13
CAR on Fock spaces

This chapter is devoted to the study of *Fock representations of the canonical anti-commutation relations*, a basic tool of quantum many-body theory and quantum field theory. It is parallel to Chap. 9, where Fock CCR representations were studied.

The basic framework is almost the same as in Chap. 9. Throughout this chapter $\mathcal{Z}$ is a Hilbert space, called the *one-particle space*. We will consider the Fock CAR representation acting on the fermionic Fock space $\Gamma_{\mathrm{a}}(\mathcal{Z})$.

As in Sect. 1.3, we introduce the space

$$\mathcal{Y} = \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}}) := \{(z, \overline{z}) \;:\; z \in \mathcal{Z}\},$$

which will serve as the *dual phase space* of our system. Recall that in the bosonic case we equipped $\mathcal{Y}$ with the structure of the *Kähler space* consisting of the anti-involution j, the Euclidean scalar product $\cdot$ and the symplectic form ω:

$$\mathrm{j}(z, \overline{z}) := (\mathrm{i}z, \overline{\mathrm{i}z}), \tag{13.1}$$

$$(z, \overline{z}) \cdot (w, \overline{w}) := 2\mathrm{Re}(z|w), \tag{13.2}$$

$$(z, \overline{z}) \cdot \omega (w, \overline{w}) := 2\mathrm{Im}(z|w) = -(z, \overline{z}) \cdot \mathrm{j}(w, \overline{w}). \tag{13.3}$$

In our presentation of the fermionic case, we will need the *Kähler anti-involution* j. The symplectic form ω will not be used. Instead of the scalar product (13.2) we will use another scalar product,

$$(z, \overline{z}) \cdot \nu (w, \overline{w}) := \mathrm{Re}(z|w) = \frac{1}{2}(z, \overline{z}) \cdot (w, \overline{w}).$$

Again we will avoid identifing $\mathcal{Z}$ with $\mathcal{Y}$.

$\mathbb{C}\mathcal{Y}$ is identified with $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ by the map

$$\mathbb{C}\mathcal{Y} \ni (z_1 + \overline{z}_1) + \mathrm{i}(z_2 + \overline{z}_2) \mapsto (z_1 + \mathrm{i}z_2, \overline{z_1 - \mathrm{i}z_2}) \in \mathcal{Z} \oplus \overline{\mathcal{Z}}.$$

$\mathcal{Y}^{\#}$, the space dual to $\mathcal{Y}$, is canonically identified with $\mathrm{Re}(\overline{\mathcal{Z}} \oplus \mathcal{Z})$ by using the scalar product (9.2), and $\mathbb{C}\mathcal{Y}^{\#}$ is identified with $\overline{\mathcal{Z}} \oplus \mathcal{Z}$.

13.1 Fock CAR representation

Consider the fermionic Fock space $\Gamma_{\mathrm{a}}(\mathcal{Z})$. Recall that, for $z \in \mathcal{Z}$, $a^*(z)$, resp. $a(z)$ denote the corresponding creation, resp. annihilation operators defined in Sect. 3.4.

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13.1.1 Field operators

Definition 13.1 *For $y = (z_1, \overline{z}_2) \in \mathbb{C}\mathcal{Y}$, the corresponding* field operator *acts on $\Gamma_{\mathrm{a}}(\mathcal{Z})$ and is defined as*

$$\phi(z_1, \overline{z}_2) := a^*(z_1) + a(z_2). \tag{13.4}$$

Recall that $I := (-1)^N$.

Theorem 13.2 (1) *Operators $\phi(z_1, \overline{z}_2)$ are bounded and*

$$\phi(z_1, \overline{z}_2)^* = \phi(z_2, \overline{z}_1).$$

In particular, $\phi(z, \overline{z})$ are self-adjoint.

(2) $[\phi(w_1, \overline{w}_2), \phi(z_1, \overline{z}_2)]_+ = (w_2|z_1) + (z_2|w_1)$. *Hence, in particular,*

$$[\phi(w, \overline{w}), \phi(z, \overline{z})]_+ = 2(w, \overline{w})\cdot\nu(z, \overline{z})\mathbb{1}.$$

(3) $[\phi(z_1, z_2), I]_+ = 0$.

(4) *If $p \in B(\mathcal{Z})$, we have*

$$\Gamma(p)\phi(z_1, \overline{z}_2) = \phi(pz_1, \overline{p^{*-1}z}_2)\Gamma(p).$$

(5) *If $h \in B(\mathcal{Z})$, we have*

$$[\mathrm{d}\Gamma(h), \phi(z_1, \overline{z}_2)] = \phi(hz_1, -\overline{h^* z_2}).$$

(6) *We have an irreducible CAR representation,*

$$\mathcal{Y} \ni (z, \overline{z}) \mapsto \phi(z, \overline{z}) \in B_{\mathrm{h}}\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big). \tag{13.5}$$

For further reference let us record:

Proposition 13.3 *Let $A \in B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$ anti-commute with $\phi(y)$, $y \in \mathcal{Y}$. Then A is proportional to I.*

Definition 13.4 *(13.5) is called the* Fock CAR representation over $\mathcal{Y}$ in $\Gamma_{\mathrm{a}}(\mathcal{Z})$.

Remark 13.5 *Suppose that $\mathcal{Z} = \mathbb{C}^m$ and $(e_1, \dots, e_m)$ is the canonical basis of $\mathbb{C}^m$. Clearly, $\Gamma_{\mathrm{a}}(\mathbb{C})$ can be identified with $\mathbb{C}^2$. Therefore, we have the identification*

$$\otimes^m \mathbb{C}^2 \simeq \otimes^m \Gamma_{\mathrm{a}}(\mathbb{C}) \simeq \Gamma_{\mathrm{a}}(\mathbb{C}^m).$$

Under this identification, $\phi^{\mathrm{JW}}_{2j-1}$, resp. ϕ^{JW}_{2j} acting on $\otimes^m \mathbb{C}^2$ defined in (12.12) coincides with $\phi(e_j, \overline{e}_j)$, resp. $\phi(\mathrm{i}e_j, -\mathrm{i}\overline{e}_j)$ acting on $\Gamma_{\mathrm{a}}(\mathbb{C}^m)$. Note that

$$(e_1, \overline{e}_1), (\mathrm{i}e_1, -\mathrm{i}\overline{e}_1), \dots, (e_m, \overline{e}_m), (\mathrm{i}e_m, -\mathrm{i}\overline{e}_m)$$

is an o.n. basis of $(\mathcal{Y}, \nu)$. Thus, the Jordan–Wigner representation over $\mathbb{R}^{2m}$ in $\otimes^m \mathbb{C}^2$ coincides with the Fock representation over $\mathbb{R}^{2m}$ on $\Gamma_{\mathrm{a}}(\mathbb{C}^m)$.

