



Purity of exponential sums on \mathbb{A}^n

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ABSTRACT

We give a purity result for two kinds of exponential sums of the type $\sum_{x \in k^n} \psi(f(x))$, where k is a finite field of characteristic p and $\psi : k \rightarrow \mathbb{C}^*$ is a non-trivial additive character. In the first case, $f \in k[x_1, \dots, x_n]$ is a polynomial of degree divisible by p whose highest-degree homogeneous form defines a non-singular projective hypersurface, and in the second case, f is a polynomial of degree prime to p whose highest-degree homogeneous form defines a projective hypersurface with isolated singularities.

1. Introduction

Let k be a finite field of characteristic p and cardinality q , and let $f \in k[x_1, \dots, x_n]$ be a polynomial of degree d . Pick a non-trivial additive character $\psi : k \rightarrow \mathbb{C}^*$, and consider the sum $\sum_{x \in k^n} \psi(f(x))$. In [Del74], Deligne proved, as a corollary to his proof of the Riemann hypothesis for projective varieties over finite fields, the following estimate.

THEOREM 1 [Del74, Théorème 8.4]. *Suppose that:*

- (i) *the highest degree homogeneous form f_d of f defines a nonsingular hypersurface in \mathbb{P}_k^{n-1} ;*
- (ii) *d is prime to p .*

Then we have the estimate

$$\left| \sum_{x \in k^n} \psi(f(x)) \right| \leq (d-1)^n \cdot q^{n/2}.$$

Moreover, he showed that the sum is *pure* of weight n and rank $(d-1)^n$. In particular, there are $(d-1)^n$ complex algebraic numbers $\alpha_1, \dots, \alpha_{(d-1)^n}$, all pure of weight n (meaning that all their conjugates over \mathbb{Q} have absolute value $q^{n/2}$) such that, for every integer $m \geq 1$, if k_m denotes the degree m extension of k in a fixed algebraic closure \bar{k} , we have

$$(-1)^n \sum_{x \in k_m^n} \psi(\text{Trace}_{k_m/k}(f(x))) = \sum_{i=1}^{(d-1)^n} \alpha_i^m.$$

What can we say in the case where p divides d ? By perversity arguments (cf. [KL85, Kat93, Kat05]) we know that the sum is pure for *almost* all $f \in k[x_1, \dots, x_n]$. More precisely, if we add a sufficiently general linear form to f (one that is contained in a suitable Zariski dense open subset U of the dual affine space $\hat{\mathbb{A}}_k^n$ depending on ψ and q), the sum becomes pure of weight n . However, these results do not give us any information about the sum associated to a particular f . On the other hand, Adolphson and Sperber [AS00b] show, using p -adic methods, that if f satisfies certain regularity

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hypotheses the L -function associated to the exponential sum (or its inverse) is a polynomial. In this article we will use these results to give a version of Theorem 1 for the case where p divides d .

Fix a prime $\ell \neq p$ and an isomorphism $\iota : \bar{\mathbb{Q}}_\ell \rightarrow \mathbb{C}$ so that we can speak about absolute values of elements of $\bar{\mathbb{Q}}_\ell$ and weights without ambiguity. From now on we will assume that such an isomorphism has been chosen, without making any further reference to it. Thus, for every $\alpha \in \bar{\mathbb{Q}}_\ell$, $|\alpha|$ will always mean $|\iota(\alpha)|$. We will also use this isomorphism to identify the sets of \mathbb{C}^* -valued characters and of $\bar{\mathbb{Q}}_\ell^*$ -valued characters of any finite group. Consider the lisse Artin–Schreier $\bar{\mathbb{Q}}_\ell$ -sheaf \mathcal{L}_ψ on \mathbb{A}_k^1 associated to the non-trivial additive character $\psi : k \rightarrow \mathbb{C}^*$ (cf. [SGA4, Définition 1.7]). For every finite extension k'/k and every $t \in \mathbb{A}^1(k') = k'$, the trace of the geometric Frobenius element in $\text{Gal}(\bar{k}/k')$ acting on the stalk of \mathcal{L}_ψ at a geometric point \bar{t} over t is $\psi(\text{Trace}_{k'/k}(t))$. In particular, as ψ takes its values among the roots of unity, \mathcal{L}_ψ is pure of weight 0.

Let $\mathcal{L}_{\psi(f)}$ denote the pull-back $f^*\mathcal{L}_\psi$ on \mathbb{A}_k^n . The cohomology groups with compact support $H_c^i(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)})$ are endowed with an action of the absolute Galois group $\text{Gal}(\bar{k}/k)$ and, in particular, of the geometric Frobenius element $F \in \text{Gal}(\bar{k}/k)$. By the Grothendieck trace formula we have

$$\sum_{x \in k^n} \psi(f(x)) = \sum_{i=0}^{2n} (-1)^i \text{Trace}(F | H_c^i(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)})).$$

Our first result is the following.

THEOREM 2. *Let d be divisible by p . Write $f = f_d + f_{d'} + f'$, where f_d is the degree d homogeneous component of f , d' is the degree of $f - f_d$ and $f_{d'}$ is the degree d' homogeneous component of f . Suppose that:*

- (a) $d'/d > p/(p + (p - 1)^2)$ and d' is prime to p ;
- (b) the equation $f_d = 0$ defines a non-singular hypersurface in \mathbb{P}_k^{n-1} ;
- (c) the hypersurface defined in \mathbb{P}_k^{n-1} by $f_{d'} = 0$ does not contain any of the common zeroes of $\partial f_d / \partial x_1, \dots, \partial f_d / \partial x_n$ in \mathbb{P}_k^{n-1} .

Then:

- (1) $H_c^i(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)}) = 0$ for $i \neq n$;
- (2) $H_c^n(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)})$ has dimension $(d'(d - 1)^n + (-1)^n(d - d'))/d$ and is pure of weight n ;
- (3) we have the estimate

$$\left| \sum_{x \in k^n} \psi(f(x)) \right| \leq \frac{d'(d - 1)^n + (-1)^n(d - d')}{d} \cdot q^{n/2}.$$

For $d' = d - 1$ (the generic case) the inequality in part (a) holds as long as $d \geq 3$, and we get the following.

COROLLARY 3. *Assume that $d \geq 3$ is divisible by p . Let $f = f_d + f_{d-1} + f'$ be as above. Suppose that:*

- (a) the equation $f_d = 0$ defines a non-singular hypersurface in \mathbb{P}_k^{n-1} ;
- (b) the equation $f_{d-1} = 0$ defines a hypersurface in \mathbb{P}_k^{n-1} that does not contain any of the common zeroes of $\partial f_d / \partial x_1, \dots, \partial f_d / \partial x_n$ in \mathbb{P}_k^{n-1} .

