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Low Frequency Estimates for Long Range Perturbations in Divergence Form

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Abstract. We prove a uniform control as $z \to 0$ for the resolvent $(P-z)^{-1}$ of long range perturbations P of the Euclidean Laplacian in divergence form by combining positive commutator estimates and properties of Riesz transforms. These estimates hold in dimension $d \geq 3$ when P is defined on $\mathbb{R}^d$ and in dimension $d \geq 2$ when P is defined outside a compact obstacle with Dirichlet boundary conditions.

1 Introduction and Main Results

Consider an elliptic self-adjoint operator in divergence form on $L^2(\mathbb{R}^d)$, $d \geq 2$,

$$P = -\mathrm{div}\,(G(x)\nabla)\,, \tag{1.1}$$

where $G(x)$ is a $d \times d$ matrix with real entries satisfying, for some $\Lambda_+ \geq \Lambda_- > 0$,

$$G(x)^T = G(x), \qquad \Lambda_+ \geq G(x) \geq \Lambda_-, \qquad x \in \mathbb{R}^d. \tag{1.2}$$

Throughout the paper, we shall assume that G belongs to $C_b^\infty(\mathbb{R}^d)$ *i.e.,* that $\partial^\alpha G$ has bounded entries for all multiindices α, but this is mostly for convenience, and much weaker assumptions on the regularity of G could actually be considered. For instance, in polar coordinates $x = |x|\omega$, Theorem 1.1 will not use any regularity in the angular variable ω.

We mainly have in mind long range perturbations of the Euclidean Laplacian, namely the situation where, for some $\mu > 0$,

$$\left|\partial^\alpha\left(G(x) - I_d\right)\right| \leq C_\alpha \langle x\rangle^{-\mu-|\alpha|}, \qquad x \in \mathbb{R}^d, \tag{1.3}$$

I_d being the identity matrix and $\langle x\rangle = (1+|x|^2)^{1/2}$ the usual Japanese bracket. In this case, it is well known that the resolvent $(P-z)^{-1}$ satisfies the limiting absorption principle, *i.e.,* that the limits

$$(P - \lambda \mp i0)^{-1} := \lim_{\delta \to 0^+} (P - \lambda \mp i\delta)^{-1}$$

exist at all positive energies $\lambda > 0$ (the frequencies being $\lambda^{1/2}$) in weighted L^2 spaces (see the historical papers [1, 27], the references therein and the references below on

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quantitative bounds). Typically, for all $\lambda_2 > \lambda_1 > 0$ and all $s > 1/2$, we have bounds of the form

$$\big\| \langle x \rangle^{-s} (P - \lambda - i0)^{-1} \langle x \rangle^{-s} \big\|_{L^2 \to L^2} \leq C(s, \lambda_1, \lambda_2), \qquad \lambda \in [\lambda_1, \lambda_2], \tag{1.4}$$

and the same holds for $(P - \lambda + i0)^{-1}$ by taking the adjoint. The behaviour of the constant $C(s, \lambda_1, \lambda_2)$ is very well known as long as λ_1 does not go to 0. For a fixed energy window, the results follow essentially from the Mourre theory ([27]), since one knows that there are no embedded eigenvalues for such operators ([24]). At large energies, $\lambda_1 \sim \lambda_2 \to \infty$, $C(s, \lambda_1, \lambda_2)$ is at worst of order $e^{C\lambda_2^{1/2}}$ (see [9]) but can be taken of order $\lambda_1^{-1/2}$ if there are no trapped geodesics (*i.e.*, all geodesics escape to infinity); see [8, 18, 30, 32, 33].

Weights of the form $\langle x \rangle^{-s}$ are of interest since they give a quantitative notion of spatial localization. They are also more general and more robust than compactly supported localizations. However, we point out that the limiting absorption principle can be justified for other kinds of weights. In particular, we can use the following well known generator of dilations,

$$A = \frac{x \cdot \nabla + \nabla \cdot x}{2i} = \frac{x \cdot \nabla}{i} + \frac{d}{2i}, \tag{1.5}$$

so called because it is the self-adjoint generator of the unitary group on $L^2(\mathbb{R}^d)$ given by

$$\left(e^{itA} \varphi\right)(x) = e^{\frac{td}{2}} \varphi(e^t x). \tag{1.6}$$

We know indeed, from the Mourre theory, that the limiting absorption principle can be justified for

$$\langle A \rangle^{-s} (P - \lambda \mp i0)^{-1} \langle A \rangle^{-s}, \tag{1.7}$$

for any $s > 1/2$ ($s = 1$ in [27] and $s > 1/2$ in [29] using an idea of Mourre or, by a different method, in [17]). We note that estimates on operators of the form (1.7) are more general, to the extent that they imply those on $\langle x \rangle^{-s} (P - \lambda \mp i0)^{-1} \langle x \rangle^{-s}$ by fairly classical and simple arguments. Furthermore, the weights $\langle A \rangle^{-s}$ commute with scalings (*i.e.*, with e^{itA}), which is not the case for $\langle x \rangle^{-s}$ and which can be interesting in situations where the coefficients of P behave nicely under scaling.

