}(T^hB_h - I)HR(I-HR)^{-1}E|_0 \to 0 \quad \text{as } N \to 0,
\end{aligned}$$

since E is compact, and $T^hB_h \to I$ pointwise on H^0.

The operator $I + (I - T^hB_hHR)^{-1}T^hB_hE$ therefore tends in norm to the invertible operator $I + (I - HR)^{-1}E$ and thus has an inverse which is uniformly bounded, as $N \to \infty$. By (4.14), then

$$|(I - T^hB_h(HR - E))^{-1}|_0 \le C, \quad \text{as } N \to \infty,$$

and so, by (4.12),

$$|(I - B_h(HR - E)T^h)^{-1}|_0 \le C, \quad \text{as } N \to \infty. \tag{4.15}$$

Hence for sufficiently large N, (4.11) is uniquely solvable, and

$$\begin{aligned}
w - w_h &= (I - B_h(HR-E)T^h)^{-1}(w - B_h((HR-E)T^hw + f)) \\
&= (I - B_h(HR-E)T^h)^{-1}[(w - B_hw) + B_h(HR-E)(w - T^hw)]
\end{aligned}$$

where we have used (3.8). Recall that $w \in H^0$. Taking $|\cdot|_0$ and using the triangle inequality with Theorem 4 and (4.15) yields the theorem.

The rate of convergence of w_h to w can now easily be deduced from Theorem 7. It is known that when $\gamma(s_l) = \mathbf{x}_{2l}$ then, for s near s_l, we have, in general, $w(s) = O(|s - s_l|^{\beta_l - 1})$, where $\beta_l = (1 + |\chi_l|)^{-1}$. (If we are using the "direct" boundary integral method on a convex polygon, the singularity will not be as severe as this: here we consider the worst case which may arise.) Since $(I - B_h)$ annihilates V_h, we have, by Theorem 4, that $|w - B_h w|_0$ is of the order of the error in the best approximation to w from V_h. It follows (see, e.g. [22]) that $|w - B_h w|_0 = O(N^{1/2-\beta})$, where $\beta = \min\{\beta_l : l = 1, \dots, r\} \in (1/2, 1)$. Since i^* is fixed, $|w - T^h w|_0$ is of the same order. So, overall,

$$|w - w_h|_0 = O(N^{1/2-\beta}) \quad \text{as } N \to \infty. \tag{4.16}$$

Thus, for any polygon, the modified method (with appropriate fixed i^*) is stable, and its rate of convergence as $N \to \infty$ is precisely the same as that of the unmodified method [22] (for which the stability theory is angle-dependent). This result is reminiscent of the role played by modification in collocation methods for second-kind equations [6]. In contrast to [6], however, we have not been able here to demonstrate any example in which the modification is *necessary* for stability. Nevertheless, the results here show conclusively that, even for a domain with very fine angles, not very much can go wrong with the collocation method. Moreover the modification has a more practical application to the acceleration of convergence of iterative methods for solving the collocation equations, and we develop this theme in Section 5 below.

Unfortunately all the theoretical results presented here are for uniform subdivisions of (each side of) the polygon Γ. This is because our theory for collocation on the polygon involves "preconditioning" the collocation matrix with the discretisation of the operator A, and needs the results of Section 2, which depend very much on uniform grids. Nevertheless, it is well known that graded meshes are necessary to obtain optimal convergence rates in practice, and we now report a numerical experiment demonstrating this fact.

In this experiment, we solved (1.1), with $g(\mathbf{x}) = (x_1^2 + x_2^2)/\pi$, and with Γ the boundary of the L-shaped domain with vertices (in anticlockwise order): $(0, 0)$, $(1, 0)$, $(1, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 1)$, and $(0, 1)$. The sides of length 1 are subdivided into N segments (N even), while the other sides (all of length $\frac{1}{2}$) are subdivided into $N/2$ segments. Then $n = 4N$, and $h = 1/(4N)$. The mesh is graded near each corner in the usual way (e.g. [5], [6], [8], [20]). For example, for given $q > 1$, the mesh on the side joining $(0, 0)$ to $(1, 0)$ consists of the points $((\frac{1}{2})(2i/N)^q, 0)$, $i = 1, \dots, N/2$, plus their reflections in the line $x_1 = \frac{1}{2}$. The mesh is similarly graded near all the other corners, using the same grading exponent q. The modification was not needed to ensure the stability of the method (i.e. the method used was simply (1.2)).

