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AN IMPROVEMENT OF A TRANSCENDENCE MEASURE OF GALOCHKIN AND MAHLER'S S-NUMBERS

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Abstract

We give a transcendence measure of special values of functions satisfying certain functional equations. This improves an earlier result of Galochkin, and gives a better upper bound of the type for such a number as an S-number in the classification of transcendental numbers by Mahler.

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1. Introduction

Let K be an algebraic number field of finite degree. Let $f(z)$ be a function which is transcendental over $\mathbb{C}(z)$ and holomorphic in some neighborhood U of the origin, and satisfies the functional equation

$$f(Tz) = \frac{A_1(z, f(z))}{A_2(z, f(z))}, \quad Tz = z^r (r \in \mathbb{N}, r \geq 2), \tag{1.1}$$

where $A_i(z, y) = a_{i1}(z)y + a_{i2}(z) \in K[z, y] (i = 1, 2)$. Suppose that the coefficients of $f(z)$ in its Taylor series expansion at the origin all lie in the field K.

Let $\alpha \in U$ be an algebraic number with $0 < |\alpha| < 1$ satisfying

$$\det(a_{ij}(T^k \alpha))_{i,j=1,2} \neq 0 \tag{1.2}$$

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for any k $(k=0, 1, 2, \dots)$. This condition allows us that $A_2(T^k\alpha, f(T^k\alpha)) \neq 0$ for any $k(k = 0, 1, 2, \dots)$.

In the notation as above, Mahler proved in [4] that the number $f(\alpha)$ is transcendental. In [2], Galochkin considered a quantitative version of this result and gave the following transcendence measure of $f(\alpha)$:

THEOREM (Galochkin [2]). *In the notation as above, further, let* $P(x) \in \mathbb{Z}[z]$ *be any nonzero polynomial whose degree is at most* d *and whose height is at most* H. *Put*

$$b = \chi_1^{-1} \log L(\alpha), \ c = \log|\alpha|^{-1} \ \textit{and} \ \chi_0 = [K(\alpha) : \mathbb{Q}],$$

where χ_1 *is the degree of* α *and* $L(\alpha)$ *is the length of* α. *Then we have*

$$|P(f(\alpha))| > H^{-(2r+1)^2 bc^{-1}\chi_0^2 d}$$

for all sufficiently large H.

Our main purpose is to sharpen this estimate. To state our results, we recall usual notation and the definition of Mahler's S-numbers (cf. Schneider [6]).

For any algebraic number α with minimal defining polynomial $Q(x) = a_0(x-\alpha)(x-\alpha')\cdots(x-\alpha^{(\chi-1)}) \in \mathbb{Z}[x](a_0 > 0)$, we denote by $\operatorname{den}(\alpha)$ the *denominator* of α, that is, the least positive integer d such that $d\alpha$ is an algebraic integer, by $\overline{|\alpha|}$ the *house* of α, that is, the maximum of the absolute values of the roots of $Q(x)$, and by $M(\alpha)$ the *Mahler measure* of α, that is the number which is defined by

$$M(\alpha) = a_0 \prod_{i=0}^{\chi-1} \operatorname{Max}(1, |\alpha^{(i)}|), \ \alpha^{(0)} = \alpha.$$

For any polynomial P (in any number of variables) whose coefficients are algebraic numbers, we denote by $\deg_x P$ the *degree* of P *in the variable* x, by $H(P)$ the *height* of P, that is, the maximum of the houses of the coefficients of P, and by $L(P)$ the *length* of P, that is, the sum of the houses of the coefficients of P. For any algebraic number α with minimal defining polynomial Q, we put $\deg\alpha = \deg Q$, $H(\alpha) = H(Q)$ and $L(\alpha) = L(Q)$.