In this paper, we address the problem of the behaviour of such estimates as the spectral parameter goes to 0, typically when $\lambda_1 \downarrow 0$ in (1.4). Let us recall that a quick look at the kernel of the resolvent in the flat case ($P = -\Delta$), whose kernel is given for $d = 3$ (for simplicity) by

$$K_{\mathrm{flat}}(x, y, z) = \frac{e^{iz^{1/2}|x-y|}}{4\pi |x - y|}, \qquad \mathrm{Im}(z^{1/2}) \geq 0,$$

suggests that, if one has no oscillation, *i.e.*, if $z = 0$, choosing $s > 1/2$ in (1.4) is not sufficient. One sees easily that $s > 2$ will be enough by the Schur lemma and, more sharply, that $s > 1$ will work too, using the Hardy–Littlewood–Sobolev inequality. This (natural) restriction is however essentially irrelevant for us: our point in this paper is not to get the sharpest weights (*e.g.*, work in optimal Besov spaces) but only to get a control on $w(A)(P-\lambda-i0)^{-1}w(A)^*$ and $\langle x\rangle^{-s}(P-\lambda-i0)^{-1}\langle x\rangle^{-s}$ as $\lambda \to 0$, for some $s > 0$ or some function w.

The very natural question of low frequency asymptotics for the resolvent of Schrödinger type operators has been considered in many papers. However, the situation is not as clear as for the positive energies. For perturbations of the flat Laplacian by potentials, we refer to [16, 22, 23, 25, 28, 34, 36], to the references therein, and also to the recent very detailed study [14]. In a sense, perturbations by potentials are harder to study due to the possible resonances or (accumulation of) eigenvalues at 0.

For compactly supported perturbations of the flat Laplacian by metrics and obstacles, the behaviour of the resolvent at 0 is obtained fairly quickly in [7,26] but making strong use of the compact support assumption.

In the more general case of asymptotically conical manifolds, low frequency estimates have been obtained by Christiansen [12] and Carron [10], with motivations in the study of the scattering phase near 0. Recently, Guillarmou and Hassell have investigated carefully the low energy asymptotics of Schrödinger operators on asymptotically conical manifolds ([20, 21]). Using the sophisticated pseudo-differential calculus of Melrose, they were able to describe accurately the kernel of the Green function at low energies. In particular, they derive optimal L^p bounds for the Riesz transform. This technology is also used in [11], again for the study of the range of p for which the Riez transform is L^p bounded. In a close geometric context, for very short range perturbations of exact conical metrics, Wang [35] also proves asymptotic expansion of the resolvent at low energies.

All the above papers dealing with metrics use a relatively strong decay of the perturbation at infinity or assume at least certain asymptotic expansions that, in any case, exclude most long-range perturbations.

The first message of this paper is that nothing nasty can happen for long range perturbations of the metric. More precisely, we will show that, if the perturbation is uniformly small on $\mathbb{R}^d$ (but arbitrarily long range at infinity), we have uniform bounds on the resolvent at low frequency. The second message is that, for arbitrary long range perturbations, we can use certain properties of the Riesz transform to handle the non-small compact part of the perturbation and get low energy estimates. In a sense, this is the opposite point of view to [11, 20, 21], to the extent that we use the Riesz transform to analyze the resolvent instead of using information on the resolvent to study the Riesz transform.

We think that the method described in this paper is quite simple (at least on $\mathbb{R}^d$). More importantly, we hope that it is rather flexible. For instance the analysis of the present paper could be extended to more general operators, for instance by allowing potentials decaying like $\langle x\rangle^{-2-\epsilon}$ at infinity, since the latter can be put under divergence form if we allow non-local G. We focus on the case (1.1) to avoid such technicalities. Furthermore, our method can be adapted to other geometries. To illustrate this fact, we have devoted Section 6 to the situation where P is defined on the exterior of a bounded obstacle with Dirichlet boundary conditions. In particular, for weights of the form $\langle x\rangle^{-s}$, we obtain uniform estimates on the resolvent at low energies in dimension ≥ 2, whereas in $\mathbb{R}^d$ we need to consider $d \geq 3$.

Before stating our results, we recall that, basically, the spirit of resolvent estimates (like many other results in scattering theory) is to consider that we are close to the flat Laplacian. This is true near infinity, and also to a certain extent in bounded sets, by using certain compactness arguments. We therefore start by giving our results in the case of small perturbations on $\mathbb{R}^d$. The proofs in this situation are simpler and thus more pedagogic. Furthermore, a large part of the proofs in the general case follow exactly the same scheme, and we feel that it is worth considering first globally small perturbations and then arbitrary ones.