Then:

- (1) $H_c^i(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)}) = 0$ for $i \neq n$;
- (2) $H_c^n(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)})$ has dimension $((d - 1)^{n+1} - (-1)^{n+1})/d$ and is pure of weight n ;

(3) we have the estimate

$$\left| \sum_{x \in k^n} \psi(f(x)) \right| \leq \frac{(d-1)^{n+1} - (-1)^{n+1}}{d} \cdot q^{n/2}.$$

As usual, part (3) is a consequence of the vanishing of the cohomology together with Deligne’s theorem on weights (cf. [Del80, Corollaire 3.3.4]).

The second result deals with another kind of sum studied by Adolphson and Sperber in [AS00b] and is a generalization of [Gar98, Theorem 0.4]. Let $f \in k[x_1, \dots, x_n]$ be a polynomial of degree d , which we will now assume to be prime to p . We will show the following.

THEOREM 4. Write $f = f_d + f_{d'} + f'$ as in Theorem 2. Suppose that:

- (a) $d'/d > p/(p + (p - 1)^2)$ and d' is prime to p ;
- (b) the hypersurface defined by $f_d = 0$ in \mathbb{P}_k^{n-1} has at worst weighted homogeneous isolated singularities of total degrees d_1, \dots, d_s prime to p (cf. [AS00b, § 2], or [Gar98, § 0.3], for the definitions);
- (c) the hypersurface defined by $f_{d'} = 0$ in \mathbb{P}_k^{n-1} does not contain any of these singularities.

Let μ_1, \dots, μ_s be the Milnor numbers corresponding to the singularities of $f_d = 0$. Then:

- (1) $H_c^i(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)}) = 0$ for $i \neq n$;
- (2) $H_c^n(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)})$ has dimension $(d - 1)^n - (d - d') \sum_{i=1}^s \mu_i$ and is pure of weight n ;
- (3) we have the estimate

$$\left| \sum_{x \in k^n} \psi(f(x)) \right| \leq \left((d - 1)^n - (d - d') \sum_{i=1}^s \mu_i \right) \cdot q^{n/2}.$$

2. A cohomological vanishing result

In this section we will begin the proof of Theorem 2. We will first use the method of pencils to show the vanishing of $H_c^i(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)})$ for $i > n + 1$. This requires studying the fibers of the map f , so the first thing we need to do is find a suitable compactification of f . Unfortunately, the compactification defined in [Kat99] by embedding \mathbb{A}^n as a dense open subset of the subscheme of $\mathbb{P}^n \times \mathbb{A}^1$ given by the vanishing of $F - \lambda X_0^d$ no longer works in this case. The reason is that we are compactifying a map of degree divisible by p , and this may introduce some wild ramification at infinity in the higher direct images of the constant sheaf with respect to the compactified map.

Therefore, instead of directly compactifying f , the idea is to first write f as the composition of a closed embedding of \mathbb{A}^n in $\mathbb{A}^n \times \mathbb{A}^1$ (given by the graph of f) followed by the projection, and then compactify the projection restricted to the image of \mathbb{A}^n . As we are compactifying a map of degree 1, we do not run into any problems caused by wild ramification. However, one disadvantage of this compactification is that the fiber at infinity will always have a singular point, so we will only be able to deduce the vanishing of the cohomology groups for $i > n + 1$.

PROPOSITION 5. Suppose that the equation $f_d = 0$ defines a non-singular hypersurface in \mathbb{P}_k^{n-1} . Then $H_c^i(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)}) = 0$ for $i > n + 1$.

Proof. Define Z to be the hypersurface in \mathbb{P}_k^{n+1} (where we take coordinates X_0, \dots, X_n, T) defined by the vanishing of $F - TX_0^{d-1}$, where F is the homogenization of f with respect to the variable X_0 (i.e. $F(X_0, \dots, X_n) = X_0^d \cdot f(X_1/X_0, \dots, X_n/X_0)$). The affine space \mathbb{A}_k^n is naturally an open subscheme of Z (just by embedding it in \mathbb{A}_k^{n+1} using the graph of f , and then identifying \mathbb{A}_k^{n+1} with \mathbb{P}_k^{n+1} minus the hyperplane $X_0 = 0$).

Next, we define the incidence variety \tilde{Z} as a divisor of $Z \times \mathbb{P}_k^1$, given (with coordinates X_0, \dots, X_n, T for the first factor and λ_0, λ_1 for the second factor) by the zero locus of $\lambda_0 T - \lambda_1 X_0$. Thus,

$$\tilde{Z}(\bar{k}) = \{((x_0, \dots, x_n, t), (\lambda_0, \lambda_1)) \in Z(\bar{k}) \times \mathbb{P}^1(\bar{k}) : \lambda_0 t = \lambda_1 x_0\}.$$

Let $\tilde{f} : \tilde{Z} \rightarrow \mathbb{P}_k^1$ be the restriction to \tilde{Z} of the canonical projection $\pi_2 : Z \times \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$. It is a proper map, being the composite of a closed immersion and a proper projection (as Z is projective).

The open subset $\mathbb{A}_k^n \hookrightarrow Z$ can be embedded as an open subscheme of \tilde{Z} in the obvious way. Namely, we identify the point $x \in \mathbb{A}^n(\bar{k})$ with $(x, f(x)) \in \tilde{Z}(\bar{k})$. In this way we get a commutative diagram

$$\begin{array}{ccc} \mathbb{A}_k^n & \longrightarrow & \tilde{Z} \\ f \downarrow & & \downarrow \tilde{f} \\ \mathbb{A}_k^1 & \longrightarrow & \mathbb{P}_k^1 \end{array}$$

where the horizontal arrows are open embeddings. The image of \mathbb{A}_k^n in \tilde{Z} can be described as the set of $(x, \lambda) \in \tilde{Z}$ such that $x \notin Z \cap \{X_0 = 0\}$.

Before going any further we need to show that \tilde{f} is a flat map.

LEMMA 6. *The map $\tilde{f} : \tilde{Z} \rightarrow \mathbb{P}_k^1$ is flat.*

Proof. By [Har77, Proposition III.9.9] it suffices to show that all geometric fibers of \tilde{f} have the same Hilbert polynomial. The fiber over a finite point $\lambda \in \mathbb{A}^1(\bar{k})$ is easily seen to be the complete intersection of the degree d hypersurface $F - \lambda X_0^d = 0$ and the hyperplane $T - \lambda X_0 = 0$. Similarly, the fiber over infinity is the complete intersection of the hypersurface $F = 0$ and the hyperplane $X_0 = 0$. As the Hilbert polynomial of a complete intersection only depends on its multidegree, we conclude that it is the same for all geometric fibers of \tilde{f} . □

We extend by zero the sheaf \mathcal{L}_ψ to the whole \mathbb{P}_k^1 , and take its pull-back by \tilde{f} to \tilde{Z} , which we will also denote by $\mathcal{L}_{\psi(f)}$. This is compatible with the previous notation, as its restriction to \mathbb{A}_k^n is just the pull-back of \mathcal{L}_ψ by f .

