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The Ramification Polygon for Curves over a Finite Field

John Scherk

Abstract. A Newton polygon is introduced for a ramified point of a Galois covering of curves over a finite field. It is shown to be determined by the sequence of higher ramification groups of the point. It gives a blowing up of the wildly ramified part which separates the branches of the curve. There is also a connection with local reciprocity.

1 Introduction

Let k be a finite field of characteristic p with q elements and let L/K be a totally ramified Galois extension of local fields over k with Galois group G. Denote by ν_L (respectively ν_K) their valuations. Let z (respectively y) be a local parameter for L (respectively K). Then z satisfies an Eisenstein equation

$$f(z) = z^e + \cdots + a_1 z + a_0 = 0 \tag{1}$$

where $a_i \in \mathcal{O}_K = k[[y]]$, $\nu_K(a_i) \geq 1$ for all i, $\nu_K(a_0) = 1$, $a_e = 1$, and $e = e_0 p^r$ with $(e_0, p) = 1$. G has a filtration

$$G = G_0 \supset G_1 \supset \cdots \supset G_i \supset \cdots$$

given as follows: for $i \geq -1$,

$$\gamma \in G_i \iff \gamma z - z \equiv 0 \pmod{z^{i+1}}.$$

This note studies "Puiseux expansions" for γz, where $\gamma \in G_1$, *i.e.*, for the wildly ramified part of the extension. A neat way to do this is to use a Newton polygon, the ramification polygon. While writing this paper, the author discovered that a similar Newton polygon was introduced by Krasner in [2] for local extensions of number fields. He obtained results analogous to those in Section 2 in this case.

In Section 2 the ramification polygon is introduced and its basic properties derived. The polygon determines a blowing-up of $\mathbb{A}^2$. This is discussed in Section 3. In Section 4 a connection with local reciprocity is explained. This result holds in the number field case as well and seems to be unknown there.

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2 The Ramification Polygon

For $i \geq 1$, let
$$U_L^i = 1 + (z^i) \subset \mathcal{O}_L^*, \quad U_K^i = 1 + (y^i) \subset \mathcal{O}_K^*.$$
For $\gamma \in G_i$, $i \geq 1$, write
$$\gamma z = z + s_\gamma z, \quad s_\gamma \in (z^i).$$
Then we can define a homomorphism
$$t_i \colon G_i/G_{i+1} \longrightarrow U_L^i/U_L^{i+1} \cong (z^i)/(z^{i+1}) \cong k$$
by
$$t_i(\bar{\gamma}) = \overline{1 + s_\gamma},$$
which is injective. Under the identification with k, $\overline{1 + s_\gamma}$ corresponds to $\bar{s}_\gamma$, the leading coefficient of s_γ as a power series in z.

Now the local parameter y can be written as a power series in z. Regarding f then as a polynomial with coefficients in $\mathcal{O}_L$, set
$$g(x) = g(x, z) := f(zx + z) \in \mathcal{O}_L[x].$$
Notice that
$$g(0) = f(z) = 0.$$
If $\gamma \in G_1$, then s_γ is a root of g, and if $s \in L$ is a root of g, then $sz + z$ is a root of f.

The *ramification polygon* Δ of L/K is defined to be the Newton polygon of g: write
$$g(x) = \sum_{i=1}^{e} b_i x^i, \quad b_i \in \mathcal{O}_L,$$
and let
$$P_i = \big(i, \nu_L(b_i)\big), \quad i = 1, \ldots, e.$$
Then Δ is the boundary of the convex hull of
$$\bigcup_{i=1}^{e} (P_i + \mathbf{R}_+^2).$$
Corollary 1 shows that Δ does not in fact depend on the choice of f.