Now we recall the definition of Mahler's S-numbers. Let ω be any complex number. Then we define a function $w_d(\omega, h)$ by

$$w_d(\omega, h) = \operatorname{Min}\{|P(\omega)|;\ P(x) \in \mathbb{Z}[x], \deg P \le d, \ H(P) \le h \text{ and } P(\omega) \neq 0\}.$$

Further, we define $w_d(\omega)$ and $w(\omega)$ by

$$w_d(\omega) = \limsup_{h\to\infty} \frac{-\log w_d(\omega, h)}{\log h} \text{ and } w(\omega) = \limsup_{d\to\infty} \frac{w_d(\omega)}{d}.$$

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Then a number ω is transcendental if and only if $w(\omega)$ is positive. Then, according to the classification of Mahler, a transcendental number ω is called an S-number if $w(\omega)$ is finite (that is, $w_d(\omega)/d$ is bounded as a function of d). For any S-number ω, we define the *type* of ω by the supremum of the sequence $\{w_d(\omega)/d\}_{d\in\mathbb{N}}$. In this terminology, Galochkin's theorem states that the number $f(\alpha)$ is an S-number of type at most $(2r+1)^2bc^{-1}\chi_0^2$.

In the present paper, we shall prove the following theorems.

THEOREM 1. *Let K be an algebraic number field of finite degree. Let $f(z)$ be a function which is transcendental over $\mathbb{C}(z)$ and holomorphic in some neighborhood U of the origin, and satisfies the functional equation* (1.1) *with $a_{ij}(z) \in K[z]$. Suppose that the coefficients of $f(z)$ in its Taylor series expansion at the origin all lie in the field K. Let $\alpha \in U$ be an algebraic number with $0 < |\alpha| < 1$ such that* (1.2) *holds for any $k (k = 0, 1, 2, \ldots)$. Put*

$$b = \chi_1^{-1}\log M(\alpha),\ c = \log|\alpha|^{-1} \text{ and } \chi_0 = [K(\alpha):\mathbb{Q}], \tag{1.3}$$

where χ_1 is the degree of α and $M(\alpha)$ is the Mahler measure of α. Then, for any positive integer d, we have

$$w_d(f(\alpha)) \le \{r(1+1/\sqrt{r})^2bc^{-1}\chi_0^2+1\}d-1. \tag{1.4}$$

In particular, the number $f(\alpha)$ is an S-number of type at most

$$r(1+1/\sqrt{r})^2bc^{-1}\chi_0^2+1.$$

COROLLARY. *In the above theorem, suppose $K = \mathbb{Q}$ and $\alpha = 1/a$ $(a \in \mathbb{Z},\ |a| \ge 2)$. Then, for any positive integer d, we have*

$$w_d(f(\alpha)) \le \{r(1+1/\sqrt{r})^2+1\}d-1.$$

In particular, the number $f(\alpha)$ is an S-number of type at most $r(1+1/\sqrt{r})^2+1$.

By specializing our situation, we can also give good lower bounds of the values $w_d(f(\alpha))$ for small d. Namely,we can prove the following theorem.

THEOREM 2. *Let $F_r(z)$ be the function defined by*

$$F_r(z) = \sum_{v=0}^{\infty} z^{r^v} \qquad (r \in \mathbb{Z},\ r \ge 2),$$

and a be an integer with $|a| \ge 2$. Put

$$d_0 = \begin{cases} (r-2)/2 & \text{if } r \text{ is even}, \\ (r-1)/2 & \text{if } r \text{ is odd}. \end{cases} \tag{1.5}$$

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Then we have

$$w_d(F_r(1/a)) = r - 1 \text{ for } d = 1, \ldots, d_0, \tag{1.6}$$

$$r - 1 \le w_d(F_r(1/a)) \le \frac{rd}{r-d} \text{ for } d = d_0 + 1, \ldots, r - 1, \tag{1.7}$$

and

$$w_d(F_r(1/a)) \le \{r(1 + 1/\sqrt{r})^2 + 1\}d - 1 \text{ for } d \ge r. \tag{1.8}$$

In particular, the number $F_r(1/a)$ *is an* S*-number of type at least* $r - 1$ *and at most* $r(1 + 1/\sqrt{r})^2 + 1$.