To state the results on small perturbations, we introduce the space $S_{\rm dil}(\mathbb{R}^d)$ defined by

$$a \in S_{\rm dil}(\mathbb{R}^d) \Longleftrightarrow a \in C_b^\infty(\mathbb{R}^d) \qquad \text{and} \qquad (x \cdot \nabla)^n a \in L^\infty(\mathbb{R}^d) \text{ for all } n,$$

and the related (semi-)norms

$$\|a\|_{N,\rm dil} := \max_{n \leq N} \|(x \cdot \nabla)^n a\|_{L^\infty}.$$

For matrices $H = (b_{jk})$ with entries in $S_{\rm dil}(\mathbb{R}^d)$, we shall denote $\|H\|_{N,\rm dil}$ for $\max_{1 \leq j,k \leq d} \|b_{jk}\|_{N,\rm dil}$.

As mentioned above, the condition $a \in C_b^\infty(\mathbb{R}^d)$ is mainly for convenience, to simplify certain algebraic manipulations. For instance, it ensures that the resolvent $(P - z)^{-1}$ maps the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ into itself if $z \notin \mathbb{R}$, which is useful to compute commutators.

This space is obviously closely related to the generator of dilations (1.5).

Theorem 1.1 *Assume that $d \geq 2$. Let G be of the form $G(x) = I_d + H(x)$, with H symmetric and with real entries in $S_{\rm dil}(\mathbb{R}^d)$. Then for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for all H satisfying*

$$2G(x) - (x \cdot \nabla)H(x) \geq \varepsilon, \qquad x \in \mathbb{R}^d, \tag{1.8}$$

and all h such that $0 < h \leq C_\varepsilon^{-1}(1 + \|H\|_{4,\rm dil})^{-1}$, we have

$$\big\| |D|(hA + i)^{-1}(P - z)^{-1}(hA - i)^{-1}|D| \big\|_{L^2 \to L^2} \leq \frac{C_\varepsilon}{h}, \qquad z \in \mathbb{C} \setminus \mathbb{R}. \tag{1.9}$$

Here $|D|$ is the usual Fourier multiplier by $|\xi|$.

The main point in this theorem is the uniform control in z of the resolvent under condition (1.8), which is essentially a smallness condition because it clearly holds if $\|H\|_{1,\rm dil}$ is small enough. We note in passing that the role of (1.8) is to ensure that $i[P, A]$ is globally elliptic and thus to get positive commutator estimates without compact remainder.

The main novelty is that we get bounds for small z, say $|z| < 1$. We also obtain bounds for large z, but these are essentially well known since the condition (1.8) implies that the metric G (or rather G^{-1}) is non trapping ($x \cdot \xi$ is a global escape function - see for instance [18, 30]).

We also point out that the regularity $\|H\|_{4,\mathrm{dil}}$ is probably not sharp. We have not tried to get the optimal regularity in order to avoid technicalities in the proofs and to focus on the main simple algebraic ideas; we thus might have made some relatively crude estimates at certain steps (in particular in Proposition 4.2). One may however hope to improve the regularity condition by changing $\|H\|_{4,\mathrm{dil}}$ into $\|H\|_{2,\mathrm{dil}}$.

We finally mention that we consider weights of the form $w(hA) = (hA - i)^{-1}$ since, in the calculation of the relevant commutator (see Section 3), one needs to consider the Fourier transform of $|w(a)|^2$, that is of $(a^2+1)^{-1}$ which leads to very explicit formulas. However, in principle, the present methods would allow to consider $w_s(hA) = (1 + h^2A^2)^{s/2}$ with $s > 1/2$ and h small enough.

We now derive weighted estimates of the same form as (1.4). For $d \geq 3$, recall the standard notation for the usual conjugate Sobolev exponents

$$2_* = \frac{2d}{d+2}, \qquad 2^* = \frac{2d}{d-2}.$$

Corollary 1.2 *If* $d \geq 3$, *under the same assumptions as in Theorem* 1.1, *we have*

$$\left\| (hA+i)^{-1}(P-z)^{-1}(hA-i)^{-1} \right\|_{L^{2_*} \to L^{2^*}} \leq \tfrac{C}{h}, \qquad z \in \mathbb{C} \setminus \mathbb{R}.$$

This in turn leads to weighted estimates for long range perturbations of the Euclidean metric.

Corollary 1.3 *Let* $d \geq 3$. *If* $G = I_d + H$ *satisfies* (1.2), (1.3) *and* (1.8), *then for all* $\epsilon > 0$, (1.10) *holds for all* $z \in \mathbb{C} \setminus \mathbb{R}$.

Note that the difference between this corollary and Theorem 1.4 below is that the estimates hold for $z \in \mathbb{C} \setminus \mathbb{R}$. The latter is natural since the assumption (1.8) implies the non-trapping condition, which gives the uniform control at high energies.