Now
$$b_i = \sum_{j=i}^{e} \binom{j}{i} a_j z^j.$$
Since $e | \nu_L(a_j)$, we have that
$$\nu_L(a_j z^j) \equiv j \pmod{e}, \quad j = i, \ldots, e$$
and thus they are all distinct. So
$$\nu_L(b_i) = \min_{i \leq j \leq e} \nu_L\left(\binom{j}{i} a_j z^j\right). \tag{2}$$

Lemma 1

(i) *For all i,* $\nu_L(b_i) \geq e$*;*
(ii) $\nu_L(b_e) = \nu_L(b_{p^r}) = e$*;*
(iii) $\nu_L(b_1) = \nu_L(\mathcal{D}) + 1$*, where* $\mathcal{D}$ *is the different of* L/K*;*
(iv) *for* $p^s < i < p^{s+1}$*,* $s < r$*,* $\nu_L(b_i) \geq \nu_L(b_{p^s})$*.*

Proof We have that

$$\nu_L\left(\binom{j}{i} a_j z^j\right) \geq e\nu_K(a_j) + j$$

for all j. Since $\nu_K(a_j) \geq 1$ for all $j < e$, (i) follows. Notice that

$$(zx + z)^e = z^e(x^{p^r} + 1)^{e_0} = z^e(x^e + \cdots + e_0 x^{p^r} + 1).$$

So (2) implies that

$$\nu_L(b_{p^r}) = e.$$

According to [3, III, §6, Cor. 2,],

$$\mathcal{D} = \left(f'(z)\right).$$

As

$$b_1 = zf'(z),$$

this proves (iii). Lastly, suppose that $s < r$, $p^s < i < p^{s+1}$. Let ν_p denote the p-adic valuation. Then, as is well known,

$$\nu_p\binom{j}{i} \geq \nu_p\binom{j}{p^s}$$

for all $j \geq i$. In particular, if $\binom{j}{p^s} \equiv 0 \pmod{p}$, then $\binom{j}{i} \equiv 0 \pmod{p}$. Therefore by (2)

$$\nu_L(b_i) \geq \nu_L(b_{p^s}).$$ ■

Thus P_i lies on Δ only if $i = p^s$ for some s. So let $Q_i = P_{p^{s_i}}$, $i = 1, \ldots, m$ be the vertices of Δ, where $0 = s_1 < \cdots < s_m = p^r$. Set $\nu_i = \nu_L(b_{p^{s_i}})$. Let $L_i = \overline{Q_{i-1}Q_i}$, $1 < i \leq m$ be the edges, and let $-\mu_i \in \mathbb{Q}$ be the slope of L_i.

Theorem 1 *The slopes* $-\mu_j$ *of the edges* $L_j = \overline{Q_{j-1}Q_j}$*,* $1 < j \leq m$*, are integral. The jumps in the sequence of higher ramification groups* $G_0 \supseteq G_1 \supseteq \cdots$ *are* $\mu_m < \cdots < \mu_2$*. The orders of the groups are* $|G_{\mu_j}| = p^{s_j}$*,* $1 < j \leq m$*.*

Proof We show that g has $p^{s_j} - p^{s_{j-1}}$ roots of order μ_j. Let $\bar{b}_i \in k$ be the coefficient of the lowest order term of $b_i \in \mathcal{O}_L = k[[z]]$. Now

$$\begin{aligned} g(z^{\mu_j}x) &= \sum_i b_i (z^{\mu_j}x)^i \\ &= \sum_i \bar{b}_i z^{\nu_L(b_i) + i\mu_j} x^i + \text{higher order terms.} \end{aligned}$$

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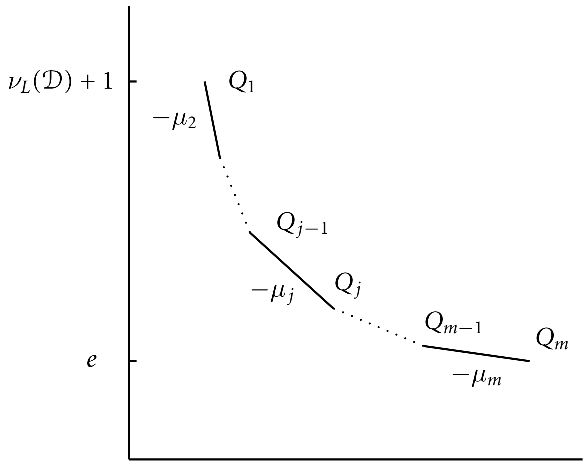