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13.1.2 Extended Fock representation

Note that I implements the parity transformation, since

$$I\phi(z,\overline{z})I^{-1} = -\phi(z,\overline{z}).$$

Let us extend the scalar product ν to the space $\mathcal{Y} \oplus \mathbb{R}$ by

$$(z_1,\overline{z_1})\cdot\nu(z_2,\overline{z_2}) + t_1t_2 := \mathrm{Re}(z_1|z_2) + t_1t_2.$$

Clearly,

$$\mathcal{Y} \oplus \mathbb{R} \ni (z,\overline{z},t) \mapsto \phi(z,\overline{z}) + tI \in B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big) \tag{13.6}$$

is also an irreducible representation of the CAR.

Definition 13.6 *(13.6) is called the* extended Fock CAR representation over $\mathcal{Y} \oplus \mathbb{R}$ in $\Gamma_{\mathrm{a}}(\mathcal{Z})$.

Remark 13.7 *Extending Remark 13.5 in an obvious way, we note that the representation (12.14) over $\mathbb{R}^{2m+1}$ in $\otimes^m \mathbb{C}^2$ can be identified with the extended Fock representation over $\mathbb{R}^{2m+1}$ in $\Gamma_{\mathrm{a}}(\mathbb{C}^m)$.*

13.1.3 Slater determinants

Let $\mathcal{W}$ be a finite-dimensional *oriented* subspace of $\mathcal{Z}$. (For the definition of an oriented complex space see Subsect. 3.6.8.) Let $(w_1,\dots,w_n)$ be an o.n. basis of $\mathcal{W}$ compatible with the orientation. Then

$$a^*(w_1)\cdots a^*(w_n)\Omega = \sqrt{n!}\, w_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} w_n \tag{13.7}$$

is a normalized vector.

Definition 13.8 *Vectors of the form (13.7) are called* Slater determinants. *If $\mathcal{W} = \mathcal{Z}$, then (13.7) is called a* ceiling vector.

If $u \in U(\mathcal{W})$, then

$$a^*(uw_1)\cdots a^*(uw_n)\Omega = (\det u)a^*(w_1)\cdots a^*(w_n)\Omega.$$

Thus a Slater determinant depends only on the oriented subspace $\mathcal{W}$.

13.2 Real-wave and complex-wave CAR representation on Fock spaces

In Subsects. 12.4.2 and 12.5.3 we introduced the concept of a real-wave CAR representation by using the GNS representation for the canonical tracial state. There exists a convenient alternative description of this representation that uses the Fock CAR representation, which we will discuss in this section.

We will also introduce the complex-wave CAR representation – an analog of the complex-wave CCR representation, which we discussed in Subsect. 9.2.1.

13.2.1 Real-wave CAR representation on Fock spaces

Let $\mathcal{Y}$ be a real Hilbert space. Clearly, $\mathbb{C}\mathcal{Y}$ is a complex Hilbert space possessing a natural conjugation. For typographical reasons, this conjugation will be sometimes denoted χ.

In this subsection we continue to discuss the real-wave representation in an arbitrary dimension.

We will consider the Fock space $\Gamma_{\mathrm{a}}(\mathbb{C}\mathcal{Y})$ equipped with the corresponding conjugation. $\Gamma_{\mathrm{a}}(\mathcal{Y})$ is its real subspace of elements fixed by the conjugation $\Gamma(\chi)$. Linear operators that preserve $\Gamma_{\mathrm{a}}(\mathcal{Y})$ are called real.

Introduce the following operators on the fermionic Fock space $\Gamma_{\mathrm{a}}(\mathbb{C}\mathcal{Y})$:

$$\begin{aligned}\phi^{\mathrm{l}}(y) &:= a^*(y) + a(y),\\ \phi^{\mathrm{r}}(y) &:= \Lambda\big(a^*(y) + a(y)\big)\Lambda, \quad y \in \mathcal{Y},\end{aligned}$$

where we recall that $\Lambda = (-1)^{N(N-1)/2}$.

Theorem 13.9 (1) *We have two mutually commuting CAR representations:*

$$\mathcal{Y} \ni y \mapsto \phi^{\mathrm{l}}(y) \in B_{\mathrm{h}}\big(\Gamma_{\mathrm{a}}(\mathbb{C}\mathcal{Y})\big), \tag{13.8}$$

$$\mathcal{Y} \ni y \mapsto \phi^{\mathrm{r}}(y) \in B_{\mathrm{h}}\big(\Gamma_{\mathrm{a}}(\mathbb{C}\mathcal{Y})\big). \tag{13.9}$$

That means, for $y_1, y_2 \in \mathcal{Y}$,

$$[\phi^{\mathrm{l}}(y_1), \phi^{\mathrm{l}}(y_2)]_+ = [\phi^{\mathrm{r}}(y_1), \phi^{\mathrm{r}}(y_2)]_+ = 2y_1\cdot\nu y_2 \mathbb{1}, \qquad [\phi^{\mathrm{l}}(y_1), \phi^{\mathrm{r}}(y_2)] = 0.$$

(2) *We have*

$$\phi^{\mathrm{l}}(w) = a^*(w) + a(\chi w), \quad \phi^{\mathrm{l}}(w)^* = \phi(\chi w), \quad w \in \mathbb{C}\mathcal{Y}. \tag{13.10}$$

(3) *Let* $\pi^{\mathrm{l}} : \mathrm{CAR}^{C^*}(\mathcal{Y}) \to B\big(\Gamma_{\mathrm{a}}(\mathbb{C}\mathcal{Y})\big)$ *be the* $*$*-homomorphism obtained by Prop. 12.31 from the CAR representations (13.8). Then* Ω *is a cyclic vector representative for the state* tr *and the representation* π^{l}. *Therefore,* π^{l} *is the GNS representation of* $\mathrm{CAR}^{C^*}(\mathcal{Y})$ *for the state* tr *and it extends to a* $*$*-isomorphism of* $\mathrm{CAR}^{W^*}(\mathcal{Y})$ *onto* $\pi^{\mathrm{l}}\Big(\mathrm{CAR}^{C^*}(\mathcal{Y})\Big)''$.

(4) *Let* J *be the modular conjugation for the state* tr. *Then* $J = \Lambda\Gamma(\chi)$. *We have*

$$J\phi^{\mathrm{l}}(y)J = \phi^{\mathrm{r}}(y), \quad y \in \mathcal{Y}.$$

(5) *We have*

$$\pi^{\mathrm{l}}\big(\mathrm{c}(A)\big) = \Gamma(\chi)\pi^{\mathrm{l}}(A)\Gamma(\chi), \quad A \in \mathrm{CAR}(\mathbb{R}^n).$$

Consequently, $\pi^{\mathrm{l}}\big(\mathrm{Cliff}(\mathcal{Y})\big)$ *consists of real elements of* $\pi^{\mathrm{l}}\big(\mathrm{CAR}(\mathcal{Y})\big)$.

Proof Statements (1) and (2) are simple computations.