Rather than calculating the rate of convergence of u_h, we look at the following functionals of u_h:

$$(a) \int_\Gamma u_h(\zeta)\, d\Gamma(\zeta); \quad (b)\ \varphi_h(0.25, 0.25); \quad (c)\ \varphi_h(-0.25, -0.25),$$

where

$$\varphi_h(z) = -\int_\Gamma \log|z-\xi|u_h(\xi)\,d\Gamma(\xi),$$

a potential which is often required in practice. (For example if the indirect boundary integral method is used to solve Laplace's equation in a region exterior or interior to Γ.) The integrals required for implementation of the collocation method are computed analytically. Those required in the calculation of the potentials are computed by Simpson's rule. The results, given in Table 1 below, show an estimated order of convergence (EOC) which climbs to a maximum of about 3 as the grading is increased. Since $O(N^{-3})$ is the rate of convergence which would be expected for potentials when Γ is smooth [16], the results with the graded meshes are very encouraging. However at present the theory does not explain properly the observed success of the method.

TABLE 1

q	N	(a)	EOC	(b)	EOC	(c)	EOC
1	16	1.1251		0.25651		−0.34490	
	32	1.1299		0.25621		−0.34604	
	64	1.1318	1.37	0.25611	1.59	−0.34649	1.33
	128	1.1325	1.36	0.25608	1.43	−0.34668	1.33

q	N	(a)	EOC	(b)	EOC	(c)	EOC
2	16	1.0903		0.26113		−0.33704	
	32	1.1258		0.25710		−0.34457	
	64	1.1318	2.59	0.25622	2.19	−0.34642	2.56
	128	1.1328	2.56	0.25608	2.66	−0.34673	2.56

q	N	(a)	EOC	(b)	EOC	(c)	EOC
3	16	1.1273		0.25749		−0.34491	
	32	1.1322		0.25632		−0.34652	
	64	1.1329	2.83	0.25609	2.37	−0.34676	2.78
	128	1.1329	2.92	0.25606	2.86	−0.34679	2.91

Even though the modification was not necessary for stability in this example, it is still of interest to know how well the modified method calculates

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linear functionals of u. To complete this section, we now use Theorem 7 to prove a "superconvergence" result for the error in approximating the linear functional (w, ν) by (w_h, ν), (where ν is a smooth given function). Note that the uniformity of the meshes demanded by Theorem 7 restricts the rates of convergence which may be proved. Better rates will be obtainable when the stability theory is extended to graded meshes.

Theorem 8. *Let* $\nu \in H^1$. *Then, when* $i^* \geq 1$, *the uniquely defined* $w_h \in S_h$ *in* (4.1) *satisfies*

$$|(w_h, \nu) - (w, \nu)| \leq CN^{-1/2} d_M(w, h)|\nu|_1.$$

Remark. When $i^* = 0$, the modification is not in use, and an analogous (but smaller) error bound is given in [22].

Proof. In the proof we use the formulae

$$K - K^h = (K - A)(I - T^h),$$
$$|(I - T^h)w_h|_0 \leq Cd_M(w, h)$$

without further appeal. Let $\nu \in H^1$, and write $\nu = K\nu_1$, where $\nu_1 \in H^0$. Then

$$\begin{aligned}(w_h - w, \nu) &= (w_h - w, K\nu_1) = (K(w_j - w), \nu_1)\\ &= ((I - Q_h)K(w_h - w), \nu_1) + (Q_hK(w_h - w), \nu_1).\end{aligned}$$

Since $Q_hK^hw_h = Q_hKw$, we have

$$\begin{aligned}Q_hK(w_h - w) &= Q_h(K - K^h)w_h\\ &= -(I - Q_h)(K - K^h)w_h + (K - K^h)w_h.\end{aligned}$$