REMARK. In the above theorem, we have the equality $w_1(F_r(1/a)) = r - 1$ for any $r \ge 3$. But according to a theorem of Shallit [7], the number $F_r(1/a)$ has the continued fraction expansion with bounded partial quotients, and hence we have also the equality $w_1(F_2(1/a)) = 1$. We note that the above mentioned equality $w_1(F_r(1/a)) = r - 1$ for any $r \ge 3$ is also deduced from a theorem of Shallit [7].

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2. Preliminaries

In this section, we give two estimates for $w_d(\omega)$ (Lemmas 2 and 3 below). The following lemma is Lemma 5 of Galochkin [2] (cf. also Güting [3, Theorem 6]).

LEMMA 1. *Let* $\alpha_1, \ldots, \alpha_s$ *be algebraic numbers of degrees* $\chi_1, \ldots, \chi_s$. *Let* K *be an algebraic number field, and* I_K *be its integer ring. Put* $\chi_0 = [K(\alpha_1, \ldots, \alpha_s): \mathbb{Q}]$. *Let* $A(x_1, \ldots, x_s) \in I_K[x_1, \ldots, x_s]$ *be a polynomial of* $\deg_{x_i} A \le d_i$ *for each* i. *If* $A(\alpha_1, \ldots, \alpha_s) \ne 0$, *then we have*

$$|A(\alpha_1, \ldots, \alpha_s)| \ge L(A)^{1-\chi_0} \prod_{i=1}^{s} M(\alpha_i)^{-(\chi_0/\chi_i)d_i}.$$

REMARK. Checking the proof of Theorem 6 of Güting [3], it is found that we may use $M(\alpha_i)$ in the above inequality instead of $L(\alpha_i)$ which is used by Galochkin [2, Lemma 5] (and also used by Güting [3, Theorem 6]). Note that we have the inequality $M(\alpha_i) \le L(\alpha_i)$ because of an inequality of Mahler [5].

The following lemma follows from the arguments of Galochkin [2].

LEMMA 2. *Let ω be a complex number, and α be an algebraic number of $\deg\alpha = \chi_1$ with $0 < |\alpha| < 1$. Let $\varphi(k)$ be a function on $\mathbb{N}$ such that, for sufficiently large $k \in \mathbb{N}$, $\varphi(k)$ is a strictly increasing positive valued function tending to infinity, and such that there exists a positive number δ satisfying*

$$\limsup_{k\to\infty} \frac{\varphi(k+1)}{\varphi(k)} = \delta < \infty. \tag{2.1}$$

Let K be an algebraic number field, and denote by I_K its integer ring. Put $\chi_0 = [K(\alpha): \mathbb{Q}]$. Let d be a positive integer, and $E > 1$ be a real number satisfying

$$\log E > b\chi_0 d \qquad (b = \chi_1^{-1} \log M(\alpha)). \tag{2.2}$$

Suppose that there exists a sequence of polynomials $\{R_k(z, y)\}_{k\in\mathbb{N}}$ such that $R_k(z, y) \in I_K[z, y]$ and $\deg_y R_k \leq m$ for any k with a certain positive integer m, and such that $R_k(z, y)$ satisfies

$$\log L(R_k) = o(\varphi(k)), \deg_z(R_k) \leq \varphi(k)(1 + o(1)), \tag{2.3}$$
$$|R_k(\alpha, \omega)| = E^{-\varphi(k)(1+o(1))}$$

as $k \to \infty$. Then we have

$$w_d(\omega) \leq \frac{\delta m \chi_0 \log E}{\log(E/M(\alpha)^{\chi_0 d/\chi_1})} + d - 1. \tag{2.4}$$

PROOF. It is convenient for our purpose to work with Koksma's function w_d^* instead of Mahler's function w_d. Here we recall the definition (cf. Schneider [6]). For a complex number ω, we define a function $w_d^*(\omega, h)$ by

$$w_d^*(\omega, h) = \mathrm{Min}\{|\omega - \beta|;\ \beta \in \overline{\mathbb{Q}}, \deg\beta \leq d, H(\beta) \leq h \text{ and } \omega \neq \beta\},$$

where $\overline{\mathbb{Q}}$ is the field of all algebraic numbers. Then we define $w_d^*(\omega)$ by

$$w_d^*(\omega) = \limsup_{h\to\infty} \frac{-\log(h w_d^*(\omega, h))}{\log h}.$$

In what follows, we shall prove

$$w_d^*(\omega) \leq \frac{\delta m \chi_0 \log E}{\log(E/M(\alpha)^{\chi_0 d/\chi_1})}, \tag{2.5}$$

where ω is a complex number which satisfies all the conditions in the lemma. Since we have $w_d(\omega) \leq w_d^*(\omega) + d - 1$ (cf. Schneider [6, Hilfssatz 19]), this proves the lemma.