The equation of L_j is

$$\eta + \mu_j \xi - (\nu_j + \mu_j p^{s_j}) = 0.$$

Since Δ is convex,

$$\nu_L(b_i) + \mu_j i - (\nu_j + \mu_j p^{s_j}) > 0$$

for all P_i not on L_j. Therefore

$$g(z^{\mu_j} x) \equiv z^{\nu_j + \mu_j p^{s_j}} \sum_{P_{p^s} \text{ on } L_j} \bar{b}_{p^s} x^{p^s} \pmod{z^{\nu_j + \mu_j p^{s_j} + 1}}. \tag{3}$$

So let

$$h_j(x) = \sum_{P_{p^s} \text{ on } L_j} \bar{b}_{p^s} x^{p^s} \in k[x].$$

This is an additive polynomial. Its degree is p^{s_j} and the lowest order term has degree $p^{s_{j-1}}$. Therefore the number of non-zero roots of h_j in $\bar{k}$ is $p^{s_j} - p^{s_{j-1}}$. A non-zero root $\bar{s}$ of h_j in k determines a root s of g in $\mathcal{O}_L$ of order μ_j and vice versa. Since g has e roots in $\mathcal{O}_L$, all the roots of h_j must lie in k and μ_j must be an integer. This also tells us that the sequence

$$0 \longrightarrow G_{\mu_j}/G_{\mu_j+1} \xrightarrow{t_{\mu_j}} U_L^{\mu_j}/U_L^{\mu_j+1} \cong k \xrightarrow{h_j} k \tag{4}$$

is exact. Therefore

$$|G_{\mu_j}/G_{\mu_j+1}| = p^{s_j} - p^{s_{j-1}},$$

and

$$|G_{\mu_j}| = \sum_{i=2}^{j} (p^{s_i} - p^{s_{i-1}}) + 1 = p^{s_j}.$$

To see that $\mu_2, \dots, \mu_m$ are precisely the jumps in the sequence of ramification groups, let s be a root of g of order μ. Suppose μ is not one of $\mu_2, \dots, \mu_m$. Since Δ is

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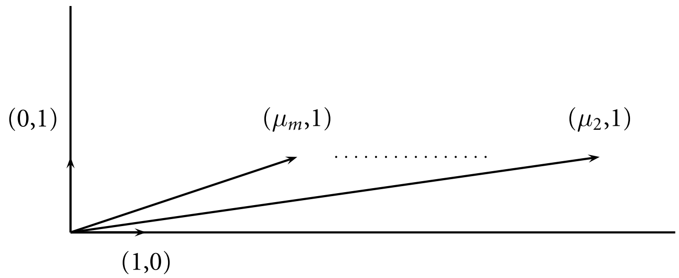

convex there will be a line with slope $-\mu$ which meets it at a single vertex, say Q_j, for some j, $1 \le j \le m$. And the rest of Δ will lie on one side of this line. Now expand $g(z^\mu x)$ as in (3):

$$g(z^\mu x) \equiv z^{\nu_j+\mu p^{s_j}} \bar{b}_{p^{s_j}} x^{p^{s_j}} \pmod{z^{\nu_j+\mu p^{s_j}+1}}.$$

But writing $s = z^\mu \tilde{s}$, with $\tilde{s} \in \mathcal{O}_L$, $\tilde{s}(0) \ne 0$, we have

$$0 = g(s) = g(z^\mu \tilde{s}) \equiv z^{\nu_j+\mu p^{s_j}} \bar{b}_{p^{s_j}} \tilde{s}^{p^{s_j}},$$

which is impossible since $\bar{b}_{p^{s_j}} \ne 0$. Therefore $\mu_2, \dots, \mu_m$ are the jumps. ■