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Consider the GNS representation of $\mathrm{CAR}^{C^*}(\mathcal{Y})$ w.r.t. the state tr, denoted $(\mathcal{H}_{\mathrm{tr}}, \pi_{\mathrm{tr}}, \Omega_{\mathrm{tr}})$. The Hilbert space $\mathcal{H}_{\mathrm{tr}}$ contains $\mathrm{CAR}^{C^*}(\mathcal{Y})$ as a dense subspace, equipped with the scalar product

$$\mathrm{tr} A^* B, \quad A, B \in \mathrm{CAR}^{C^*}(\mathcal{Y}).$$

Let us define a linear operator

$$\overset{\mathrm{al}}{\Gamma}_{\mathrm{a}}(\mathbb{C}\mathcal{Y}) \ni a \mapsto Ua := \mathrm{Op}(a) \in \mathrm{CAR}^{C^*}(\mathcal{Y}) \subset \mathcal{H}_{\mathrm{tr}}.$$

The identity

$$\mathrm{tr}\mathrm{Op}(b)^* \mathrm{Op}(c) = (b|c) \tag{13.11}$$

implies that U extends to a unitary operator

$$U : \Gamma_{\mathrm{a}}(\mathbb{C}\mathcal{Y}) \to \mathcal{H}_{\mathrm{tr}}.$$

U maps $\Omega \in \Gamma_{\mathrm{a}}(\mathbb{C}\mathcal{Y})$ onto $\mathrm{Op}(1) = \mathbb{1} = \Omega_{\mathrm{tr}}$.

We have

$$\pi_{\mathrm{tr}}(A)B = AB, \quad A \in \mathrm{CAR}^{C^*}(\mathcal{Y}), \quad B \in \mathrm{CAR}^{C^*}(\mathcal{Y}) \subset \mathcal{H}_{\mathrm{tr}}.$$

In particular, consider $A = \mathrm{Op}(y) = \phi(y)$, $y \in \mathcal{Y}$, and $B = \mathrm{Op}(b)$. Adding up (12.27) and (12.28) we obtain

$$\pi_{\mathrm{tr}}\big(\phi(y)\big)\mathrm{Op}(b) = \mathrm{Op}\big(y{\cdot}b + (\nu y){\cdot}\nabla_v b\big).$$

Therefore,

$$\begin{aligned} U^* \pi_{\mathrm{tr}}\big(\phi(y)\big)U &= y{\cdot}v + (\nu y){\cdot}\nabla_v \\ &= a^*(y) + a(y) = \pi^{\mathrm{l}}(y). \end{aligned}$$

This proves (3). □

By the above theorem, we can identify the representation π^{l} with the real-wave representation π_{tr} considered in Subsect. 12.5.3.

13.2.2 Operators in the real-wave CAR representation

This subsection is parallel to Subsect. 9.3.5, where we studied operators in the real-wave CCR representation. For brevity, we will write $\mathfrak{R}$ for $\mathrm{CAR}^{W^*}(\mathcal{Y})$.

Let us use the terminology of non-commutative probability spaces, introduced in Sect. 6.5. By Thm. 13.9, we have a canonical unitary identification

$$L^2\big(\mathfrak{R}, \mathrm{tr}\big) \simeq \Gamma_{\mathrm{a}}(\mathbb{C}\mathcal{Y}). \tag{13.12}$$

Thus the real-wave representation acts on a "non-commutative L^2 space", which we view as a justification for the name "real-wave representation of CAR" for the constructions described above.

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Definition 13.10 *Let a be a contraction on $\mathcal{Y}$. We define an operator $\Gamma_{\mathrm{rw}}(a)$ on $L^2(\mathfrak{R}, \mathrm{tr}) = \Gamma_{\mathrm{a}}(\mathbb{C}\mathcal{Y})$ by*

$$\Gamma_{\mathrm{rw}}(a) = \Gamma(a_{\mathbb{C}}).$$

Proposition 13.11 (1) *Let a be a contraction on $\mathcal{Y}$, $b, c \in \Gamma_{\mathrm{a}}(\mathbb{C}\mathcal{Y})$ and $c = \Gamma_{\mathrm{rw}}(a)b$. Then, if we use the identification (13.12), we have*

$$\mathrm{Op}(c) = \mathrm{Op}\big(\Gamma(a)b\big).$$

(2) *Let $\mathcal{Y}_1$ be a closed subspace of $\mathcal{Y}$ and e_1 the orthogonal projection onto $\mathcal{Y}_1$. Then*

$$\Gamma_{\mathrm{rw}}(e_1) = E_{\mathcal{Y}_1},$$

where $E_{\mathcal{Y}_1}$ is the conditional expectation introduced in Subsect. 12.5.4.

(3) *Let $r \in O(\mathcal{Y})$. Then $\hat{r}$, defined originally as an automorphism of $\mathfrak{R}$, can be extended to a unitary operator on $L^2(\mathfrak{R}, \mathrm{tr})$. If we denote this extension also by $\hat{r}$, we have*

$$\Gamma_{\mathrm{rw}}(r) = \hat{r}.$$

The following fermionic analog of Prop. 9.29 is due to Gross (1972):

Proposition 13.12 *Let $a \in B(\mathcal{Y})$. Then if $\|a\| \leq 1$, $\Gamma_{\mathrm{rw}}(a)$ is positivity preserving. It follows that $\Gamma_{\mathrm{rw}}(a)$ extends to a contraction on $L^p(\mathfrak{R}, \mathrm{tr})$ for all $1 \leq p \leq \infty$.*

Proof We follow the proof in Prop. 9.29, writing a as $j^* u j$. The map $\Gamma_{\mathrm{rw}}(j)$ becomes

$$L^2(\mathfrak{R}, \mathrm{tr}) \ni A \mapsto A \otimes \mathbb{1} \in L^2(\mathfrak{R}, \mathrm{tr}) \otimes L^2(\mathfrak{R}, \mathrm{tr}),$$

which is positivity preserving, as well as $\Gamma_{\mathrm{rw}}(j^*) = \Gamma_{\mathrm{rw}}(j)^*$. If $A \in \mathfrak{R} \otimes \mathfrak{R}$, then $\Gamma_{\mathrm{rw}}(u)A$ as an operator on $\Gamma_{\mathrm{a}}(\mathbb{C}\mathcal{Y} \oplus \mathbb{C}\mathcal{Y})$ equals $\Gamma(u_{\mathbb{C}})A\Gamma(u_{\mathbb{C}})^{-1}$, which belongs to $\mathfrak{R} \otimes \mathfrak{R}$ and is positive if A is. Hence, $\Gamma(u_{\mathbb{C}})$ is positivity preserving. The second statement then follows from Thm. 6.81. □

The following fermionic version of Nelson's hyper-contractivity theorem is due to Gross (1972) and Carlen–Lieb (1993):

Theorem 13.13 *Let $a \in B(\mathcal{Y})$, $1 < p \leq q < \infty$ and*

$$\|a\| \leq (p-1)^{\frac{1}{2}}(q-1)^{-\frac{1}{2}}.$$

Then $\Gamma_{\mathrm{rw}}(a)$ is a contraction from $L^p(\mathfrak{R}, \mathrm{tr})$ to $L^q(\mathfrak{R}, \mathrm{tr})$.