Let β be any algebraic number with $\deg\beta \leq d$ and $H(\beta) \leq h$. Put $\Delta = |\omega - \beta|$. We must give a good lower bound for Δ which leads to (2.5).

We may assume $\Delta \leq 1$ without loss of generality. Let ε be any (small) positive number. Put

$$R = E^{1-2\varepsilon} M(\alpha)^{-(\chi_0 d/\chi_1)(1+\varepsilon)}.$$

We may assume that $R > 1$ because of (2.2). Choose a positive integer k such that

$$R^{\varphi(k-1)/\chi_0 m} \leq h < R^{\varphi(k)/\chi_0 m}. \tag{2.6}$$

By taking a sufficiently large h as an upper bound for $H(\beta)$, we may assume that k is also sufficiently large. We claim that

$$\Delta \geq E^{-\varphi(k)(1+2\varepsilon)}. \tag{2.7}$$

Indeed, if (2.7) is false, then by (2.3), we have

$$\begin{aligned} |R_k(\alpha, \beta)| &\geq |R_k(\alpha, \omega)| - |R_k(\alpha, \omega) - R_k(\alpha, \beta)| \\ &\geq E^{-\varphi(k)(1+\varepsilon)} - L(R_k) m(|\omega| + 1)^m \Delta \\ &\geq E^{-\varphi(k)(1+\varepsilon)} - E^{-\varphi(k)(1+3\varepsilon/2)} > 0. \end{aligned}$$

Then, by Lemma 1 and (2.3), we have

$$\begin{aligned} |R_k(\alpha, \omega)| &\geq |R_k(\alpha, \beta)| - |R_k(\alpha, \omega) - R_k(\alpha, \beta)| \\ &\geq L(R_k)^{1-\chi_0 d} M(\alpha)^{-(\chi_0 d/\chi_1)\varphi(k)(1+\varepsilon/2)} M(\beta)^{-\chi_0 m} - E^{-\varphi(k)(1+3\varepsilon/2)} \\ &\geq M(\alpha)^{-(\chi_0 d/\chi_1)\varphi(k)(1+\varepsilon)} h^{-\chi_0 m} - E^{-\varphi(k)(1+3\varepsilon/2)}. \end{aligned}$$

Comparing this lower bound with an upper bound

$$|R_k(\alpha, \omega)| \leq E^{-\varphi(k)(1-\varepsilon)},$$

we conclude

$$h \geq (E^{1-2\varepsilon} M(\alpha)^{-(\chi_0 d/\chi_1)(1+\varepsilon)})^{\varphi(k)/\chi_0 m} = R^{\varphi(k)/\chi_0 m}.$$

Since this inequality contradicts (2.6), our claim is proved.

Now, by (2.1) and (2.6), we have

$$\varphi(k) \leq \delta(1+\varepsilon)\varphi(k-1) \leq \frac{(1+\varepsilon)\delta\chi_0 m \log h}{\log R}.$$

Hence, by (2.7), we obtain

$$\Delta \geq h^{-(1+4\varepsilon)\delta\chi_0 m(\log E)/\log(E^{1-2\varepsilon} M(\alpha)^{-(\chi_0 d/\chi_1)(1+\varepsilon)})}.$$

Since we can take ε arbitrarily small, this leads (2.5). The lemma is proved.

We need the following lemma to prove Theorem 2.