Corollary 1 *The ramification polygon is independent of the choice of Eisenstein polynomial f.*

Proof The sequence of jumps and the orders of the ramification groups determine the numbers $\mu_2, \dots, \mu_m, p^{s_1}, \dots, p^{s_m}$ which in turn determine Δ. ■

3 Blowing Up

Equation (3) can be interpreted as a "blowing up" of the curve $g(x, z) = 0$, which separates the branches γz, $\gamma \in G_1$. The fan Σ associated with Δ [1, Section 8.2] consists of the cones generated by $\{(0, 1), (\mu_m, 1)\}$, $\{(\mu_{j+1}, 1), (\mu_j, 1)\}$ for $1 < j < m$ and $\{(\mu_2, 1), (1, 0)\}$.

This fan defines a variety $X(\Sigma)$ over k and a proper map $\phi\colon X(\Sigma) \to \mathbb{A}^2$. Let Σ' be the fan consisting of the cones generated by $\{(\mu, 1), (\mu + 1, 1)\}$ for $0 \le \mu < \mu_2$ and $\{(\mu_2, 1), (1, 0)\}$. Then Σ' is a simple fan subordinate to Σ, and $X' := X(\Sigma')$ is smooth.

For each j, $1 < j \le m$, let Π_j be the cone spanned by $\{(\mu_j, 1), (1, 0)\}$. We have the canonical map

$$\phi_j\colon X(\Pi_j) \cong \mathbb{A}^2 \longrightarrow \mathbb{A}^2$$

defined by $\phi_j(x, y) = (xy^{\mu_j}, y)$. Then $\phi_j^* g$ is given by (3). Let D_j denote the exceptional curve $\{y = 0\}$ in $X(\Pi_j)$. Equation (3) can be interpreted geometrically as follows. Under the blow-up ϕ_j precisely the components of $g^{-1}(0)$ of order μ_j meet D_j. Their points of intersection are given by the roots of h_j.

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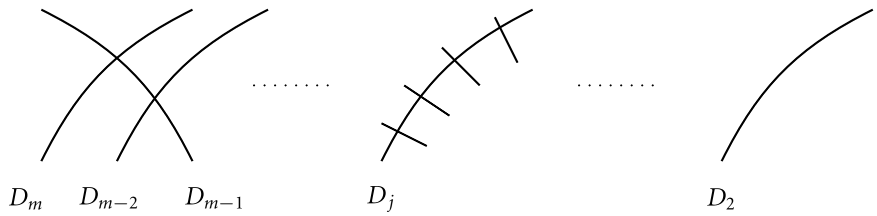

Let Σ_j be the fan consisting of the cones generated by $\{(\mu_{i+1}, 1), (\mu_i, 1)\}$ for $1 < i < j$ and $\{(\mu_2, 1), (1, 0)\}$. Then $X(\Sigma_j)$ is an open subvariety of $X(\Sigma)$, and also a blow-up of $X(\Pi_j)$:

$$\begin{tikzcd} & X(\Sigma') \arrow[d] \\ X(\Sigma_j) \arrow[r] \arrow[d] & X(\Sigma) \\ X(\Pi_j) & \end{tikzcd}$$

In $X(\Sigma_j)$, the proper pre-image of D_j (also denoted by D_j) lies in the chart corresponding to the cone generated by $\{(\mu_j, 1), (\mu_{j-1}, 1)\}$. The components of the exceptional divisor of $\phi\colon X(\Sigma) \to \mathbb{A}^2$ are then the curves $D_2, \dots, D_m$. In the exceptional divisor of $X(\Sigma')$ there are $\mu_j - \mu_{j+1} - 1$ rational curves interpolated between D_{j+1} and D_j.

4 Local Reciprocity

In this section we point out how the polynomials h_j are connected with local reciprocity. Assume that G is abelian. The local reciprocity map

$$w\colon K^*/NL^* \longrightarrow G$$

respects the natural filtrations on both sides and induces maps w_i on the quotients. We first recall the description of these maps given in [3].