It is also worth noticing that the assumptions of Theorem 1.1 and the scale invariant space $S_{\mathrm{dil}}(\mathbb{R}^d)$ are very close to the context of [31] where the time dependent Schrödinger equation is studied. Among other dispersive estimates, Tataru proves in [31] L^2-space-time bounds, usually referred to as global smoothing effect, for small long range perturbations of the Euclidean metric, possibly time dependent, by using also positive commutator techniques. In the time independent case, our (weighted) resolvent estimates (1.10) combined with the usual ones at high energy also imply this smoothing effect. From the point of view of space-time bounds, the results of [31] are stronger, since they allow time dependent metrics. But on the other hand, in the time independent case, our resolvent estimates (which are L^∞_{loc} in term of the spectral parameter z) are stronger than L^2-space-time bounds on the evolution group.

We next state the results on $\mathbb{R}^d$ for general long range perturbations of the metric.

Theorem 1.4 *Let* $d \geq 3$. *Assume that* G *satisfies* (1.2) *and* (1.3). *Then for some* $h > 0$ *small enough and* $\lambda_0 > 0$ *small enough we have*

$$\left\| |D|(hA+i)^{-1}(P-z)^{-1}(hA-i)^{-1}|D| \right\|_{L^2 \to L^2} \lesssim 1, \qquad |\mathrm{Re}(z)| < \lambda_0,\ z \notin \mathbb{R}.$$

Furthermore, for all $\epsilon > 0$,

$$\| \langle x \rangle^{-2-\epsilon} (P - z)^{-1} \langle x \rangle^{-2-\epsilon} \|_{L^2 \to L^2} \leq C_{\epsilon,G}, \qquad |z| < 1, \ z \notin \mathbb{R}. \tag{1.10}$$

In Section 6, a similar theorem is obtained in the exterior of a compact obstacle. One may notice that, since we use Dirichlet boundary conditions, it holds in dimension $d \geq 2$.

Very recently, after a first version of this paper was posted, Bony-Häfner obtained results similar to (1.10) for $P^{1/2}$, which can be adapted to derive low frequency estimates for P as well [4, 5]. Their results give estimates with weights of the form $\langle x \rangle^{-s}$, $s > 1$. However their method does not clearly allow uniform bounds with weights of the form $w(A)$ nor the treatment of obstacles. Furthermore, it holds only in dimension ≥ 3.

Our estimates rely on a very simple observation. To state it and for further use in this paper, we give the following definition.

Definition 1.5 A differential operator B is of *div-grad* type if it is of the form

$$B = \sum_{j,k=1}^{d} D_j \big(b_{jk}(x) D_k \big) ,$$

with coefficients such that $b_{jk} \in S_{\text{dil}}(\mathbb{R}^d)$. As usual, we have set $D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$.

The first ingredient of the proof of Theorem 1.1 is the following trivial remark.

Lemma 1.6 *If B is of div-grad type then $[A, B]$ is of div-grad type. More precisely, if*

$$B = \sum D_j (b_{jk}(x) D_k),$$

then

$$i[B, A] = \sum_{jk} D_j \big(2b_{jk}(x) - (x \cdot \nabla b_{jk})(x) \big) D_k.$$

We omit the proof, which follows from an elementary computation (see also (2.5) below). Note that the formal computations are justified by the assumption that the coefficients b_{jk} are smooth.

The second ingredient is the Mourre theory (see for instance [27]). Basically, the Mourre theory allows us to derive a priori bounds on the solutions to $(P - z)u = f$, (or more general Schrödinger operators), by exploiting a positive commutator estimate of the form $\chi(P) i[P, A] \chi(P) \geq c\chi^2(P)$, with $c > 0$ and $\chi \in C_0^\infty(\mathbb{R})$ real valued and equal to 1 in a neighborhood of $\text{Re}(z)$. For operators of div-grad type as in this paper, such estimates hold only if χ is supported in $\mathbb{R}^+$, *i.e.,* away from the 0 threshold. This is due to the fact that $i[P, A]$ is close to $2P$ (at least for globally small perturbations or near infinity), so one can essentially bound from below the (spectrally localized) commutator by $2\chi(P)P\chi(P)$. The latter is only positive definite (on the range of $\chi(P)$) if χ is supported in $\mathbb{R}^+$, and one then has $\|P^{1/2}\chi(P)v\|_{L^2} \approx$

$\|\chi(P)v\|_{L^2}$ by the spectral theorem. If 0 belongs to the support of χ, we lose this equivalence. Rather than getting lower bounds by L^2 norms, we shall use the weaker observation that (in the simple case of small perturbations)

$$\big(i[P,A]v,v\big) \geq \|\nabla v\|_{L^2}^2 \gtrsim \|v\|_{L^{2*}}^2$$

by the homogeneous Sobolev embedding

$$\|v\|_{L^{2*}} \leq C\big\| |D|v\big\|_{L^2}. \tag{1.11}$$

In other words, we keep the $P^{1/2}$ factor to bound $2(\chi(P)P\chi(P)v,v)$ from below by $\|P^{1/2}\chi(P)v\|^2$. By combining this remark with techniques due basically to Mourre, we shall derive (weighted) $L^{2_*} \to L^{2^*}$ bounds for the resolvent of P.