LEMMA 7. *There is a quasi-isomorphism*

$$R\Gamma_c(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)}) \xrightarrow{\sim} R\Gamma_c(\tilde{Z} \otimes \bar{k}, \mathcal{L}_{\psi(f)}).$$

Proof. To simplify the notation, we will identify each homogeneous form with the projective hypersurface defined by its vanishing. It is clear that $\tilde{Z}_1 := (Z \cap T \cap X_0) \times \mathbb{P}_k^1$ is contained in \tilde{Z} as a closed subscheme. Let \tilde{Z}_0 be its complement. The restriction of \tilde{f} to \tilde{Z}_1 is just the second projection. From the decomposition

$$\tilde{Z}_0 \xrightarrow{j} \tilde{Z} \xleftarrow{i} \tilde{Z}_1$$

we get an exact sequence of sheaves

$$0 \rightarrow j_! j^* \mathcal{L}_{\psi(f)} \rightarrow \mathcal{L}_{\psi(f)} \rightarrow i_* i^* \mathcal{L}_{\psi(f)} \rightarrow 0$$

from which we get a distinguished triangle in $\mathcal{D}^b(\bar{\mathbb{Q}}_\ell - \text{vector spaces})$

$$R\Gamma_c(\tilde{Z}_0 \otimes \bar{k}, \mathcal{L}_{\psi(f)}) \rightarrow R\Gamma_c(\tilde{Z} \otimes \bar{k}, \mathcal{L}_{\psi(f)}) \rightarrow R\Gamma_c(\tilde{Z}_1 \otimes \bar{k}, \mathcal{L}_{\psi(f)}) \rightarrow .$$

Now in $\tilde{Z}_1 \cong (Z \cap T \cap X_0) \times \mathbb{P}_k^1$ the sheaf $\mathcal{L}_{\psi(f)}$ is just the external tensor product $\bar{\mathbb{Q}}_\ell \boxtimes \mathcal{L}_\psi$. Therefore, by the Künneth formula, we have

$$R\Gamma_c(\tilde{Z}_1 \otimes \bar{k}, \mathcal{L}_{\psi(f)}) = R\Gamma_c((Z \cap T \cap X_0) \otimes \bar{k}, \bar{\mathbb{Q}}_\ell) \otimes R\Gamma_c(\mathbb{P}_k^1, \mathcal{L}_\psi) = 0$$

as $R\Gamma_c(\mathbb{P}_{\bar{k}}^1, \mathcal{L}_\psi) = R\Gamma_c(\mathbb{A}_{\bar{k}}^1, \mathcal{L}_\psi) = 0$ (cf. [SGA4, Théorème 2.7*]). Hence, we get a quasi-isomorphism

$$R\Gamma_c(\tilde{Z}_0 \otimes \bar{k}, \mathcal{L}_{\psi(f)}) \xrightarrow{\sim} R\Gamma_c(\tilde{Z} \otimes \bar{k}, \mathcal{L}_{\psi(f)}).$$

The image of the open immersion $h : \mathbb{A}_{\bar{k}}^n \hookrightarrow \tilde{Z}_0$ is the set of $(x, \lambda) \in \tilde{Z}$ such that $x \notin Z \cap X_0$. Its complement in \tilde{Z}_0 is the set of $(x, \lambda) \in \tilde{Z}$ such that $x \in Z \cap X_0$ and $x \notin Z \cap T$, so it maps to the point at infinity under \tilde{f} . As the stalk of \mathcal{L}_ψ at infinity is zero, we have an equality $h_! h^* \mathcal{L}_{\psi(f)} = \mathcal{L}_{\psi(f)}$, and therefore a quasi-isomorphism

$$R\Gamma_c(\mathbb{A}_{\bar{k}}^n, \mathcal{L}_{\psi(f)}) \xrightarrow{\sim} R\Gamma_c(\tilde{Z}_0 \otimes \bar{k}, \mathcal{L}_{\psi(f)}) \xrightarrow{\sim} R\Gamma_c(\tilde{Z} \otimes \bar{k}, \mathcal{L}_{\psi(f)}). \quad \square$$

We will also denote by $\tilde{f} : \tilde{Z} \otimes \bar{k} \rightarrow \mathbb{P}_{\bar{k}}^1$ the map deduced from $\tilde{f} : \tilde{Z} \rightarrow \mathbb{P}_{\bar{k}}^1$ by extension of scalars to \bar{k} . As \tilde{f} is proper, we have (by composition of derived functors)

$$R\Gamma_c(\tilde{Z} \otimes \bar{k}, \mathcal{L}_{\psi(f)}) = R\Gamma_c(\mathbb{P}_{\bar{k}}^1, R\tilde{f}_* \mathcal{L}_{\psi(f)}).$$

On the other hand, by the projection formula we have

$$R\tilde{f}_* \mathcal{L}_{\psi(f)} = R\tilde{f}_*(\bar{Q}_\ell \otimes \tilde{f}^* \mathcal{L}_\psi) = R\tilde{f}_* \bar{Q}_\ell \otimes \mathcal{L}_\psi$$

so Proposition 5 is equivalent to the following.

PROPOSITION 8. *Under the previous hypotheses, the cohomology group $H_c^i(\mathbb{P}_{\bar{k}}^1, R\tilde{f}_* \bar{Q}_\ell \otimes \mathcal{L}_\psi)$ vanishes for $i > n + 1$.*

Therefore, we will prove Proposition 8 instead.

PROPOSITION 9. *The sheaves $R^i \tilde{f}_* \bar{Q}_\ell$ on $\mathbb{P}_{\bar{k}}^1$ are lisse for $i \geq n + 1$. For $i = n$ it is the extension of a lisse sheaf by a punctual sheaf.*

Proof. The fiber of \tilde{f} at a point $\lambda \in \mathbb{A}^1(\bar{k})$ is defined in $\mathbb{P}_{\bar{k}}^{n+1}$ (with the usual coordinates X_0, \dots, X_n, T) by the homogeneous ideal $(F - TX_0^{d-1}, T - \lambda X_0) = (F - \lambda X_0^d, T - \lambda X_0)$. Its intersection with the hyperplane $X_0 = 0$ is then defined by the ideal (F, X_0, T) , and is therefore isomorphic to the hypersurface defined in $\mathbb{P}_{\bar{k}}^{n-1}$ by $f_d = 0$, which is non-singular by hypothesis. Therefore, the fiber itself has at worst isolated singularities. On the other hand, the fiber at $\lambda = \infty$ is defined in $\mathbb{P}_{\bar{k}}^{n+1}$ by the ideal (F, X_0) . This is the projective cone over the hypersurface defined in $\mathbb{P}_{\bar{k}}^{n-1}$ by $f_d = 0$, so it has only one singular point (the vertex).