Hence

$$\begin{aligned}(w_h - w, \nu) = ((I - Q_h)K(w_h - w), \nu_1) &- ((I - Q_h)(K - K^h)w_h, \nu_1)\\ &+ ((K - K^h)w_h, \nu_1).\end{aligned} \tag{4.17}$$

Now, it is clear that

$$\begin{aligned}|((I - Q_h)K(w_h - w), \nu_1)| &\leq |(I - Q_h)K|_0|w_h - w|_0|\nu_1|_0\\ &\leq Chd_M(w, h)|\nu|_1,\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}|((I - Q_h)(K - K^h)w_h, \nu_1)| &\leq |(I - Q_h)(K - A)|_0|(I - T^h)w_h|_0|\nu_1|_0\\ &\leq Chd_M(w, h)|\nu|_1.\end{aligned} \tag{4.19}$$

However,

$$((K - K^h)w_h, \nu_1) = ((K - A)(I - T^h)w_h, \nu_1)$$
$$= ((I - T^h)w_h, (I - T^h)(K - A)\nu_1).$$

So,

$$|((K - K^h)w_h, \nu_1)| \le |(I - T^h)w_h|_0 |(I - T^h)(K - A)|_0 |\nu_1|_0$$
$$\le Ch^{\frac{1}{2}} d_M(w, h)|\nu|_1. \qquad (4.20)$$

The next follows by substitution of (4.18)–(4.20) into (4.17), and using $h = O(N^{-1})$.

5. An iterative solver

The theory of the previous section has an interesting application to the acceleration of multiple-grid methods for solving the collocation equations (1.7). Such multiple-grid methods apply most naturally to second-kind equations rather than first kind, but recall from Section 3 that we may write (1.7) as

$$(Q_h A_h)(I + B_h A^{-1}(K - A))w_h = (Q_h A_h)B_h f$$

where $f = A^{-1}g$. The matrix of $Q_h A_h$ is inverted in $O(N \log N)$ operations as described in Section 2, and so it is realistic to premultiply by $(Q_h A_h)^{-1}$, obtaining the non-standard second-kind numerical method

$$(I + B_h A^{-1}(K - A))w_h = B_h f. \qquad (5.1)$$

This can be rearranged as

$$w_h = B_h f - B_h A^{-1}(K - A)w_h. \qquad (5.2)$$

A multiple-grid procedure, e.g. [11], [18], can be written down for (5.1), using (5.2) as a smoothing step, and this procedure can be shown to converge provided $A^{-1}(K - A)$ is sufficiently smooth (e.g. compact on H^0). Most importantly, the multiple-grid method for (5.1) boils down to nothing more than a preconditioned multiple-grid procedure for (1.7), where the preconditioner is the circulant matrix corresponding to $Q_h A_h$. Hence the operators in (5.1) (which are unpleasantly obscure) never have to be calculated in practice.

If Γ is smooth, $A^{-1}(K-A)$ is compact on H^0, and the convergence proof mentioned above follows. (See [18] for related observations for a multigrid Galerkin-type method.) If Γ is a polygon, this proof fails. A similar failure of proof occurs for second-kind equations on polygons, and indeed this theoretical failure manifests itself in the actual practical failure of the iterative method when the polygon has very sharp corners [2]. In [2], these failures were corrected by an appropriate modification of the equations near the corners. Here we show that a similar modification leads to the generation of pleasant convergence proofs for (preconditioned) multiple-grid methods applied to (1.7).

We remark in passing that the idea of preconditioning with a circulant matrix has received recent attention in a different context (see [4] and the references therein). We also mention that the Galerkin equations for a different first-kind integral equation are solved iteratively in [12].

To simplify the discussion here, we shall restrict attention to a popular two-grid algorithm (e.g. [11, p. 309]). This algorithm avoids the recursion present in full multigrid schemes, but in many circumstances is powerful enough to solve (1.7) efficiently. We first give an algorithm description and then we discuss the theory. The idea is to solve (1.7) for $h = 2\pi/n$, $n = MN$, N large iteratively by a process involving direct inversion of (1.7) only for some larger h, say $\tilde{h} = 2\pi/\tilde{n}$, $\tilde{n} = M\widetilde{N}$, with $\widetilde{N}$ small. For convenience we assume

$$N = k\widetilde{N}, \tag{5.3}$$

for some $k \in \mathbf{N}$, although the algorithm does not depend strongly on this assumption. Note that (5.3) implies $V_{\tilde{h}} \subseteq V_h$.