2 Properties of the Generator of Dilations

In this section we collect some elementary formulas for the generator of dilations (1.5) and its resolvent. For further purposes, it will be convenient to consider its semiclassical version, *i.e.*, hA with $0 < h < 1$. All the properties will follow from the usual formula

$$(hA-z)^{-1} = \frac{1}{i}\int_0^{\pm\infty} e^{-itz}e^{ithA}dt, \qquad \pm\mathrm{Im}(z) < 0 \tag{2.1}$$

combined with the explicit form of the unitary group (1.6).

Observe first that, since

$$\|e^{ithA}\varphi\|_{L^p} = e^{ht\left(\frac{d}{2}-\frac{d}{p}\right)}\|\varphi\|_{L^p}$$

for $p \in [1,\infty]$ and, for instance, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the formula (2.1) implies that

$$\|(hA-z)^{-1}\varphi\|_{L^p} \leq \frac{1}{|\mathrm{Im}(z)| - h|\frac{d}{2}-\frac{d}{p}|}\|\varphi\|_{L^p},$$

provided that $|\mathrm{Im}(z)| > h|\frac{d}{2}-\frac{d}{p}|$. For the applications in this paper, this will always be the case since z will be close to $\pm i$, and h will be small.

Next, if ρ is a measurable function of polynomial growth, one readily checks that

$$e^{ithA}\rho(D)e^{-ithA} = \rho(e^{-th}D), \qquad e^{ithA}\rho(x)e^{-ithA} = \rho(e^{ht}x). \tag{2.2}$$

Also, if ρ is C^1 with gradient of polynomial growth, we have

$$i\big[\rho(D),A\big] = (\xi\cdot\nabla_\xi\rho)(D), \qquad i\big[\rho(x),A\big] = -(x\cdot\nabla_x\rho)(x).$$

In the special case where $\rho = \rho_s$ is homogeneous of real degree $s \geq 0$, we have

$$e^{ithA}\rho_s(D)e^{-ithA} = e^{-sth}\rho_s(D), \tag{2.3}$$

from which one easily deduces that

$$(2.4)\qquad (hA-z)^{-1}\rho_s(D)=\rho_s(D)(hA-z+ihs)^{-1},\qquad |\mathrm{Im}(z)|>hs$$

using (2.1).

Finally, we consider the action on differential operators. If $B=\sum_{jk}D_j(b_{jk}(x)D_k)$ is of div-grad type, (2.2) and (2.3) readily imply that

$$(2.5)\qquad e^{ithA}Be^{-ithA}=e^{-2ht}\sum_{jk}D_j\big(b_{jk}(e^{ht}x)D_k\big).$$

Operators of this form will be of great importance in this paper. Let us record the following simple property.

Proposition 2.1 *Let $b\in S_{\mathrm{dil}}(\mathbb{R}^d)$ and set $b_{(\tau)}(x)=b(e^\tau x)$, i.e., $b_{(\tau)}=e^{i\tau A}be^{-i\tau A}$ as multiplication operators. Then for all $k,n\in\mathbb{N}$,*

$$\partial_\tau^k(x\cdot\nabla)^n\big(b_{(\tau)}\big)=\big((x\cdot\nabla)^{k+n}b\big)_{(\tau)}.$$

In particular, for all N, $\|b\|_{N,\mathrm{dil}}=\|b_{(\tau)}\|_{N,\mathrm{dil}}$.

The proof is a straightforward calculation that we omit.

For further purposes, it will be convenient to use the following definition.

Definition 2.2 (Admissible operators) Let $m\in\mathbb{N}$. We say that a family $(b_\tau)_{\tau\in\mathbb{R}}$ is m-admissible in $S_{\mathrm{dil}}(\mathbb{R}^d)$ if, for all integers k,n

$$\|\partial_\tau^k(x\cdot\nabla)^n b_\tau\|_{L^\infty}\le C_{kn}e^{m|\tau|}.$$

A family of differential operators $(B_\tau)_{\tau\in\mathbb{R}}$ is m-admissible if

$$B_\tau=\sum_{j,k=1}^d D_j\big(b_{jk,\tau}(x)D_k\big),$$

with $(b_{jk,\tau})_{\tau\in\mathbb{R}}$ m-admissible families in $S_{\mathrm{dil}}(\mathbb{R}^d)$.

Example With the notation of Proposition 2.1, $b_\tau^\pm:=e^{\pm2\tau}b_{(\tau)}$ are two 2-admissible families in $S_{\mathrm{dil}}(\mathbb{R}^d)$.

Proposition 2.3 *Let $(B_\tau)_{\tau\in\mathbb{R}}$ be an m-admissible family of differential operators. Then if $w\colon[0,1]\to\mathbb{C}$ is continuous, the operators*

$$\frac{d}{d\tau}B_\tau,\qquad e^{i\tau A}B_\tau e^{-i\tau A},\qquad \text{and}\qquad \int_0^1 w(s)B_{s\tau}ds$$

are respectively m, $m+2$, and m-admissible.