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LEMMA 3. *Let ω be a real number, $Q > 1$ be a real number, and $r \geq 2$ be an integer. Put $E = Q^{r-1}$. Suppose that there exists a sequence of rational numbers $\{p_k/q_k\}_{k\in\mathbb{N}}$ such that p_k and $q_k > 0$ are relatively prime integers, and satisfy*

$$q_k = Q^{r^k(1+o(1))} \quad \text{and} \quad |q_k\omega - p_k| = E^{-r^k(1+o(1))}$$

as $k \to \infty$. Let d_0 be the number defined by (1.5) *in Theorem* 2. *Then we have*

$$w_d(\omega) = r - 1 \text{ for } d = 1, \dots, d_0, \tag{2.8}$$

and

$$r - 1 \leq w_d(\omega) \leq \frac{rd}{r-d} \text{ for } d = d_0 + 1, \dots, r-1. \tag{2.9}$$

This is a special case of Lemma 1 of Amou [1].

3. Proof of the theorems

PROOF OF THEOREM 1. Put $\omega = f(\alpha)$. Note that we may assume without loss of generality that $A_i(z, y) \in I_K[z, y] (i = 1, 2)$. Let m and n be any positive integers. By the theory of homogeneous linear equations, we can construct an auxiliary polynomial $R_0(z, y) \in I_K[z, y]$, $R_0(z, y) \neq 0$, such that

$$\deg_z R_0 \leq n, \ \deg_y R_0 \leq m \quad \text{and} \quad \operatorname{ord} R_0(z, f(z)) > (m+1)n, \tag{3.1}$$

where $\operatorname{ord} R_0(z, f(z))$ is the order of zeros of the function $R_0(z, f(z))$ at $z = 0$. Since $f(z)$ is transcendental over $\mathbb{C}(z)$, we have $R_0(z, f(z)) \neq 0$, and hence we can write $\operatorname{ord} R_0(z, f(z)) = \lambda n(m+1)$ for some $\lambda > 1$. Then, because of the functional equation (1.1) for $f(z)$, for any positive integer k, we can construct $R_k(z, y) \in I_K[z, y]$ inductively by taking

$$R_k(z, f(z)) = A_2(z, f(z))^m R_{k-1}(Tz, f(Tz)).$$

We can easily show that

$$\deg_z R_k \leq e(k) := [nr^k(1 + \varepsilon(m, n))] \text{ and } \deg_y R_k \leq m, \tag{3.2}$$

where $\varepsilon(m, n)$ is a positive valued function of $m, n \in \mathbb{N}$ satisfying $\varepsilon(m, n) \to 0$ as $m/n \to 0$. Further, by Lemma 3 of Galochkin [2], we have

$$L(R_k) \leq (2L)^{mk} L(R_0) \quad \text{and} \quad |R_k(\alpha, \omega)| = e^{-c\lambda n(m+1)r^k(1+o(1))} \tag{3.3}$$

as $k \to \infty$, where $L = \operatorname{Max}\{L(a_{ij}(z)); i, j = 1, 2\}$ and $c = \log|\alpha|^{-1}$.

We fix the following notation. Let $S(m, n)$ be the set of all polynomials $R_0(z, y) \in I_K[z, y]$, $R_0(z, y) \neq 0$, satisfying (3.1). Put

$$\lambda(m, n) = \sup\left\{\frac{1}{(m+1)n}\operatorname{ord} R_0(z, f(z));\ R_0(z, y) \in S(m, n)\right\}.$$

Let $\lambda(m)$ be the number defined by

$$\lambda(m) = \limsup_{n\to\infty} \lambda(m, n).$$

This number plays an essential role in our proof.

Put

$$m = [bc^{-1}\chi_0 d(1+\tau)], \qquad \tau = 1/\sqrt{r},$$

where b, c and χ_0 are the numbers defined by (1.3). In the following argument, we consider two cases.