13.2.3 Complex-wave CAR representation in finite dimensions

One can reformulate the Fock CAR representation so that it becomes analogous to the complex-wave CCR representation considered in Subsect. 9.2.1. For simplicity, at first we restrict ourselves to finite-dimensional spaces $\mathcal{Z}$. We identify $\mathcal{Z}^{\#}$ with $\overline{\mathcal{Z}}$ using the scalar product.

Recall that an alternative notation for $\Gamma_{\mathrm{a}}(\mathcal{Z})$ is $\mathrm{Pol}_{\mathrm{a}}(\overline{\mathcal{Z}})$. Elements of $\mathrm{Pol}_{\mathrm{a}}(\overline{\mathcal{Z}})$ are treated as sequences whose n-th element is an anti-symmetric n-linear form on $\overline{\mathcal{Z}}$. Thus to define $F \in \mathrm{Pol}_{\mathrm{a}}(\overline{\mathcal{Z}})$ we need to specify

$$F(\overline{z}_1, \dots, \overline{z}_n), \qquad z_1, \dots, z_n \in \mathcal{Z}, \quad n = 0, 1, 2, \dots. \tag{13.13}$$

In algebraic formulas we write $F(\overline{z})$ instead of (13.13), treating $\overline{z}$ as the "generic variable" in $\overline{\mathcal{Z}}$, as discussed in Subsect. 3.5.1.

Likewise, an alternative notation for $\Gamma_{\mathrm{a}}(\overline{\mathcal{Z}})$ is $\mathrm{Pol}_{\mathrm{a}}(\mathcal{Z})$. Applying the complex conjugation to $F \in \mathrm{Pol}_{\mathrm{a}}(\overline{\mathcal{Z}})$, we obtain $\overline{F} \in \mathrm{Pol}_{\mathrm{a}}(\mathcal{Z})$ such that

$$\overline{F}(z_1, \dots, z_n) = \overline{F(\overline{z}_1, \dots, \overline{z}_n)}, \qquad z_1, \dots, z_n \in \mathcal{Z}, \quad n = 0, 1, 2, \dots. \tag{13.14}$$

We will commonly write $\overline{F}(z)$ or $\overline{F(\overline{z})}$ instead of (13.14), treating z as the "generic variable" in $\mathcal{Z}$.

Let us fix a (complex) volume form $\mathrm{d}\overline{z}$ on $\mathcal{Z}$ compatible with the scalar product of $\mathcal{Z}$, and let $\mathrm{d}z$ be the dual volume form on $\overline{\mathcal{Z}}$. As in Subsect. 7.2.1, if $A \in \mathbb{C}\mathrm{Pol}_{\mathrm{a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$, we can define its Berezin integral,

$$\int A(z, \overline{z}) \mathrm{d}z \mathrm{d}\overline{z}.$$

Equip $\mathrm{Pol}_{\mathrm{a}}(\overline{\mathcal{Z}})$ with the scalar product

$$(F|G) := \int \overline{F(\overline{z})} G(\overline{z}) \mathrm{e}^{z \cdot \overline{z}} \mathrm{d}z \mathrm{d}\overline{z}.$$

We define the map $T^{\mathrm{cw}} : \Gamma_{\mathrm{a}}(\mathcal{Z}) \to \mathrm{Pol}_{\mathrm{a}}(\overline{\mathcal{Z}})$ by

$$T^{\mathrm{cw}} \Psi(\overline{z}_1, \dots, \overline{z}_n) := \frac{1}{\sqrt{n!}} (z_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} z_n | \Psi), \quad \Psi \in \Gamma_{\mathrm{a}}(\mathcal{Z}), \; z_1, \dots, z_n \in \mathcal{Z}.$$

Applying Thm. 7.23 (2) to $\mathcal{Y} = \mathcal{Z}$, $\mathcal{Y}^{\#} = \overline{\mathcal{Z}}$, we obtain the following theorem:

Theorem 13.14 (1) *The operator T^{cw} is unitary, that is, for $\Phi, \Psi \in \Gamma_{\mathrm{a}}(\mathcal{Z})$,*

$$(\Phi|\Psi) = \int \overline{T^{\mathrm{cw}} \Phi(\overline{z})} T^{\mathrm{cw}} \Psi(\overline{z}) \mathrm{e}^{z \cdot \overline{z}} \mathrm{d}z \mathrm{d}\overline{z}.$$

(2) *For $w \in \mathcal{Z}$ we have*

$$\begin{aligned} T^{\mathrm{cw}} \Omega &= 1, \\ T^{\mathrm{cw}} a^*(w) &= w \cdot \overline{z}\, T^{\mathrm{cw}}, \\ T^{\mathrm{cw}} a(w) &= \overline{w} \cdot \nabla_{\overline{z}}\, T^{\mathrm{cw}}, \\ \big(T^{\mathrm{cw}} \Gamma(p) \Psi\big)(\overline{z}) &= (T^{\mathrm{cw}} \Psi)(p^{\#} \overline{z}), \quad p \in B(\mathcal{Z}), \quad \Psi \in \Gamma_{\mathrm{a}}(\mathcal{Z}). \end{aligned}$$

Proposition 13.15 *For $w \in \mathcal{Z}$, define an operator on $\mathrm{Pol}_{\mathrm{a}}(\overline{\mathcal{Z}})$ by*

$$\phi^{\mathrm{cw}}(w, \overline{w}) := w \cdot \overline{z} + \overline{w} \cdot \nabla_{\overline{z}}.$$

The map

$$\mathrm{Re}(\mathcal{Z}\oplus\overline{\mathcal{Z}})\ni(w,\overline{w})\mapsto\phi^{\mathrm{cw}}(w,\overline{w})\in B_{\mathrm{h}}\big(\mathrm{Pol}_{\mathrm{a}}(\overline{\mathcal{Z}})\big)$$

is a CAR representation unitarily equivalent to the Fock representation:

$$\phi^{\mathrm{cw}}(w,\overline{w})=T^{\mathrm{cw}}\big(a^*(w)+a(w)\big)T^{\mathrm{cw}*}. \tag{13.15}$$

Definition 13.16 *(13.15) is called the* basic form of the complex-wave CAR representation.

13.2.4 Complex-wave CAR representation: the general case

If $\mathcal{Z}$ is infinite-dimensional, the Berezin integral does not exist anymore. Therefore, strictly speaking, the definition of the complex-wave CAR representation has to be modified. We will need to use the formalism of non-commutative probability spaces.