In this proposition, the derivative $\frac{d}{d\tau}$ (resp. integration) means that one considers the operator with coefficients differentiated (resp. integrated) with respect to τ.

Proof The case of $(d/d\tau)B_\tau$ is obvious. For the second operator, the result follows from (2.5) (with $th=\tau$), and the fact that m-admissible coefficients are stable by conjugation by $e^{i\tau A}$ (due to Proposition 2.1). The last case is simply a consequence of the fact that $\int_0^1|w(s)|s^ke^{m|s\tau|}ds\lesssim e^{m|\tau|}$, for all non negative integer k. ■

3 A Representation Formula for the Commutator

As indicated in the introduction, we shall use the commutator techniques of Mourre to get lower bounds. It will be convenient to use the recent energy estimates approach proposed by Gérard [17]. The purpose of this section is to compute the relevant commutator relatively explicitly.

In the sequel we denote by F the bounded function $F(\lambda) = \arctan(\lambda)$, $\lambda \in \mathbb{R}$, whose final interest will be that it is positive (or negative) up to an additive constant and has a positive derivative.

We also introduce

$$P_\tau = e^{-i\tau A} i[P, A] e^{i\tau A}, \tag{3.1}$$

and standardly denote

$$\big(i[P, F(hA)]u_1, u_2\big) = \big(iF(hA)u_1, Pu_2\big) - \big(iPu_1, F(hA)u_2\big). \tag{3.2}$$

The purpose of this section is to prove a representation formula for this commutator. Rather than using the Helffer–Sjöstrand formula as in [19], we use a functional calculus based on Fourier transform that is more convenient, since we have an explicit formula for the unitary group e^{itA}.

Proposition 3.1 *For all $u_1, u_2 \in \mathcal{S}(\mathbb{R}^d)$ and all $0 < h < 1$, we have*

$$\big(i[P, F(hA)]u_1, u_2\big) = \frac{h}{2}\int_{\mathbb{R}} e^{-|t|}\Big(\frac{1}{t}\int_0^t (e^{ithA}P_{sh}u_1, u_2)ds\Big)\, dt. \tag{3.3}$$

In the spirit of [17], we use a semiclassical parameter h thanks to which the derivation of a positive estimate will be fairly transparent.

The rest of the section is devoted to the proof of this proposition. Recall first that

$$\arctan(\lambda) = \int_0^{+\infty} \frac{\sin(t\lambda)}{t} e^{-t} dt,$$

which we are going to approximate by

$$F_\nu(\lambda) = \int_0^{+\infty} \sin(t\lambda)\frac{t}{t^2+\nu^2} e^{-t} dt = \frac{1}{2i}\int_{\mathbb{R}} e^{it\lambda}\frac{t}{t^2+\nu^2} e^{-|t|} dt,$$

with $\nu > 0$. For future reference, we record the following lemma.

Lemma 3.2 *There exists $C > 0$ such that $|F_\nu(\lambda)| \le C|\lambda|$, $\nu > 0$, $\lambda \in \mathbb{R}$. Furthermore, for all $\lambda \in \mathbb{R}$, $F_\nu(\lambda) \to F(\lambda)$, $\nu \to 0$.*

We omit the very simple proof.

Lemma 3.3 *For all $v, w \in L^2(\mathbb{R}^d)$, all $\nu > 0$, and all $h > 0$, we have*

$$\big(F_\nu(hA)v, w\big) = \frac{i}{2}\int_{\mathbb{R}} \frac{te^{-|t|}}{t^2+\nu^2}(e^{ithA}v, w)dt. \tag{3.4}$$

Proof If $(E_\lambda^{hA})_{\lambda\in\mathbb{R}}$ denotes the spectral resolution of hA, we have by definition

$$\big(F_\nu(hA)v, w\big) = \int_{\mathbb{R}} F_\nu(\lambda)d(E_\lambda^{hA}v, w),$$

and then by Parseval's identity

$$\big(F_\nu(hA)v, w\big) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{F}_\nu(t)(e^{-ithA}v, w)dt, = \frac{i}{2}\int_{\mathbb{R}} \frac{te^{-|t|}}{t^2+\nu^2}(e^{ithA}v, w)dt.$$

This identity can be justified by a standard density argument, assuming first that v and w are spectrally localized (*i.e.,* of the form $\chi(A)v$, $\chi(A)w$ with $\chi \in C_0^\infty$) and approximating (for fixed ν) F_ν by Schwartz functions by adding a cutoff vanishing close to $t = 0$ in the definition of F_ν. These Schwartz functions converge pointwise to F_ν with uniform bound of order $C|\lambda|$, which is harmless if we consider spectrally localized v and w. Their Fourier transforms converge dt almost everywhere (pointwise on $\mathbb{R}_t \setminus 0$) to $\widehat{F}_\nu$ with uniform bound by $C|t|e^{-|t|}$, and the result follows easily. ■

Since F_ν is real valued, we have $(F_\nu(hA)v, w) = (v, F_\nu(hA)w)$, and thus

$$\big(v, F_\nu(hA)w\big) = \frac{i}{2}\int_{\mathbb{R}} \frac{te^{-|t|}}{t^2+\nu^2}(v, e^{-ithA}w)dt.$$

From the latter identity and (3.4), we deduce that

$$\text{(3.5)}\qquad \big(i[P, F_\nu(hA)]u_1, u_2\big) = \frac{1}{2}\int \frac{te^{-|t|}}{t^2+\nu^2}\big((e^{ithA}u_1, Pu_2) - (Pu_1, e^{-ithA}u_2)\big)\,dt,$$

where the commutator in the left hand side is understood in the sense of (3.2) (*i.e.,* the form sense).