CASE I. $\lambda(m) > \sqrt{r} = \tau^{-1}$. Let ε be any (small) positive number. In this case, there are infinitely many n satisfying $\lambda(m, n) > \tau^{-1}$. We take and fix such an n with $1+\varepsilon(m, n) \leq \{\tau(\tau^{-1}-\varepsilon)\}^{-1}$, where $\varepsilon(m, n)$ is the quantity in (3.2). Then we have a sequence of polynomials $R_k(z, y) \in I_K[z, y]$ for $k \in \mathbb{N}$ satisfying (3.1), (3.2) and (3.3) with $\lambda > \tau^{-1}$. Put

$$E = e^{c(\tau^{-1}-\varepsilon)(m+1)} \text{ and } \varphi(k) = \lambda(\tau^{-1}-\varepsilon)^{-1} n r^k$$

for $k \in \mathbb{N}$. Because of our choice of m, we may assume that $\log E > b\chi_0 d$ by taking ε small enough. Then, all of the conditions in Lemma 2 are satisfied. Put $\gamma = bc^{-1}\chi_0$. Since we can take ε arbitrarily small, applying Lemma 2 to this situation and letting $\varepsilon \to 0$, we obtain from (2.4) and from our choice of m that

$$\begin{aligned} w_d(\omega) - d + 1 &\leq \frac{rm\chi_0 c\tau^{-1}(m+1)}{\{c\tau^{-1}(m+1) - b\chi_0 d\}} \leq \frac{r(1+\tau)\gamma\chi_0 d}{1 - \frac{\gamma d}{\tau^{-1}(m+1)}} \\ &\leq \frac{r(1+\tau)\gamma\chi_0 d}{1 - \frac{\gamma d}{\tau^{-1}\gamma d(1+\tau)}} = r(1+\tau)^2\gamma\chi_0 d = r(1+1/\sqrt{r})^2 bc^{-1}\chi_0^2 d. \end{aligned}$$

CASE II. $\lambda(m) \leq \tau^{-1}$. Let ε be any (small) positive number. We shall construct a finite sequence of positive integers $\{n_i\}_{1\leq i\leq t}$ which satisfies suitable conditions. First we take a positive integer n_1 with $n_1 \geq \tau/\varepsilon$ such that, for any $n \in \mathbb{N}$ with $n \geq n_1$, we have

$$1 + \varepsilon(m, n) \leq (1-\varepsilon)^{-1} \text{ and } \lambda(m, n) \leq \tau^{-1}(1+2\varepsilon)/(1+\varepsilon).$$

Next we take the least positive integer satisfying $\lambda(m, n_1)n_1(1+\varepsilon) \leq n_2$. Then we have

$$1 + \varepsilon \leq \frac{\lambda(m, n_2)n_2}{\lambda(m, n_1)n_1} \leq \tau^{-1}(1+3\varepsilon).$$

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Further we can take positive integers $n_3, n_4, \ldots$ such that n_{i+1} is the least positive integer satisfying $\lambda(m, n_i)n_i(1+\varepsilon) \leq n_{i+1} (i = 2, 3, \ldots)$. Thus we obtain a sequence of positive integers $\{n_i\}_{i\in\mathbb{N}}$ satisfying

$$1+\varepsilon \leq \frac{\lambda(m, n_{i+1})n_{i+1}}{\lambda(m, n_i)n_i} \leq \tau^{-1}(1+3\varepsilon)$$

for any $i \in \mathbb{N}$. Let t be the least positive integer satisfying

$$\frac{\lambda(m, n_1)n_1 r}{\lambda(m, n_t)n_t} \leq \tau^{-1}(1+3\varepsilon).$$

Note that the left-hand side of the above inequality is greater than 1. We now have a finite sequence $\{n_i\}_{1\leq i\leq t}$ which can be used below to define a sequence of polynomials $R_k(z, y) \in I_K[z, y]$ $(k \in \mathbb{N})$ and a function $\varphi(k)$ of $k \in \mathbb{N}$.