By [SGA7, Exposé I, Corollaire 4.3] we deduce that for every $\lambda \in \mathbb{P}^1(\bar{k})$ the I_λ -invariant specialization map $(R^i \tilde{f}_* \bar{Q}_\ell)_\lambda \rightarrow (R^i \tilde{f}_* \bar{Q}_\ell)_{\bar{\eta}}$ (where $\bar{\eta}$ is a geometric generic point of $\mathbb{P}_{\bar{k}}^1$ and I_λ the inertia group at λ) is an isomorphism for $i > n$ and surjective for $i = n$. As a consequence, $R^i \tilde{f}_* \bar{Q}_\ell$ is lisse at λ for $i > n$. For $i = n$ we have an exact sequence (cf. [Kat99, Theorem 13])

$$0 \rightarrow (\text{punctual sheaf}) \rightarrow R^n \tilde{f}_* \bar{Q}_\ell \rightarrow j_* j^* R^n \tilde{f}_* \bar{Q}_\ell \rightarrow 0$$

where j is the inclusion of an open subset of $\mathbb{P}_{\bar{k}}^1$ on which $R^n \tilde{f}_* \bar{Q}_\ell$ is lisse. However, as the specialization map $(R^n \tilde{f}_* \bar{Q}_\ell)_\lambda \rightarrow (R^n \tilde{f}_* \bar{Q}_\ell)_{\bar{\eta}}$ is surjective and I_λ -equivariant, the action of I_λ on $(R^n \tilde{f}_* \bar{Q}_\ell)_{\bar{\eta}}$ is trivial. As a consequence, the sheaf $j_* j^* R^n \tilde{f}_* \bar{Q}_\ell$ is lisse at λ . \square

PROPOSITION 10. *The cohomology group $H_c^a(\mathbb{P}_{\bar{k}}^1, R^b \tilde{f}_* \bar{Q}_\ell \otimes \mathcal{L}_\psi)$ vanishes for:*

- (i) $a > 2$, all b ;
- (ii) $b > n$, all a ;
- (iii) $b = n$, $a > 0$.

Proof. Part (i) is clear for cohomological dimension reasons. For $b > n$, the sheaf $R^b \tilde{f}_* \bar{Q}_\ell$ is lisse on \mathbb{P}_k^1 by Proposition 9. As \mathbb{P}_k^1 is simply connected, it must be constant. Then, if $\bar{\eta}$ is a geometric generic point of \mathbb{P}_k^1 , we get

$$R\Gamma_c(\mathbb{P}_k^1, R^b \tilde{f}_* \bar{Q}_\ell \otimes \mathcal{L}_\psi) = (R^b \tilde{f}_* \bar{Q}_\ell)_{\bar{\eta}} \otimes R\Gamma_c(\mathbb{P}_k^1, \mathcal{L}_\psi) = 0$$

as $R\Gamma_c(\mathbb{P}_k^1, \mathcal{L}_\psi) = 0$. This proves (ii).

To prove (iii), let $j : V \hookrightarrow \mathbb{P}_k^1$ be as in Proposition 9, where V is a dense open set on which $R^n \tilde{f}_* \bar{Q}_\ell$ is lisse, and let $\mathcal{H} = j_* j^* R^n \tilde{f}_* \bar{Q}_\ell$. Then \mathcal{H} is lisse on \mathbb{P}_k^1 by Proposition 9, so exactly as above we get $R\Gamma_c(\mathbb{P}_k^1, \mathcal{H} \otimes \mathcal{L}_\psi) = 0$. From the exact sequence

$$0 \rightarrow \mathcal{I} (= \text{punctual sheaf}) \rightarrow R^n \tilde{f}_* \bar{Q}_\ell \rightarrow \mathcal{H} \rightarrow 0$$

we get, after tensoring with \mathcal{L}_ψ ,

$$0 \rightarrow \mathcal{I} \otimes \mathcal{L}_\psi \rightarrow R^n \tilde{f}_* \bar{Q}_\ell \otimes \mathcal{L}_\psi \rightarrow \mathcal{H} \otimes \mathcal{L}_\psi \rightarrow 0.$$

Now $\mathcal{I} \otimes \mathcal{L}_\psi$ is punctual, so $H_c^i(\mathbb{P}_k^1, \mathcal{I} \otimes \mathcal{L}_\psi) = 0$ for $i > 0$. From the long exact sequence of cohomology associated to the exact sequence above we get isomorphisms

$$H_c^a(\mathbb{P}_k^1, R^n \tilde{f}_* \bar{Q}_\ell \otimes \mathcal{L}_\psi) \xrightarrow{\sim} H_c^a(\mathbb{P}_k^1, \mathcal{H} \otimes \mathcal{L}_\psi) = 0$$

for $a > 0$. This proves (iii). □

We can now complete the proof of Proposition 8. We have a spectral sequence

$$H_c^a(\mathbb{P}_k^1, R^b \tilde{f}_* \bar{Q}_\ell \otimes \mathcal{L}_\psi) \Rightarrow H_c^{a+b}(\mathbb{P}_k^1, R \tilde{f}_* \bar{Q}_\ell \otimes \mathcal{L}_\psi).$$

Suppose that $a + b > n + 1$. Then either:

- $a > 2$, so $H_c^a(\mathbb{P}_k^1, R^b \tilde{f}_* \bar{Q}_\ell \otimes \mathcal{L}_\psi) = 0$ by part (i) of Proposition 10;
- $b > n$, so $H_c^a(\mathbb{P}_k^1, R^b \tilde{f}_* \bar{Q}_\ell \otimes \mathcal{L}_\psi) = 0$ by part (ii) of Proposition 10; or
- $a = 2$ and $b = n$, so $H_c^a(\mathbb{P}_k^1, R^b \tilde{f}_* \bar{Q}_\ell \otimes \mathcal{L}_\psi) = 0$ by part (iii) of Proposition 10.

Therefore, the spectral sequence implies that $H_c^i(\mathbb{P}_k^1, R \tilde{f}_* \bar{Q}_\ell \otimes \mathcal{L}_\psi)$ vanishes for $i > n + 1$. □

3. A sum of Milnor numbers computation

Consider the L -function associated to the sheaf $\mathcal{L}_{\psi(f)}$ on \mathbb{A}_k^n :

$$L(T, \mathcal{L}_{\psi(f)}) = \exp \sum_{m=1}^{\infty} \frac{S_m}{m} T^m$$

where

$$S_m = \sum_{x \in k_m^n} \psi(\text{Trace}_{k_m/k}(f(x)))$$

and k_m is the extension of degree m of k in \bar{k} . By the Grothendieck trace formula, we have

$$L(T, \mathcal{L}_{\psi(f)}) = \prod_{i=0}^{2n} \det(1 - T \cdot F | H_c^i(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)}))^{(-1)^{i+1}}$$

where $F \in \text{Gal}(\bar{k}/k)$ is the geometric Frobenius element.