Lemma 3.4 *For all $t \in \mathbb{R}$, $h > 0$ and $u_1, u_2 \in \mathcal{S}(\mathbb{R}^d)$,*

$$\text{(3.6)}\qquad (e^{ithA}u_1, Pu_2) - (Pu_1, e^{-ithA}u_2) = h\int_0^t \big(e^{ithA}P_{sh}u_1, u_2\big)\,ds.$$

In addition, for each pair u_1, u_2, there is a constant C such that

$$\text{(3.7)}\qquad \big|(e^{ithA}u_1, Pu_2) - (Pu_1, e^{-ithA}u_2)\big| \le C|t|e^{h|t|}, \qquad t \in \mathbb{R}.$$

Proof The formula (3.6) is equivalent to the same one with u_1 replaced by $e^{-ithA}u_1$, and the corresponding identity is then a consequence of Duhamel's formula, *i.e.,* is obtained by checking that the derivatives of both sides coincide using (2.5). To get (3.7), we use (3.6) and observe that, since the coefficients of P_{sh} are of order e^{2sh} (see (2.5) and (3.1)), we have

$$\begin{aligned}|(e^{ithA}P_{sh}u_1, u_2)| &\le Ce^{2sh}\|\nabla u_1\|_{L^2}\|\nabla e^{-ithA}u_2\|_{L^2}\\ &\le e^{(2s-t)h}\|\nabla u_1\|_{L^2}\|\nabla u_2\|_{L^2},\end{aligned}$$

where $|2s - t| \le |t|$, since s is between 0 and t. The conclusion follows easily. ■

Proof of Proposition 3.1 By Lemma 3.2 and the Spectral Theorem, we have

$$F_\nu(hA)u_j \rightharpoonup F(hA)u_j, \qquad \nu \to 0,\ j = 1, 2.$$

Thus the left-hand side of (3.3) is the limit as $\nu \to 0$ of the left-hand side of (3.5). To compute the limit of the right-hand side of (3.5), we simply insert (3.6) therein and let $\nu \to 0$ by dominated convergence using (3.7) and the fact that $h < 1$. The limit is clearly the right-hand side of (3.3), and this completes the proof. ■

4 Semiclassical Expansion of the Commutator

In this section, we establish the first order asymptotic expansion in h of (3.3). To state this result, we introduce the following notation. Write first

$$P_{sh} = P_0 + shQ_{sh}, \quad \text{with} \quad Q_s = \int_0^1 \frac{d}{d\tau} P_{\tau|_{\tau=\sigma s}} d\sigma.$$

Write next

$$\frac{1}{t}\int_0^t hsQ_{sh}ds = th\int_0^1 sQ_{tsh}ds,$$

and set $B_\tau := \tau\int_0^1 sQ_{s\tau}ds$. Notice that $(P_\tau)_{\tau\in\mathbb{R}}$ given by (3.1) is a 2-admissible family of differential operators (see Definition 2.2), and hence so are $(Q_\tau)_{\tau\in\mathbb{R}}$ and $(B_\tau)_{\tau\in\mathbb{R}}$ by Proposition 2.3.

Observe that

$$\frac{h}{2}\int e^{-|t|}(e^{ithA}P_0u_1, u_2)dt = h\big(P_0u_1, (h^2A^2+1)^{-1}u_2\big),$$

as follows easily from the spectral theorem and the Fourier transform

$$\frac{1}{1+\lambda^2} = \frac{1}{2}\int_{\mathbb{R}} e^{-it\lambda}e^{-|t|}dt.$$

It can also be seen as a consequence of (2.1). Define

$$\mathcal{A}_{h,H}(u_1, u_2) := \big(P_0(hA+i)^{-1}u_1, (hA+i)^{-1}u_2\big),$$

and

$$\mathcal{B}_{H,h}(u_1, u_2) = \tfrac{1}{h}\big\{\big(P_0u_1, (h^2A^2+1)^{-1}u_2\big) - \big(P_0(hA+i)^{-1}u_1, (hA+i)^{-1}u_2\big)\big\},$$

so that

$$h\big(P_0u_1, (h^2A^2+1)^{-1}u_2\big) = h\mathcal{A}_{h,H}(u_1, u_2) + h^2\mathcal{B}_{h,H}(u_1, u_2).$$

If we finally set

$$(\mathcal{C}_hu_1, u_2) := \frac{1}{2}\int_{\mathbb{R}} e^{-|t|}t(e^{ithA}B_{th}u_1, u_2)dt,$$

we have

$$\big(i[P, F(hA)]u_1, u_2\big) = h\mathcal{A}_{h,H}(u_1, u_2) + h^2\mathcal{B}_{h,H}(u_1, u_2) + h^2\mathcal{C}_{h,H}(u_1, u_2).$$

The purpose of this section is thus to estimate $\mathcal{B}_{h,H}$ and $\mathcal{C}_{h,H}$.