Then we propose the following (preconditioned) two-grid algorithm for (1.7). Choose $w_h^{(0)}$. (Usually $w_h^{(0)} = w_{\tilde{h}}$, or $w_h^{(0)} = 0$.) Then, for $v \geq 0$, until $|w_h^{(v+1)} - w_h^{(v)}|_0$ is sufficiently small, perform the following steps.

$$\begin{array}{ll}
\textbf{Input} & w_h^{(v)} \\
\textbf{Compute residual} & d_h^{(v)} := Q_h(g - Kw_h^{(v)}) \\
\textbf{Precondition} & d_h^{(v)} := (Q_hA_h)^{-1}d_h^{(v)} \\
\textbf{Smoothing step} & \delta_h^{(v)} := Q_h(K - A)d_h^{(v)} \\
\textbf{Precondition} & \delta_h^{(v)} := (Q_hA_h)^{-1}\delta_h^{(v)} \\
\textbf{Coarse grid correction} & \delta_{\tilde{h}}^{(v)} := -(Q_{\tilde{h}}K_{\tilde{h}})^{-1}(Q_{\tilde{h}}A)\delta_h^{(v)} \\
\textbf{Output} & w_h^{(v+1)} = w_h^{(v)} + d_h^{(v)} + \delta_{\tilde{h}}^{(v)}.
\end{array} \tag{5.4}$$

Some elementary algebra shows (5.4) is equivalent to the following algorithm for (5.1).

$$\begin{array}{ll}
\textbf{Input} & w_h^{(v)} \\
\textbf{Compute residual} & d_h^{(v)} := B_hf - (I + L_h)w_h^{(v)} \\
\textbf{Smoothing step} & \delta_h^{(v)} := L_hd_h^{(v)} \\
\textbf{Coarse grid correction} & \delta_{\tilde{h}}^{(v)} := -(I + L_{\tilde{h}})^{-1}B_{\tilde{h}}\delta_h^{(v)} \\
\textbf{Output} & w_h^{(v+1)} := w_h^{(v)} + d_h^{(v)} + \delta_h^{(v)},
\end{array} \tag{5.5}$$

where $L_h := B_h A^{-1}(K - A)$. Standard arguments (e.g. [11]) applied to (5.5) then yield the error estimate

$$|w_h - w_h^{(v+1)}|_0 \le C_{\widetilde{N},k} |w_h - w_h^{(v)}|_0, \tag{5.6}$$

where

$$C_{\widetilde{N},k} = C^1_{\widetilde{N},k} C^2_{\widetilde{N},k}, \tag{5.7}$$

and

$$\begin{aligned} C^1_{\widetilde{N},k} &= |(I + L_{\tilde{h}})^{-1}|_0 \\ C^2_{\widetilde{N},k} &= |(B_{\tilde{h}} - I)L_h + (B_{\tilde{h}} L_h - L_{\tilde{h}})L_h|_0. \end{aligned} \tag{5.8}$$

Observe now that the operators $B_{\tilde{h}} B_h - B_h$ and $B_{\tilde{h}} B_h - B_{\tilde{h}}$ both annihilate $V_{\tilde{h}}$, and hence, as $\widetilde{N} \to \infty$,

$$B_{\tilde{h}} B_h - B_h \to 0, \qquad B_{\tilde{h}} B_h - B_{\tilde{h}} \to 0, \tag{5.9}$$

with the convergence in each case being pointwise on H^0. Thus, when $A^{-1}(K - A)$ is compact, $C^2_{\widetilde{N},k}$ may be made arbitrarily small and $C^1_{\widetilde{N},k}$ is uniformly bounded as $\widetilde{N} \to \infty$. So for $\widetilde{N}$ sufficiently large $C_{\widetilde{N},k} < 1$, and (5.4) converges linearly. Then $w_h^{(v)}$ converges to within $O(1/n) = O(h)$ tolerance of w_h in $O(\log n)$ iterations. Each iteration costs $O(n^2)$ multiplications, provided the coarse grid correction is considered negligible (recall that each preconditioning step costs $O(n \log n)$ multiplications). Convergence is thus obtained in $O(n^2 \log n) = O(N^2 \log N)$ multiplications.