Let us start by defining an appropriate real-wave CAR representation with a tracial state that will replace the Berezin integral. Consider the space $\mathcal{Z}\oplus\overline{\mathcal{Z}}$ equipped with a natural conjugation

$$\chi(z_1,\overline{z}_2):=(z_2,\overline{z}_1),\quad (z_1,\overline{z}_2)\in\mathcal{Z}\oplus\overline{\mathcal{Z}},$$

whose real subspace is $\mathrm{Re}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$. $\mathrm{Re}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$ is equipped with the symmetric form

$$(z,\overline{z})\cdot\nu(z',\overline{z}'):=2\mathrm{Re}(z|z'),\quad (z,\overline{z}),(z',\overline{z}')\in\mathrm{Re}(\mathcal{Z}\oplus\overline{\mathcal{Z}}).$$

Following Thm. 13.9, consider the real-wave CAR representation

$$\mathrm{Re}(\mathcal{Z}\oplus\overline{\mathcal{Z}})\ni(z,\overline{z})\mapsto\phi^{\mathrm{rw}}(z,\overline{z}):=a^*(z,\overline{z})+a(z,\overline{z})\in B_{\mathrm{h}}\left(\Gamma_{\mathrm{a}}(\mathcal{Z}\oplus\overline{\mathcal{Z}})\right).$$

The fields $\phi^{\mathrm{rw}}(z,\overline{z})$, $z\in\mathcal{Z}$, generate a von Neumann algebra $\mathfrak{R}$ isomorphic to $\mathrm{CAR}^{W^*}\big(\mathrm{Re}(\mathcal{Z}\oplus\overline{\mathcal{Z}})\big)$. As in (13.10), we extend these fields from $\mathrm{Re}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$ to $\mathcal{Z}\oplus\overline{\mathcal{Z}}$ by complex linearity, setting for $(z_1,\overline{z}_2)\in\mathcal{Z}\oplus\overline{\mathcal{Z}}$

$$\phi^{\mathrm{rw}}(z_1,\overline{z}_2):=a^*(z_1,\overline{z}_2)+a(z_2,\overline{z}_1)\in\mathfrak{R}.$$

We have

$$\begin{aligned}\phi^{\mathrm{rw}}(z_1,\overline{z}_2)^*&=\phi^{\mathrm{rw}}(z_2,\overline{z}_1),\\ [\phi^{\mathrm{rw}}(z_1,\overline{z}_2),\phi^{\mathrm{rw}}(z_1',\overline{z}_2')]_+&=2(z_1|z_2')+2(z_2|z_1').\end{aligned} \tag{13.16}$$

Denote by $\mathfrak{R}_{\widetilde{\mathrm{cw}}}$ the σ-weakly closed (but non-self-adjoint) sub-algebra of $\mathfrak{R}$ generated by $\phi^{\mathrm{rw}}(z,0)$, $z\in\mathcal{Z}$.

Theorem 13.17 *There exists a unique bounded linear map*

$$T^{\widetilde{\mathrm{cw}}}:\Gamma_{\mathrm{a}}(\mathcal{Z})\to L^2(\mathfrak{R},\mathrm{tr})$$

such that

$$T^{\widetilde{\mathrm{cw}}}\Omega := \mathbb{1},$$
$$T^{\widetilde{\mathrm{cw}}}a^*(z) = \phi^{\mathrm{rw}}(z,0)T^{\widetilde{\mathrm{cw}}}, \quad z \in \mathcal{Z}.$$

The map is isometric, i.e.

$$(\Phi|\Psi) = \mathrm{tr}\big(T^{\widetilde{\mathrm{cw}}}\Phi\big)^* T^{\widetilde{\mathrm{cw}}}\Psi, \quad \Phi, \Psi \in \Gamma_{\mathrm{a}}(\mathcal{Z}). \tag{13.17}$$

It satisfies

$$T^{\widetilde{\mathrm{cw}}}a(z) = \phi^{\mathrm{rw}}(0,\overline{z})T^{\widetilde{\mathrm{cw}}}, \quad z \in \mathcal{Z},$$
$$T^{\widetilde{\mathrm{cw}}}\Gamma(p) = \Gamma_{\mathrm{rw}}(p \oplus \overline{p})T^{\widetilde{\mathrm{cw}}}, \quad p \in B(\mathcal{Z}).$$

The image of $T^{\widetilde{\mathrm{cw}}}$ *is a commutative sub-algebra of* $L^2(\mathfrak{R}, \mathrm{tr})$.

Definition 13.18 *The image of* $T^{\widetilde{\mathrm{cw}}}$ *is denoted by* $L^2(\mathfrak{R}_{\widetilde{\mathrm{cw}}}, \mathrm{tr})$.

Proposition 13.19 *For* $z \in \mathcal{Z}$*, the multiplication by* $\phi^{\mathrm{rw}}(z,\overline{z})$ *preserves* $L^2(\mathfrak{R}_{\widetilde{\mathrm{cw}}}, \mathrm{tr})$*. Therefore,*

$$\phi^{\widetilde{\mathrm{cw}}}(z,\overline{z})A := \phi^{\mathrm{rw}}(z,\overline{z})A, \quad A \in L^2(\mathfrak{R}_{\widetilde{\mathrm{cw}}}, \mathrm{tr}),$$

defines an operator on $L^2(\mathfrak{R}_{\widetilde{\mathrm{cw}}}, \mathrm{tr})$*. The map*

$$\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}}) \ni (z,\overline{z}) \mapsto \phi^{\widetilde{\mathrm{cw}}}(z,\overline{z}) \in B_{\mathrm{h}}\big(L^2(\mathfrak{R}_{\widetilde{\mathrm{cw}}}, \mathrm{tr})\big)$$

is a CAR representation unitarily equivalent to the Fock representation:

$$\phi^{\widetilde{\mathrm{cw}}}(z,\overline{z}) = T^{\widetilde{\mathrm{cw}}}\big(a^*(z) + a(z)\big)T^{\widetilde{\mathrm{cw}}*}. \tag{13.18}$$

Definition 13.20 *(13.18) is called the* alternate form of the complex-wave CAR representation*.*

13.3 Wick and anti-Wick fermionic quantization

This section is parallel to Sect. 9.4, where the bosonic Wick and anti-Wick quantizations were considered.

The framework of this section is the same as that of the whole chapter. Recall that $\mathcal{Z}$ is a Hilbert space, $\mathcal{Y} = \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ and we identify $\mathcal{Y}^\# \simeq \mathrm{Re}(\overline{\mathcal{Z}} \oplus \mathcal{Z})$. Recall from Subsect. 3.5.6 that $\mathbb{C}\mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}^\#)$ is identified with $\mathrm{Pol}_{\mathrm{a}}(\overline{\mathcal{Z}} \oplus \mathcal{Z})$.

We consider the Fock CAR representation

$$\mathcal{Y} \ni y \mapsto \phi(y) \in B_{\mathrm{h}}\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big).$$

Recall that $\mathrm{CAR}^{\mathrm{alg}}(\mathcal{Y})$ is the $*$-algebra generated by $\phi(y)$, $y \in \mathcal{Y}$. It can be represented by operators on the space $\Gamma_{\mathrm{a}}(\mathcal{Z})$. Recall that $\Lambda = (-1)^{N(N-\mathbb{1})/2}$.

We will define and study the fermionic Wick and anti-Wick quantizations.