The following result of Adolphson and Sperber [AS00b, Theorem 1.11 and Proposition 6.5] gives an important restriction on the shape of this L -function.

THEOREM 11. Write $f = f_d + f_{d'} + f'$, where f_d is the degree d homogeneous component of f , d' is the degree of $f - f_d$ and $f_{d'}$ is the degree d' homogeneous component of f . Suppose that $d'/d > p/(p + (p - 1)^2)$ and d' is prime to p . Suppose also that $\partial f_d/\partial x_1, \dots, \partial f_d/\partial x_n$ have a finite number of common zeroes in $\mathbb{P}_{\bar{k}}^{n-1}$ (which is automatic if the hypersurface $f_d = 0$ in $\mathbb{P}_{\bar{k}}^{n-1}$ is non-singular) and the hypersurface defined in $\mathbb{P}_{\bar{k}}^{n-1}$ by $f_{d'} = 0$ does not contain any of them. Then $L(T, \mathcal{L}_{\psi(f)})^{(-1)^{n+1}}$ is a polynomial of degree $(d - 1)^n - (d - d') \sum_{i=1}^s \mu_i$, where the sum is taken over the set $\{P_1, \dots, P_s\}$ of common zeroes of $\partial f_d/\partial x_1, \dots, \partial f_d/\partial x_n$ in $\mathbb{P}_{\bar{k}}^{n-1}$ and μ_i denotes the corresponding Milnor number

$$\mu_i = \dim_{\bar{k}} \mathcal{O}_{S, P_i}.$$

Here S is the zero-dimensional subscheme of $\mathbb{P}_{\bar{k}}^{n-1}$ defined by the ideal $(\partial f_d/\partial x_1, \dots, \partial f_d/\partial x_n)$, and \mathcal{O}_{S, P_i} its local ring at P_i , which is a finite \bar{k} -algebra.

We will now compute this sum of Milnor numbers explicitly in the following more general setting.

LEMMA 12. Let $F_1, \dots, F_n \in \bar{k}[x_1, \dots, x_n]$ be (possibly zero) homogeneous polynomials of degree $d - 1$. Suppose that:

- (i) F_1, \dots, F_n have a finite number of common zeroes in $\mathbb{P}_{\bar{k}}^{n-1}$;
- (ii) we have the relation

$$\sum_{i=1}^n x_i \cdot F_i = 0.$$

Let $\{P_1, \dots, P_s\}$ be the set of common zeroes of F_1, \dots, F_n in $\mathbb{P}_{\bar{k}}^{n-1}$, and for every $i = 1, \dots, s$ let μ_i be the corresponding Milnor number. Then we have

$$\sum_{i=1}^s \mu_i = \frac{(d - 1)^n - (-1)^n}{d}.$$

Proof. By induction on n , we first prove it for $n = 2$. In this case, both F_1 and F_2 must be non-zero (otherwise, by (ii) they would both be zero, and (i) would not hold). The relation $x_1 F_1 + x_2 F_2 = 0$ implies that x_1 divides F_2 and x_2 divides F_1 . Let $F_1 = x_2 G_1$ and $F_2 = x_1 G_2$. Then $x_1 x_2 (G_1 + G_2) = 0$, so $G_2 = -G_1$. Therefore, the subscheme defined by F_1 and F_2 is that defined by G_1 , which is a polynomial of degree $d - 2$. The common zeroes of F_1 and F_2 are then in one-to-one correspondence with the distinct linear factors of G_1 , and the Milnor numbers are the corresponding multiplicities. Thus, in this case we get $\sum_{i=1}^s \mu_i = d - 2 = ((d - 1)^2 - 1)/d$.

We assume now that the lemma is true for $n - 1 \geq 2$, and prove it for n . Choose $(\alpha_1, \dots, \alpha_{n-1}) \in \bar{k}^{n-1}$ such that none of the points P_1, \dots, P_s is contained in the hyperplane $x_n - \sum_{j=1}^{n-1} \alpha_j x_j = 0$. We construct the polynomials F'_1, \dots, F'_n given by

$$\begin{aligned} F'_i(x_1, \dots, x_{n-1}, x_n) &= F_i\left(x_1, \dots, x_{n-1}, x_n + \sum_{j=1}^{n-1} \alpha_j x_j\right) \\ &\quad + \alpha_i F_n\left(x_1, \dots, x_{n-1}, x_n + \sum_{j=1}^{n-1} \alpha_j x_j\right) \quad \text{for } i = 1, \dots, n - 1 \\ F'_n(x_1, \dots, x_{n-1}, x_n) &= F_n\left(x_1, \dots, x_{n-1}, x_n + \sum_{j=1}^{n-1} \alpha_j x_j\right). \end{aligned}$$

Then the schemes S defined by the ideal (F_1, \dots, F_n) and S_1 defined by (F'_1, \dots, F'_n) correspond to each other via the automorphism φ of \mathbb{P}_k^{n-1} given by $\varphi(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}, x_n + \sum_{j=1}^{n-1} \alpha_j x_j)$. In particular, the sums of the Milnor numbers at the points of S and S_1 are the same. Moreover, we have

$$\begin{aligned} \sum_{i=1}^n x_i \cdot F'_i(x_1, \dots, x_{n-1}, x_n) &= \sum_{i=1}^{n-1} x_i \cdot \left(F_i \left(x_1, \dots, x_{n-1}, x_n + \sum_{j=1}^{n-1} \alpha_j x_j \right) \right. \\ &\quad \left. + \alpha_i F_n \left(x_1, \dots, x_{n-1}, x_n + \sum_{j=1}^{n-1} \alpha_j x_j \right) \right) \\ &\quad + x_n \cdot F_n \left(x_1, \dots, x_{n-1}, x_n + \sum_{j=1}^{n-1} \alpha_j x_j \right) \\ &= \sum_{i=1}^{n-1} x_i \cdot F_i \left(x_1, \dots, x_{n-1}, x_n + \sum_{j=1}^{n-1} \alpha_j x_j \right) \\ &\quad + \left(x_n + \sum_{i=1}^{n-1} \alpha_i x_i \right) \cdot F_n \left(x_1, \dots, x_{n-1}, x_n + \sum_{j=1}^{n-1} \alpha_j x_j \right) = 0. \end{aligned}$$

If $P = (x_1, \dots, x_n)$ is a common zero of F'_1, \dots, F'_n , then $\varphi(P) = (x_1, \dots, x_{n-1}, x_n + \sum_{j=1}^{n-1} \alpha_j x_j)$ is a common zero of F_1, \dots, F_n so, by the choice of the α_i , $\varphi(P)$ is not contained in the hyperplane $x_n - \sum_{j=1}^{n-1} \alpha_j x_j = 0$. Hence, P is not contained in the hyperplane $x_n = 0$. Therefore, we can, and do, assume that none of the common zeroes of F_1, \dots, F_n is contained in the hyperplane $x_n = 0$.