When Γ is a polygon, $A^{-1}(K - A) = -HR + E$ is not compact and the above convergence proof fails. However consider the modified collocation scheme (4.1). This may be solved iteratively by applying (5.4) with K replaced by K^h. This is again equivalent to (5.5), but this time with

$$L_h = B_h(-HR + E)T^h \quad \text{and} \quad L_{\tilde{h}} = B_{\tilde{h}}(-HR + E)T^h.$$

(i.e. it is a two-grid method for (4.11).) An error estimate is again given by (5.6)–(5.8). We analyse its convergence in Theorem 9 below, under the further assumption that i^* is allowed to depend on $\widetilde{N}$. For simplicity we assume that

$$i^* = \widetilde{N}. \tag{5.10}$$

We discuss the practical implications of this at the end of the section. Recall from Section 4 that T^h truncates a function to zero on the interval I^h. Assuming sufficiently large k, assumptions (5.3), (5.10), yield

$$I^h = \left[S_0, S_0 + \frac{2\pi}{kM}\right] \cup \left(\bigcup_{l=1}^{r-1} \left[S_{2l} - \frac{2\pi}{kM}, S_{2l} + \frac{2\pi}{kM}\right]\right) \cup \left[S_{2r} - \frac{2\pi}{kM}, S_{2r}\right].$$

So I^h *does not vary* with $\tilde{N}$. Under these assumptions we have the following result.

THEOREM 9. *For sufficiently large* k, (5.4) *with* K *replaced by* K^h *converges linearly for all* $\tilde{N}$ *sufficiently large.*

PROOF. The result is obtained by proving the following assertions.

(i) For sufficiently large k, $C^1_{\tilde{N},k}$ is uniformly bounded as $\tilde{N} \to \infty$.

(ii) For all K, $C^2_{\tilde{N},k} \to 0$, as $\tilde{N} \to \infty$.

To obtain (i), recall that, for all k, $|T^h HR|_0 \le |R|_0 < 1$, which shows $(I - T^h HR)^{-1}$ uniformly bounded in k. Also $T^h \to I$ pointwise on H^0 as $k \to \infty$. Since E is compact, $(I - T^h(HR - E))$ has a uniformly bounded inverse as $k \to \infty$. Hence by (4.12), $(I - (HR - E)T^h)^{-1}$ exists for sufficiently large k. Choosing such a k, $(-HR + E)T^h$ is compact and independent of $\tilde{N}$. Since $B_{\tilde{h}} \to I$ pointwise on H^0, as $\tilde{N} \to \infty$, it follows that $|(I + L_{\tilde{h}})^{-1}|_0$ is uniformly bounded as $\tilde{N} \to \infty$, yielding (i). The proof of (ii) is trivial from (5.9) and the compactness of $(HR - E)T^h$.

Discussion

(i) If i^* is allowed to depend on $\tilde{N}$, as in (5.10), the limit function of w_h of the sequence $w_h^{(v)}$ defined by (5.4) does not converge as $\tilde{N} \to \infty$. This is because the component $|w - T^h w|_0$ in the error given by Theorem 7 does not converge as $\tilde{N} \to \infty$. However, for k chosen sufficiently large initially this error will be small compared with the other component $|w - B_h w|_0$, thus making w_h respectably close to w. In the second kind case [2], experiments suggest that even in the case of extremely sharp angles relatively small i^* is needed to achieve convergence of the iteration for moderately large $\tilde{N}$. It is hoped that the assumptions (5.10) will prove much stronger than necessary. Future numerical experiments will be needed to see how the method behaves in practice.

(ii) Although the collocation method with a uniform grid applied on the polygon has suboptimal convergence, as reported in Section 4, the fast solver described here allows much finer discretisations to be implemented in the same amount of computation time, and thus increases the practicality of the method.

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