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13.3.1 Wick and anti-Wick ordering

Recall that, with $b \in \mathrm{Pol_a}(\overline{\mathcal{Z}})$, in Subsect. 3.4.4 we defined the multiple creation and annihilation operators. We obtain two homomorphisms,

$$\mathrm{Pol_a}(\overline{\mathcal{Z}}) \ni b \mapsto a^*(b) \in \mathrm{CAR^{alg}}(\mathcal{Y}),$$
$$\mathrm{Pol_a}(\mathcal{Z}) \ni \overline{b} \mapsto a(\Lambda b) \in \mathrm{CAR^{alg}}(\mathcal{Y}).$$

Note that the possibility of unambiguously defining $a^*(b)$ and $a(b)$ follows from the fact that $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ are isotropic subspaces of $\mathbb{C}\mathcal{Y}$ for the bilinear symmetric form $\nu_{\mathbb{C}}$.

Definition 13.21 *For $b_1, b_2 \in \mathrm{Pol_a}(\overline{\mathcal{Z}})$ we set*

$$\mathrm{Op}^{a^*,a}(b_1\overline{b}_2) := a^*(b_1)a(\Lambda b_2),$$

$$\mathrm{Op}^{a,a^*}(\overline{b}_2 b_1) := a(\Lambda b_2)a^*(b_1).$$

These maps extend by linearity to maps

$$\begin{aligned}\mathbb{C}\mathrm{Pol_a}(\mathcal{Y}^{\#}) \ni b \mapsto \mathrm{Op}^{a^*,a}(b) \in \mathrm{CAR^{alg}}(\mathcal{Y}),\\ \mathbb{C}\mathrm{Pol_a}(\mathcal{Y}^{\#}) \ni b \mapsto \mathrm{Op}^{a,a^*}(b) \in \mathrm{CAR^{alg}}(\mathcal{Y}),\end{aligned} \tag{13.19}$$

called the Wick *and* anti-Wick fermionic quantizations*.*

Definition 13.22 *The inverse maps to (13.19) will be denoted by*

$$\mathrm{CAR^{alg}}(\mathcal{Y}) \ni B \mapsto \mathrm{s}_B^{a^*,a} \in \mathbb{C}\mathrm{Pol_a}(\mathcal{Y}^{\#}),$$

$$\mathrm{CAR^{alg}}(\mathcal{Y}) \ni B \mapsto \mathrm{s}_B^{a,a^*} \in \mathbb{C}\mathrm{Pol_a}(\mathcal{Y}^{\#}).$$

The anti-symmetric polynomial $\mathrm{s}_B^{a^*,a}$, resp. s_B^{a,a^*} is called the Wick*, resp.* anti-Wick symbol of the operator B*.*

Remark 13.23 *If we fix an o.n. basis $(e_i \ : \ i \in I)$ of $\mathcal{Z}$ parametrized by a totally ordered set I, and write*

$$b = \sum_{\{i_1,\dots,i_m\},\{i'_n,\dots,i'_i\}\subset I} b_{i_1,\dots,i_m;i'_n,\dots,i_1}\overline{z}_{i_1}\cdots\overline{z}_{i_m} z_{i'_n}\cdots z_{i'_1},$$
$$c = \sum_{\{i_1,\dots,i_m\},\{i'_n,\dots,i'_i\}\subset I} c_{i_1,\dots,i_m;i'_n,\dots,i_1} z_{i_1}\cdots z_{i_m}\overline{z}_{i'_n}\cdots\overline{z}_{i'_1},$$

then we have explicit formulas

$$\mathrm{Op}^{a^*,a}(b) = \sum_{\{i_1,\dots,i_m\},\{i'_n,\dots,i'_i\}\subset I} b_{i_1,\dots,i_m;i'_n,\dots,i_1} a^*_{i_1}\cdots a^*_{i_m} a_{i'_n}\cdots a_{i'_1},$$
$$\mathrm{Op}^{a,a^*}(c) = \sum_{\{i_1,\dots,i_m\},\{i'_n,\dots,i'_i\}\subset I} c_{i_1,\dots,i_m;i'_n,\dots,i_1} a_{i_1}\cdots a_{i_m} a^*_{i'_n}\cdots a^*_{i'_1}.$$

Proposition 13.24 (1) $\mathrm{Op}^{a^*,a}(b)^* = \mathrm{Op}^{a^*,a}(\Lambda\bar{b})$ *and* $\mathrm{Op}^{a,a^*}(b)^* = \mathrm{Op}^{a,a^*}(\Lambda\bar{b})$.
(2) *Let* $w \in \mathcal{Z}$, $b \in \mathbb{C}\mathrm{Pol_a}(\mathcal{Y}^\#)$. *Then*

$$\mathrm{Op}^{a^*,a}(w\cdot b) = a^*(w)\mathrm{Op}^{a^*,a}(b), \quad \mathrm{Op}^{a^*,a}(b\cdot\overline{w}) = \mathrm{Op}^{a^*,a}(b)a(w),$$

$$a^*(w)\mathrm{Op}^{a^*,a}(b) - \mathrm{Op}^{a^*,a}(Ib)a^*(w) = \mathrm{Op}^{a^*,a}(w\cdot\nabla_z b),$$

$$a(w)\mathrm{Op}^{a^*,a}(b) - \mathrm{Op}^{a^*,a}(Ib)a(w) = \mathrm{Op}^{a^*,a}(\overline{w}\cdot\nabla_{\overline{z}} b).$$

(3) *If* $\mathrm{Op}^{a,a^*}(b_-) = \mathrm{Op}^{a^*,a}(b_+)$, *then*

$$\begin{aligned} b_+(\overline{z}, z) &= \mathrm{e}^{\nabla_{\overline{z}}\cdot\nabla_z} b_-(\overline{z}, z) \\ &= \int \mathrm{e}^{(z-z_1)\cdot(\overline{z}-\overline{z}_1)} b_-(\overline{z_1}, z_1)\mathrm{d}\overline{z}_1\mathrm{d}z_1. \end{aligned}$$

(4) *If* $\mathrm{Op}^{a^*,a}(b_1)\mathrm{Op}^{a^*,a}(b_2) = \mathrm{Op}^{a^*,a}(b)$, *then*

$$\begin{aligned} b(\overline{z}, z) &= \mathrm{e}^{\nabla_{\overline{z}_1}\cdot\nabla_{z_1}} b_1(\overline{z}, z_1)b_2(\overline{z}_1, z)\big|_{z_1=z} \\ &= \int \mathrm{e}^{(z-z_1)\cdot(\overline{z}-\overline{z}_1)} b_1(\overline{z}, z_1)b_2(\overline{z}_1, z)\mathrm{d}z_1\mathrm{d}\overline{z_1}. \end{aligned}$$

(5) *The Wick quantization satisfies*

$$\big(\Omega|\mathrm{Op}^{a^*,a}(b)\Omega\big) = b(0), \quad b \in \mathbb{C}\mathrm{Pol_a}(\mathcal{Y}^\#). \tag{13.20}$$

Proof It suffices to prove (1) and (2) when b is a monomial, which is an easy computation.

To prove (3) and (4), we use the complex-wave representation. We see that the Wick, resp. anti-Wick, quantization can be seen as the $\overline{z}, \nabla_{\overline{z}}$ resp. $\nabla_{\overline{z}}, \overline{z}$ quantization. (3) and (4) follow then from Thm. 7.26. □

The following formula is the fermionic version of what is usually called Wick's theorem. We will give its diagrammatic interpretation in Chap. 20.