Under this assumption, we claim that F_1, \dots, F_{n-1} form a regular sequence in $\bar{k}[x_1, \dots, x_n]$ (compare [AS00b, Lemma 5.1]). Otherwise, the subscheme defined by them in \mathbb{P}_k^{n-1} would have an irreducible component Y of dimension at least 1. From (ii) we deduce that Y is contained in the hypersurface $x_n F_n = 0$. Being irreducible, it must be contained either in $x_n = 0$ or in $F_n = 0$. Furthermore, as it has dimension ≥ 1 , its intersections with both $x_n = 0$ and $F_n = 0$ are non-empty. So in either case, the intersection of F_1, \dots, F_{n-1}, F_n and $x_n = 0$ would be non-empty, in contradiction with the assumption made above.

Denote by S_1 the subscheme of \mathbb{P}_k^{n-1} defined by (F_1, \dots, F_{n-1}) . The support of S_1 is the disjoint union of the points P_1, \dots, P_s , which are contained in $F_n = 0$, and the points P_{s+1}, \dots, P_{s+r} , which are contained in $x_n = 0$. Let ν_1, \dots, ν_{s+r} be the corresponding Milnor numbers (i.e. $\nu_i = \dim_{\bar{k}} \mathcal{O}_{S_1, P_i}$). As F_1, \dots, F_{n-1} form a regular sequence of polynomials of degree $d - 1$, S_1 is a zero-dimensional complete intersection of degree $(d - 1)^{n-1}$, therefore

$$\sum_{i=1}^{s+r} \nu_i = \dim_{\bar{k}} \Gamma(S_1, \mathcal{O}_{S_1}) = (d - 1)^{n-1}.$$

For every $i = 1, \dots, s$, x_n is invertible in the local ring $\mathcal{O}_{\mathbb{P}^{n-1}, P_i}$. So from (ii) we deduce that F_n is contained in the ideal generated by F_1, \dots, F_{n-1} in this local ring. Therefore

$$\mathcal{O}_{S, P_i} = \mathcal{O}_{\mathbb{P}^{n-1}, P_i} / (F_1, \dots, F_{n-1}, F_n) = \mathcal{O}_{\mathbb{P}^{n-1}, P_i} / (F_1, \dots, F_{n-1}) = \mathcal{O}_{S_1, P_i}$$

and, in particular, $\nu_i = \mu_i$.

On the other hand, for $i = 1, \dots, r$, F_n is invertible in the local ring $\mathcal{O}_{\mathbb{P}^{n-1}, P_{s+i}}$, so x_n is contained in the ideal generated by F_1, \dots, F_{n-1} in this local ring. Let $G_j = F_j(x_1, \dots, x_{n-1}, 0)$, S_2 the subscheme of \mathbb{P}_k^{n-2} (which we identify with the hyperplane $x_n = 0$ in \mathbb{P}_k^{n-1}) defined by (G_1, \dots, G_{n-1}) . The points Q_1, \dots, Q_r of S_2 are in one-to-one correspondence with P_{s+1}, \dots, P_{s+r}

via the inclusion $\mathbb{P}^{n-2}(\bar{k}) \hookrightarrow \mathbb{P}^{n-1}(\bar{k})$, and

$$\begin{aligned} \mathcal{O}_{S_2, Q_i} &= \mathcal{O}_{\mathbb{P}^{n-2}, Q_i} / (G_1, \dots, G_{n-1}) = \mathcal{O}_{\mathbb{P}^{n-1}, P_{s+i}} / (F_1, \dots, F_{n-1}, x_n) \\ &= \mathcal{O}_{\mathbb{P}^{n-1}, P_{s+i}} / (F_1, \dots, F_{n-1}) = \mathcal{O}_{S_1, P_{s+i}}, \end{aligned}$$

so the Milnor numbers are the same.

Now G_1, \dots, G_{n-1} fall under the hypotheses of the lemma, so we can apply the induction hypothesis and deduce that $\sum_{i=s+1}^{s+r} \nu_i = ((d-1)^{n-1} - (-1)^{n-1})/d$. Therefore,

$$\sum_{i=1}^s \mu_i = \sum_{i=1}^s \nu_i = \sum_{i=1}^{s+r} \nu_i - \sum_{i=s+1}^{s+r} \nu_i = (d-1)^{n-1} - ((d-1)^{n-1} - (-1)^{n-1})/d = ((d-1)^n - (-1)^n)/d. \quad \square$$

Thus, under the hypotheses of Theorem 11, $L(T, \mathcal{L}_{\psi(f)})^{(-1)^{n+1}}$ is a polynomial of degree $(d-1)^n - (d-d')((d-1)^n - (-1)^n)/d = (d'(d-1)^n + (-1)^n(d-d'))/d$.

4. End of the proof of Theorem 2

Part (3) of the theorem is a direct consequence of the previous two parts via the trace formula and Deligne’s theorem. So it suffices to prove parts (1) and (2). Fix a positive integer $d' < d$ prime to p such that $d'/d > p/(p + (p-1)^2)$. Denote by $\mathcal{P}_{d,d'}$ the affine space of all polynomials in $k[x_1, \dots, x_n]$ of degree $\leq d$ whose homogeneous component of degree i is zero for all $d' < i < d$. Let $\pi_1 : \mathcal{P}_{d,d'} \times \mathbb{A}_k^n \rightarrow \mathcal{P}_{d,d'}$ be the projection and $ev : \mathcal{P}_{d,d'} \times \mathbb{A}_k^n \rightarrow \mathbb{A}_k^1$ the evaluation map. Let $K \in \mathcal{D}_c^b(\mathcal{P}_{d,d'}, \mathbb{Q}_\ell)$ be the object $R\pi_{1!} ev^* \mathcal{L}_\psi[n + \dim \mathcal{P}_{d,d'}]$.

LEMMA 13. *The object K is perverse and pure of weight $n + \dim \mathcal{P}_{d,d'}$.*

Proof. For $d' = d - 1$ (i.e. when $\mathcal{P}_{d,d'}$ is the affine space of all polynomials of degree $\leq d$) this is [Kat05, Theorem 3.1.2(1)]. We will see that the same proof works in general.