For any $i \in \mathbb{N}$ with $1 \leq i \leq t$, we take a polynomial $R_{i,0}(z, y) \in I_K[z, y]$ such that

$$\deg_z R_{i,0} \leq n_i, \ \deg_y R_{i,0} \leq m$$

and

$$\operatorname{ord} R_{i,0}(z, f(z)) = \lambda(m, n_i)n_i(m+1).$$

Then, for any positive integer j, we can construct $R_{i,j}(z, y) \in I_K[z, y]$ inductively by taking

$$R_{i,j}(z, f(z)) = A_2(z, f(z))^m R_{i,j-1}(Tz, f(Tz)).$$

Let us write any $k \in \mathbb{N}$ as $k = j(k)t + i(k)$ where $i(k)$, $j(k)$ are integers with $0 \leq i(k) < t$. In this notation, for any $k \in \mathbb{N}$, we define $R_k(z, y)$ and $\varphi(k)$ by

$$R_k(z, y) = \begin{cases} R_{i(k),j(k)}(z, y) & \text{if } i(k) \neq 0, \\ R_{t,j(k)-1}(z, y) & \text{if } i(k) = 0, \end{cases}$$

and by

$$\varphi(k) = \begin{cases} \lambda(m, n_{i(k)})n_{i(k)} r^{j(k)}(1-\varepsilon)^{-1} & \text{if } i(k) \neq 0, \\ \lambda(m, n_t)n_t r^{j(k)-1}(1-\varepsilon)^{-1} & \text{if } i(k) = 0. \end{cases}$$

Put $E = e^{c(m+1)(1-\varepsilon)}$. As in Case I, we may assume that $\log E > b\chi_0 d$ by taking ε small enough. Then $R_k(z, y)(k \in \mathbb{N})$, $\varphi(k)$ and E satisfy the conditions (2.1) with $\delta \leq \tau^{-1}(1+3\varepsilon)$, (2.2) and (2.3) in Lemma 2. Since we can take ε arbitrarily small, applying Lemma 2 to this situation and letting

$\varepsilon \to 0$, we obtain from (2.4) and from our choice of m that

$$\begin{aligned} w_d(\omega) - d + 1 &\le \frac{\tau^{-1} m\chi_0 c(m+1)}{c(m+1) - b\chi_0 d} \le \frac{\tau^{-1}(1+\tau)\gamma\chi_0 d}{1 - \frac{\gamma d}{m+1}} \\ &\le \frac{\tau^{-1}(1+\tau)\gamma\chi_0 d}{1 - \frac{1}{1+\tau}} \\ &= \tau^{-2}(1+\tau)^2\gamma\chi_0 d = r(1 + 1/\sqrt{r})^2 bc^{-1}\chi_0^2 d. \end{aligned}$$

In any case, we obtain $w_d(\omega) \le \{r(1 + 1/\sqrt{r})^2 bc^{-1}\chi_0^2 + 1\}d - 1$. This is (1.4), and we have proved the theorem.

If $K = \mathbb{Q}$ and $\alpha = 1/a (a \in \mathbb{Z}, |a| \ge 2)$, then we have $b = c$ and $\chi_0 = 1$, and hence the corollary follows.

Remark. We can easily show as a corollary of the above proof that, for any $\varepsilon > 0$, the inequality

$$w_d(f(\alpha)) \le \{(4+\varepsilon)\sqrt{r} bc^{-1}\chi_0^2 + 1\}d - 1$$

holds for infinitely many d.

Proof of Theorem 2. Since the function $f_r(z)$ satisfies the functional equation $f_r(z^r) = f_r(z) - z$, by the corollary of Theorem 1, we have (1.8). Now, we show (1.6) and (1.7). Put $\omega = f_r(1/a)$. For any $k \in \mathbb{N}$, we define a rational number p_k/q_k by

$$\frac{p_k}{q_k} = \sum_{v=0}^{k} a^{-r^v}, \quad q_k = a^{r^k}.$$

Then p_k and $q_k > 0$ are relatively prime integers, and satisfy

$$|q_k\omega - p_k| = a^{-(r-1)r^k(1+o(1))}$$

as $k \to \infty$. Put $Q = a$ and $E = a^{r-1}$. Then, by applying Lemma 3, we deduce (1.6) and (1.7) from (2.8) and (2.9) of Lemma 3 respectively. This completes the proof of the theorem.

About the number $F_r(1/a)$, we conjecture that

$$w_d(F_r(1/a)) = r - 1 \text{ for } d = 1, \dots, r-1,$$

and

$$w_d(F_r(1/a)) = d \text{ for } d \ge r.$$

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