Theorem 13.25 *Let* $b_1, \dots, b_n \in \mathbb{C}\mathrm{Pol_a}(\mathcal{Y}^\#)$. *Let* $b \in \mathbb{C}\mathrm{Pol_a}(\mathcal{Y}^\#)$ *and*

$$\mathrm{Op}^{a^*,a}(b) = \mathrm{Op}^{a^*,a}(b_1)\cdots\mathrm{Op}^{a^*,a}(b_n).$$

Then

$$\begin{aligned} &b(\overline{z}, z) \\ &= \exp\Big(\sum_{i>j}\nabla_{\overline{z}_i}\cdot\nabla_{z_j}\Big)b_1(\overline{z}_1, z_1)\cdots b_n(\overline{z}_n, z_n)\big|_{z=z_1=\cdots=z_n}, \\ &\big(\Omega|\mathrm{Op}^{a^*,a}(b_1)\cdots\mathrm{Op}^{a^*,a}(b_n)\Omega\big) \\ &= \exp\Big(\sum_{i>j}\nabla_{\overline{z}_i}\cdot\nabla_{z_j}\Big)b_1(\overline{z}_1, z_1)\cdots b_n(\overline{z}_n, z_n)\big|_{0=z_1=\cdots=z_n}. \end{aligned}$$

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13.3.2 Relation between Wick, anti-Wick and anti-symmetric quantizations

We can introduce the anti-symmetric quantization

$$\mathrm{Pol_a}(\overline{\mathcal{Z}} \oplus \mathcal{Z}) \simeq \mathbb{C}\mathrm{Pol_a}(\mathcal{Y}^{\#}) \ni b \mapsto \mathrm{Op}(b) \in B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$$

as in Sect. 12.4.

Proposition 13.26 *Let $b, b_+, b_- \in \mathbb{C}\mathrm{Pol_a}(\mathcal{Y}^{\#})$. Let*

$$\mathrm{Op}^{a^*,a}(b_+) = \mathrm{Op}(b) = \mathrm{Op}^{a,a^*}(b_-).$$

(1) *The anti-symmetric symbol is given in terms of the Wick symbol by*

$$\begin{aligned} b(\overline{z}, z) &= \mathrm{e}^{\frac{1}{2}\nabla_z \cdot \nabla_{\overline{z}}} b_+(\overline{z}, z) \\ &= 2^d \int \mathrm{e}^{2(\overline{z}-\overline{z}_1)\cdot(z-z_1)} b_+(\overline{z}_1, z_1) \mathrm{d}\overline{z}_1 \mathrm{d}z_1. \end{aligned}$$

(2) *The anti-symmetric symbol is given in terms of the anti-Wick symbol by*

$$\begin{aligned} b(\overline{z}, z) &= \mathrm{e}^{-\frac{1}{2}\nabla_z \cdot \nabla_{\overline{z}}} b_-(\overline{z}, z) \\ &= 2^d \int \mathrm{e}^{-2(\overline{z}-\overline{z}_1)\cdot(z-z_1)} b_-(\overline{z}_1, z_1) \mathrm{d}\overline{z}_1 \mathrm{d}z_1. \end{aligned}$$

Proof To prove (1) we can assume that $b_+(\overline{z}, z) = b_1(\overline{z})\overline{b}_2(z)$, so that

$$\mathrm{Op}^{a^*,a}(b_+) = a^*(b_1)a(b_2) = \mathrm{Op}(b_1)\mathrm{Op}(\overline{b}_2) = \mathrm{Op}(b),$$

using that $\mathcal{Z}, \overline{\mathcal{Z}}$ are isotropic for the scalar product ν. Using Prop. 12.42 we get that

$$\begin{aligned} b(\overline{z}, z) &= \mathrm{e}^{(\nabla_{\overline{z}_1}, \nabla_{z_1})\cdot\nu(\nabla_{\overline{z}_2}, \nabla_{z_2})} b_1(\overline{z}_1)\overline{b}_2(z_2)\big|_{z_1=z_2=z} \\ &= \mathrm{e}^{\frac{1}{2}\nabla_z \cdot \nabla_{\overline{z}}} b_1(\overline{z})\overline{b}_2(z), \end{aligned}$$

which proves (1). Statement (2) follows then from Prop. 13.24. □

13.3.3 Wick quantization: the operator formalism

This subsection is parallel to Subsect. 9.4.5 about the bosonic case.

We will now treat Wick symbols as operators acting on the Fock space. We will restrict ourselves to a rather small class of such operators.

Recall that if N is the number operator, then $\mathbb{1}_{\{n\}}(N)$ is the projection from $\Gamma_{\mathrm{s}}(\mathcal{Z})$ onto $\Gamma_{\mathrm{s}}^n(\mathcal{Z})$. Similarly to the bosonic case, for $b \in B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$ we set $b_{n,m} := \mathbb{1}_{\{n\}}(N) b \mathbb{1}_{\{m\}}(N)$ and

$$\begin{aligned} &B^{\mathrm{fin}}\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big) \\ &= \big\{ b \in B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big) \ : \ \text{there exists} \ \ n_0 \ \text{such that} \ \ b_{n,m} = 0 \ \ \text{for} \ \ n, m > n_0 \big\}. \end{aligned}$$

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Definition 13.27 *Let $b \in B^{\mathrm{fin}}\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$. The* Wick quantization of b *is defined as the quadratic form on $\Gamma_{\mathrm{a}}^{\mathrm{fin}}(\mathcal{Z})$ such that for $\Phi, \Psi \in \Gamma_{\mathrm{a}}^{\mathrm{fin}}(\mathcal{Z})$,*

$$\begin{aligned}&(\Phi|\mathrm{Op}^{a^*,a}(b)\Psi)\\ &= \sum_{n,m=0}^{\infty}\sum_{k=0}^{\infty}\frac{\sqrt{(n+k)!(m+k)!}}{k!}(\Phi|b_{n,m}\otimes \mathbb{1}_{\mathcal{Z}}^{\otimes k}\Psi).\end{aligned} \tag{13.21}$$

The above definition can be viewed as a generalization of Def. 13.21.

Proposition 13.28 *Let $b \in \mathbb{C}\mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}^{\#}) \simeq \mathrm{Pol}_{\mathrm{a}}(\overline{\mathcal{Z}}\oplus\mathcal{Z})$ be identified with $b \in B^{\mathrm{fin}}\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$ by*

$$\begin{aligned}&(z_n\otimes_{\mathrm{a}}\cdots\otimes_{\mathrm{a}} z_1|b_{n,m}\, z_m'\otimes_{\mathrm{a}}\cdots\otimes_{\mathrm{a}} z_1') \\ &= \frac{(n+m)!}{n!m!}b_{n,m}(\overline{z}_n\otimes_{\mathrm{a}}\cdots\otimes_{\mathrm{a}}\overline{z}_1\otimes_{\mathrm{a}} z_1'\otimes_{\mathrm{a}}\cdots z_m'),\quad z_1,\dots,z_n,z_1',\dots,z_m'\in\mathcal{Z}.\end{aligned} \tag{13.22}$$

Then $\mathrm{Op}^{a^*,a}(b)$ in the sense of Def. 13.21, which involves b in the first meaning, coincides with $\mathrm{Op}^{a^*,a}(b)$ in the sense of Def. 13.27, involving b in the second meaning.