Let φ denote the Herbrand function, and ψ its inverse. Then if $N\colon L^* \to K^*$ is the norm, we have

$$N(U_L^{\psi(i)}) \subset U_K^i, \quad N(U_L^{\psi(i)+1}) \subset U_K^{i+1}$$

[3, V, Prop. 8]. Furthermore, the induced map

$$k \cong U_L^{\psi(i)}/U_L^{\psi(i)+1} \xrightarrow{N_i} U_K^i/U_K^{i+1} \cong k \tag{5}$$

is an additive polynomial of degree $|G_{\psi(i)}|$, and the sequence

$$0 \longrightarrow G_{\psi(i)}/G_{\psi(i)+1} \xrightarrow{t_{\psi(i)}} U_L^{\psi(i)}/U_L^{\psi(i)+1} \xrightarrow{N_i} U_K^i/U_K^{i+1}$$

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is exact [3, V, Prop. 9].

Now there belongs to this sequence a "coboundary map"

$$\delta_i\colon\ \operatorname{coker} N_i \cong U_K^i/U_K^{i+1}NU_L^{\psi(i)} \longrightarrow G_{\psi(i)}/G_{\psi(i)+1}.$$

It is constructed as follows, keeping in mind the identifications in (5) (*cf.* [3, XV, Section 1]): take $a \in k \cong U_K^i/U_K^{i+1}$ and let $b \in \bar{k}$ be a solution of

$$N_i(b) = a.$$

Set

$$c = Fb - b,$$

where F is the Frobenius homomorphism. Then $c \in \ker N_i \subset k$ and c does not depend on the choice of b. This determines a well-defined homomorphism

$$U_K^i/U_K^{i+1}NU_L^{\psi(i)} \longrightarrow \ker N_i.$$

So define

$$\delta_i(a) := t_{\psi(i)}(c^{-1}) \in G_{\psi(i)}/G_{\psi(i)+1}.$$

On the other hand, the local reciprocity map w also respects the filtrations:

$$w(U_K^i/NU_L^{\psi(i)}) \subset G_{\psi(i)},$$

and induces isomorphisms

$$w_i\colon U_K^i/U_K^{i+1}NU_L^{\psi(i)} \longrightarrow G_{\psi(i)}/G_{\psi(i)+1}.$$

Serre [3, XV, Prop. 4] proves that

$$w_i(a) = \delta_i(a^{-1}), \quad a \in U_K^i/U_K^{i+1}NU_L^{\psi(i)}.$$

Since δ_i is determined by N_i, it is therefore of interest to know more about these additive polynomials. The quotients $G_{\psi(i)}/G_{\psi(i)+1}$ are trivial unless $\psi(i) = \mu_j$ for some j, or equivalently, $i = \varphi(\mu_j)$.

Theorem 2 *For $j \geq 2$, $N_{\varphi(\mu_j)}$ and h_j coincide up to a constant.*

Proof The polynomial h_j is an additive polynomial of degree $|G_{\mu_j}|$. Its kernel is $\operatorname{im} t_{\mu_j}$ (*cf.* (4)). Set $i = \varphi(\mu_j)$ so that $\mu_j = \psi(i)$. Then the degree of N_i is also $|G_{\mu_j}|$ and its kernel is $\operatorname{im} t_{\mu_j}$ too. Therefore N_i and h_j coincide up to a constant (*cf.* [3, V, Section 5]). ■

Remark 1 The isomorphisms

$$U_L^i/U_L^{i+1} \cong k, \quad U_K^i/U_K^{i+1} \cong k$$

depend on the choice of z, respectively y. Choosing an Eisenstein polynomial f is equivalent to fixing z. The norm map N_i regarded as an additive polynomial then still depends on the choice of y. Varying y multiplies N_i by a constant.

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Department of Mathematics
University of Toronto
100 St. George Street
Toronto, Ontario
M5S 3G3
email: scherk@math.toronto.edu