There is a natural finite map $\tau : \mathbb{A}_k^n \rightarrow \hat{\mathcal{P}}_{d,d'}$. Namely, for every $t \in \mathbb{A}^n(\bar{k})$, $\tau(t) \in \hat{\mathcal{P}}_{d,d'}(\bar{k})$ is the evaluation map at t , $ev(-, t) : \mathcal{P}_{d,d'}(\bar{k}) \rightarrow \bar{k}$. As $\bar{\mathbb{Q}}_\ell[n]$ is perverse and pure of weight n on \mathbb{A}_k^n , so is $\tau_* \bar{\mathbb{Q}}_\ell[n]$ on $\hat{\mathcal{P}}_{d,d'}$. Its Fourier transform $T_\psi(\tau_* \bar{\mathbb{Q}}_\ell[n]) \in \mathcal{D}_c^b(\mathcal{P}_{d,d'}, \bar{\mathbb{Q}}_\ell)$ with respect to ψ is K (by the very definition of K). Therefore, K is perverse and pure of weight $n + \dim \mathcal{P}_{d,d'}$ (cf. [KL85, § 2], or [KW01, § III.8], for the definition and main properties of the Fourier transform). \square

Note that for every finite extension k'/k and every $f \in \mathcal{P}_{d,d'}(k')$, the trace of the geometric Frobenius element in $\text{Gal}(\bar{k}/k')$ acting on the stalk of K at a geometric point over f is the sum

$$(-1)^{n+\dim \mathcal{P}_{d,d'}} \sum_{x \in k'^n} \psi(\text{Trace}_{k'/k} f(x)).$$

Let $U \subset \mathcal{P}_{d,d'}$ be the maximal dense open set on which K has lisse cohomology sheaves. Then $\mathcal{H}^i(K)|_U = 0$ for $i \neq -\dim \mathcal{P}_{d,d'}$ and $\mathcal{F} := \mathcal{H}^{-\dim \mathcal{P}_{d,d'}}(K) = R^n \pi_{1!} ev^* \mathcal{L}_\psi$ is lisse and pure of weight n on U . Thus, for different finite extensions k'/k and polynomials $f \in U(k')$, the exponential sums $\sum_{x \in k'^n} \psi(\text{Trace}_{k'/k} f(x))$ are pure of weight n and the same rank as \mathcal{F} .

Let $V \subset \mathcal{P}_{d,d'}$ (respectively $W \subset \mathcal{P}_{d,d'}$) be the dense open set of all polynomials f such that f_d defines a non-singular hypersurface on \mathbb{P}_k^{n-1} (respectively the dense open set of all f such that $\partial f_d / \partial x_1, \dots, \partial f_d / \partial x_n$ have a finite number of common zeroes in \mathbb{P}_k^{n-1} and the hypersurface $f_{d'} = 0$ does not contain any of them). We know that:

- (i) for every $f \in V(k)$, we have $H_c^i(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)}) = 0$ for $i \neq n, n+1$; for $i > n+1$, this is Proposition 5; for $i < n$ it is just Poincaré duality, as \mathbb{A}_k^n is smooth and $\mathcal{L}_{\psi(f)}$ is lisse;

(ii) for every $f \in W(k)$, the L -function

$$L(T, \mathcal{L}_{\psi(f)})^{(-1)^{n+1}} = \prod_{i=0}^{2n} \det(1 - T \cdot F | H_c^i(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)}))^{(-1)^{n+i}}$$

is a polynomial of degree $N := (d'(d - 1)^n + (-1)^n(d - d'))/d$ (cf. § 3).

Recall that a constructible $\bar{\mathbb{Q}}_\ell$ -sheaf \mathcal{G} on a smooth connected scheme S is said to be of *perverse origin* if there is a perverse sheaf $L \in \mathcal{D}_c^b(S, \bar{\mathbb{Q}}_\ell)$ such that $\mathcal{G} = \mathcal{H}^{-\dim S}(L)$ (cf. [Kat03, § 1]). In that case, we have the following (cf. [Kat03, Proposition 12]).

THEOREM 14. *The integer valued function defined by $s \mapsto \text{rank } \mathcal{G}_{\bar{s}}$ on S (where \bar{s} is a geometric point over s) is lower semicontinuous. In other words, the rank of \mathcal{G} does not increase under specialization. In particular, the dimension of the stalk of \mathcal{G} at any geometric point of S can never exceed the generic rank of \mathcal{G} . Moreover, the largest open set on which \mathcal{G} is lisse is precisely the set where the rank of \mathcal{G} is maximal (equal to the generic rank).*

Note that on U the degree of the L -function is just the rank of \mathcal{F} . Therefore, on $U \cap W$, \mathcal{F} is lisse of rank N . In particular, the generic rank of \mathcal{F} is N . As \mathcal{F} is of perverse origin, from Theorem 14 we deduce that for every $f \in \mathcal{P}_{d,d'}(k)$ the cohomology group $H_c^n(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)})$ (which is the stalk of \mathcal{F} at a geometric point over f) has dimension at most N .

Now let $f \in V \cap W(k)$. From part (i) we have

$$L(T, \mathcal{L}_{\psi(f)})^{(-1)^{n+1}} = \frac{\det(1 - T \cdot F | H_c^n(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)}))}{\det(1 - T \cdot F | H_c^{n+1}(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)}))}.$$

On the other hand, from part (ii) we know that this is a polynomial of degree N . Therefore,

$$\dim H_c^n(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)}) - \dim H_c^{n+1}(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)}) = N.$$

As $\dim H_c^n(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)}) \leq N$ and $\dim H_c^{n+1}(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)})$ cannot be negative, we conclude that $H_c^n(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)})$ has dimension N and the group $H_c^{n+1}(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)})$ vanishes.

From Theorem 14 we deduce that \mathcal{F} is lisse on $V \cap W$, as it has maximal rank there. Furthermore, it is pure of weight n , because $K|_{V \cap W} = \mathcal{F}|_{V \cap W}[\dim \mathcal{P}_{d,d'}]$ is pure of weight $n + \dim \mathcal{P}_{d,d'}$. This completes the proof of Theorem 2.

Remark 15. When $p = d = 2$ and n is even, the sum $\sum_{x \in k^n} \psi(f(x))$ is known to be pure of weight n and rank 1 if the hypersurface defined by $f_2 = 0$ is non-singular (cf. [AS00a, § 6]).

Remark 16. We will see now that, for the rank formula in Theorem 2 to hold, the restriction $d'/d > p/(p + (p - 1)^2)$ (or at least some milder lower bound for d') is essential. More precisely, let $d = p^\alpha d_0$, where d_0 is prime to p . We claim that the formula is not true for $d' < d_0$. Let $\mathcal{P}_{d,-1}$ be the affine space of homogeneous polynomials of degree d . Let A (respectively B) be the generic rank of $R^n \pi_{1!} ev^* \mathcal{L}_\psi$ on $\mathcal{P}_{d,-1}$ (respectively $\mathcal{P}_{d,d'}$). By [Kat05, Theorem 3.6.5] we know that

$$A = \frac{(d - 1)^n + (-1)^n(d - 1)}{d} + \frac{d_0 - 1}{d}((d - 1)^n - (-1)^n).$$

On the other hand, as $\mathcal{P}_{d,-1} \subset \mathcal{P}_{d,d'}$ and $R^n \pi_{1!} ev^* \mathcal{L}_\psi$ is of perverse origin, we have the inequality $A \leq B$. However, it is easy to see that the inequality $A \leq (d'(d - 1)^n + (-1)^n(d - d'))/d$ is equivalent to $d_0 \leq d'$. Therefore, if $d' < d_0$, we cannot have $B = (d'(d - 1)^n + (-1)^n(d - d'))/d$.