Proof Choose a totally ordered o.n. basis in $\mathcal{Z}$. Let

$$b = e_{i_1}\otimes_{\mathrm{a}}\cdots\otimes_{\mathrm{a}} e_{i_n}\otimes_{\mathrm{a}}\overline{e}_{j_m}\otimes_{\mathrm{a}}\cdots\otimes_{\mathrm{a}}\overline{e}_{j_1}.$$

Then $\mathrm{Op}^{a^*,a}(b)$ in the sense of Def. 13.21 equals

$$a_{i_1}^*\cdots a_{i_n}^* a_{j_m}\cdots a_{j_1}. \tag{13.23}$$

(13.22) identifies b with the operator

$$|e_{i_1}\otimes_{\mathrm{a}}\cdots\otimes_{\mathrm{a}} e_{i_n})(e_{j_1}\otimes_{\mathrm{a}}\cdots\otimes_{\mathrm{a}} e_{j_m}|.$$

$\mathrm{Op}^{a^*,a}(b)$ in the sense of Def. 13.27 is the quadratic form on $\Gamma_{\mathrm{a}}^{\mathrm{fin}}(\mathcal{Z})$ equal to

$$\sum_{k=0}^{\infty}\frac{\sqrt{(n+k)!(m+k)!}}{k!}|e_{i_1}\otimes_{\mathrm{a}}\cdots\otimes_{\mathrm{a}} e_{i_n})(e_{j_1}\otimes_{\mathrm{a}}\cdots\otimes_{\mathrm{a}} e_{j_m}|\otimes\mathbb{1}_{\mathcal{Z}}^{\otimes k}. \tag{13.24}$$

It is easy to see that (13.23) and (13.24) are equal. □

Proposition 13.29 *Let $b \in B^{\mathrm{fin}}\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$, $h \in B(\mathcal{Z}) \subset B^{\mathrm{fin}}\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$, $p \in B(\mathcal{Z},\mathcal{Z})$. Then*

$$\begin{aligned}\mathrm{Op}^{a^*,a}(b)^* &= \mathrm{Op}^{a^*,a}(b^*),\\ \mathrm{Op}^{a^*,a}(h) &= \mathrm{d}\Gamma(h),\\ [\mathrm{d}\Gamma(h),\mathrm{Op}^{a^*,a}(b)] &= \mathrm{Op}^{a^*,a}(hb-bh^*),\\ \Gamma(p)\mathrm{Op}^{a^*,a}\big(b\Gamma(p)\big) &= \mathrm{Op}^{a^*,a}\big(\Gamma(p)b\big)\Gamma(p),\\ \Gamma(p)\mathrm{Op}^{a^*,a}(b) &= \mathrm{Op}^{a^*,a}\big(\Gamma(p)b\Gamma(p^*)\big)\Gamma(p)\qquad \textit{if } p \textit{ is isometric},\\ \Gamma(p)\mathrm{Op}^{a^*,a}(b)\Gamma(p^*) &= \mathrm{Op}^{a^*,a}\big(\Gamma(p)b\Gamma(p^*)\big)\qquad \textit{if } p \textit{ is unitary}.\end{aligned}$$

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The following proposition describes the special class of particle preserving operators. Recall that the operator $\Theta(\sigma)$ is defined in Def. 3.11.

Theorem 13.30 *If $b \in B\big(\Gamma_{\mathrm{a}}^m(\mathcal{Z})\big)$, then*

$$(\Phi|\mathrm{Op}^{a^*,a}(b)\Psi) = \sum_{k=0}^{\infty} \tfrac{(m+k)!}{k!}(\Phi|b \otimes 1\!\!1_{\mathcal{Z}}^k \Psi).$$

Thus

$$\frac{1}{m!}\mathrm{Op}^{a^*,a}(b)\big|_{\Gamma_{\mathrm{a}}^{m+k}(\mathcal{Z})} = \left(\sum_{1\leq i_1<\cdots<i_m\leq m+k} b_{i_1,\dots,i_m}^{m+k}\right)\Big|_{\Gamma_{\mathrm{a}}^{m+k}(\mathcal{Z})},$$

where the operators $b_{i_1,\dots,i_m}^{m+k} \in B\big(\Gamma_{\mathrm{a}}^{m+k}(\mathcal{Z})\big)$ are defined as follows:

$$b_{i_1,\dots,i_m}^{m+k} := \Theta(\sigma)\ b \otimes 1\!\!1_{\mathcal{Z}}^{\otimes k}\ \Theta(\sigma)^{-1}\big|_{\Gamma_{\mathrm{a}}^{m+k}(\mathcal{Z})},$$

where $\sigma \in S_{m+k}$ is any permutation that transforms $(1,\dots,m)$ onto $(i_1,\dots,i_m)$.

13.3.4 Estimates on Wick polynomials

Fermionic Wick monomials tend to be bounded more often than bosonic ones. Here is an example of this phenomenon:

Proposition 13.31 *Let $h \in B^1(\mathcal{Z})$ be positive. Then $\|\mathrm{d}\Gamma(h)\| = \mathrm{Tr}\, h$.*

We also have a fermionic analog of bosonic N_τ *estimates* described in Prop. 9.50. The proof in the fermionic case is fully analogous to that in the bosonic case.

Proposition 13.32 *Let $b \in B\big(\Gamma_{\mathrm{a}}^q(\mathcal{Z}), \Gamma_{\mathrm{a}}^p(\mathcal{Z})\big) \subset B^{\mathrm{fin}}\big(\Gamma_a(\mathcal{Z})\big)$ for $p, q \in \mathbb{N}$. Let $m > 0$ be a self-adjoint operator on $\mathcal{Z}$. Then for all Ψ_1, $\Psi_2 \in \Gamma_{\mathrm{a}}(\mathcal{Z})$ one has*

$$\begin{aligned}&\left|\left(\mathrm{d}\Gamma(m)^{-p/2}\Psi_1|\mathrm{Op}^{a^*,a}(b)\mathrm{d}\Gamma(m)^{-q/2}\Psi_2\right)\right|\\ &\leq \|\Gamma(m)^{-\frac{1}{2}} b\Gamma(m)^{-\frac{1}{2}}\|\|\Psi_1\|\|\Psi_2\|.\end{aligned}$$

13.4 Notes

The Wick theorem goes back to Wick (1950).

The fermionic real-wave representation is due to Segal (1956). Second quantized operators in the fermionic real-wave representation were studied by Gross (1972) and Carlen–Lieb (1993).

The fermionic complex-wave representation was developed by Shale–Stinespring (1964).

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