5. Proof of Theorem 4

We will use a similar procedure to prove the second result, therefore we will first show the following.

PROPOSITION 17. *Suppose that d is prime to p and the hypersurface defined in \mathbb{P}_k^{n-1} by the equation $f_d = 0$ has at worst isolated singularities. Then $H_c^i(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)}) = 0$ for $i > n + 1$.*

Proof. This is already proven, although not explicitly stated, in [Kat99, Theorem 16]. Let \tilde{X} be the incidence variety defined in $\mathbb{P}_k^n \times \mathbb{A}_k^1$ (with coordinates X_0, X_1, \dots, X_n for the first factor and λ for the second factor) by the vanishing of $F - \lambda X_0^d$, where F is again the homogenization of f with respect to the variable X_0 . Let $\tilde{f} : \tilde{X} \rightarrow \mathbb{A}_k^1$ be the restriction of the second projection. The affine space \mathbb{A}_k^n can be naturally embedded as a dense open subset of \tilde{X} and, as in Lemma 7, there is a quasi-isomorphism

$$R\Gamma_c(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)}) \xrightarrow{\sim} R\Gamma_c(\tilde{X} \otimes \bar{k}, \mathcal{L}_{\psi(f)})$$

where we also denote by $\mathcal{L}_{\psi(f)}$ the pull-back of \mathcal{L}_{ψ} to \tilde{X} by \tilde{f} . The proof of [Kat99, Theorem 16], applied to $X = \mathbb{P}_k^n$, $L = X_0$ and $H = F$ (hence $\delta = 0$, $\varepsilon = -1$) shows that

$$H_c^a(\mathbb{A}_k^1, R^b \tilde{f}_* \bar{Q}_\ell \otimes \mathcal{L}_{\psi}) = 0$$

for $a + b \geq n + 2$. In particular, the spectral sequence

$$H_c^a(\mathbb{A}_k^1, R^b \tilde{f}_* \bar{Q}_\ell \otimes \mathcal{L}_{\psi}) \Rightarrow H_c^{a+b}(\tilde{X} \otimes \bar{k}, \mathcal{L}_{\psi(f)})$$

implies that $H_c^i(\mathbb{A}_k^n, \mathcal{L}_{\psi(f)}) \cong H_c^i(\tilde{X} \otimes \bar{k}, \mathcal{L}_{\psi(f)}) = 0$ for $i > n + 1$. □

We will think of the homogeneous form f_d and the integer d' as being fixed, and the degree $\leq d'$ part of f , which we will call g , as being variable. Let $\mathcal{P}_{d'}$ be the affine space of all polynomials of degree $\leq d'$. Let $\pi_1 : \mathcal{P}_{d'} \times \mathbb{A}_k^n \rightarrow \mathcal{P}_{d'}$ be the projection and $ev_{f_d} : \mathcal{P}_{d'} \times \mathbb{A}_k^n \rightarrow \mathbb{A}_k^1$ the map $(g, x) \mapsto f_d(x) + g(x)$. Let $K \in \mathcal{D}_c^b(\mathcal{P}_{d'}, \bar{Q}_\ell)$ be the object $R\pi_{1!} ev_{f_d}^* \mathcal{L}_{\psi}[n + \dim \mathcal{P}_{d'}]$. Exactly as in Lemma 13, one shows the following.

LEMMA 18. *The object K is perverse and pure of weight $n + \dim \mathcal{P}_{d'}$.*

For every finite extension k'/k and every $g \in \mathcal{P}_{d'}(k')$, the trace of the geometric Frobenius element of $\text{Gal}(\bar{k}/k')$ acting on the stalk of K at a geometric point over g is the sum

$$(-1)^{n + \dim \mathcal{P}_{d'}} \sum_{x \in k'^n} \psi(\text{Trace}_{k'/k}(f_d(x) + g(x))).$$

Now let $V \subset \mathcal{P}_{d'}$ be the open set of all polynomials g whose homogeneous component of degree d' is non-zero and the hypersurface it defines in \mathbb{P}_k^{n-1} does not contain any of the singularities of $f_d = 0$. For every $g \in V(k)$ we have $H_c^i(\mathbb{A}_k^n, \mathcal{L}_{\psi(f_d+g)}) = 0$ for $i \neq n, n + 1$, by Proposition 17 and Poincaré duality. On the other hand, by [AS00b, Theorem 1.10 and Proposition 6.5] we know that

$$L(T, \mathcal{L}_{\psi(f_d+g)})^{(-1)^{n+1}} = \frac{\det(1 - T \cdot F | H_c^n(\mathbb{A}_k^n, \mathcal{L}_{\psi(f_d+g)}))}{\det(1 - T \cdot F | H_c^{n+1}(\mathbb{A}_k^n, \mathcal{L}_{\psi(f_d+g)})}$$

is a polynomial of degree $N' := (d - 1)^n - (d - d') \sum_{i=1}^s \mu_i$.

Let $U \subset \mathcal{P}_{d'}$ be a dense open subset where K has lisse cohomology sheaves. Then $\mathcal{H}^i(K)|_U = 0$ for $i \neq -\dim \mathcal{P}_{d'}$ and $\mathcal{F} := \mathcal{H}^{-\dim \mathcal{P}_{d'}}(K) = R^n \pi_{1!} ev_{f_d}^* \mathcal{L}_{\psi}$ is lisse of rank N' and pure of weight n on U . Being of perverse origin, by Theorem 14 this implies that for any $g \in \mathcal{P}_{d'}(k)$ the cohomology group $H_c^n(\mathbb{A}_k^n, \mathcal{L}_{\psi(f_d+g)})$ has dimension at most N' . Moreover, if $g \in V(k)$, as

$$\dim H_c^n(\mathbb{A}_k^n, \mathcal{L}_{\psi(f_d+g)}) - \dim H_c^{n+1}(\mathbb{A}_k^n, \mathcal{L}_{\psi(f_d+g)}) = N',$$

we conclude that $H_c^{n+1}(\mathbb{A}_k^n, \mathcal{L}_{\psi(f_d+g)}) = 0$ and $H_c^n(\mathbb{A}_k^n, \mathcal{L}_{\psi(f_d+g)})$ has dimension N' . In particular, $\mathcal{F}|_V$ has constant rank N' , so it is lisse by Theorem 14. Therefore, as $K|_V = \mathcal{F}|_V[\dim \mathcal{P}_{d'}]$ and K is pure of weight $n + \dim \mathcal{P}_{d'}$, the sheaf \mathcal{F} must be pure of weight n on V . This concludes the proof of Theorem 4.

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