Proposition 4.1 *There exists C such that for all $0 < h < 1$ and all H,*

$$\left|\mathcal{B}_{H,h}(u_1,u_2)\right| \leq C\left(1+\|H\|_{2,\mathrm{dil}}\right)\left\||D|(hA+i)^{-1}u_1\right\|_{L^2}\left\||D|(hA+i)^{-1}u_2\right\|_{L^2}.$$

Proof By the resolvent identity

$$(hA+i+ih)^{-1} = (hA+i)^{-1} - ih(hA+i)^{-1}(hA+i+ih)^{-1}$$

and (2.4), we have

$$(hA+i)^{-1}D_j = D_j\left(1-ih(hA+i+ih)^{-1}\right)(hA+i)^{-1}. \tag{4.1}$$

Next, we observe that

$$\left[(hA+i)^{-1}, G_{jk}\right] = -\tfrac{h}{i}(hA+i)^{-1}(x\cdot\nabla H_{jk})(hA+i)^{-1}, \tag{4.2}$$

and finally that we also have

$$(hA+i)^{-1}D_k = \left(1-ih(hA+i)^{-1}\right)D_k(hA+i)^{-1}, \tag{4.3}$$

since

$$\left[(hA+i)^{-1}, D_k\right] = -h(hA+i)^{-1}[A,D_k](hA+i)^{-1} = ih(hA+i)^{-1}D_k(hA+i)^{-1}.$$

From (4.1), (4.2), and (4.3), we see that

$$\left[(hA+i)^{-1}, P_0\right] = \sum_{jk} D_jB_{jk}(h)D_k(hA+i)^{-1},$$

with

$$\|B_{jk}(h)\|_{L^2\to L^2} \lesssim h\left(1+\|H\|_{2,\mathrm{dil}}\right).$$

The result follows. ■

Proposition 4.2 *For all $0 < h_0 < 1/4$, there exists $C > 0$ such that*

$$|\mathcal{C}_{h,H}(u_1,u_2)| \leq C(1+\|H\|_{4,\mathrm{dil}})\left\||D|(hA+i)^{-1}u_1\|_{L^2}\right\||D|(hA+i)^{-1}u_2\right\|_{L^2}$$

for all $u_1,u_2 \in \mathcal{S}(\mathbb{R}^d)$, all $0 < h < h_0$, and all H.

Proof The proof simply relies on integrations by parts. Indeed, since

$$e^{-ithA}u_2 = ie^{-ithA}(hA+i)^{-1}u_2 + i\tfrac{d}{dt}e^{-ithA}(hA+i)^{-1}u_2 \tag{4.4}$$

we can write

$$\mathcal{C}_{h,H}(u_1,u_2) = i\mathcal{C}_{h,H}(u_1,(hA+i)^{-1}u_2) + \frac{i}{2}\int te^{-|t|}\left(B_{ht}u_1, \frac{d}{dt}e^{-ithA}(hA+i)^{-1}u_2\right)dt,$$

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where the second term in the right-hand side reads

$$-\frac{i}{2}\int e^{-|t|}\Big(\big\{ thB'_{ht} + (1-|t|)B_{ht}\big\}\, u_1, e^{-ithA}(hA+i)^{-1}u_2\Big)\, dt,$$

if $B'_\tau = (d/d\tau)B_\tau$. Recall that $(B'_\tau)_\tau$ is still a 2-admissible family of operators, so that

$$\tilde{B}_\tau := e^{i\tau A}B_\tau e^{-i\tau A} \quad \text{and} \quad \hat{B}_\tau := e^{i\tau A}B'_\tau e^{-i\tau A},$$

define 4-admissible families of operators by Proposition 2.3. Then, using again (4.4) with u_1 instead of u_2 and integrating by parts (observe that the functions $e^{-|t|}$ and $(1-|t|)$ are not C^1 at $t=0$ but are continuous, and therefore there are no boundary terms), we obtain a sum of integrals of the form

$$\int_0^{\pm\infty} w_\pm(t)e^{-|t|}\big(e^{ithA}C^\pm_{ht}(hA+i)^{-1}u_1, (hA+i)^{-1}u_2\big)\, dt$$

with $w_\pm$ polynomial and $(C^\pm_\tau)_{\tau\in\mathbb{R}}$ 4-admissible families of operators whose coefficients are bounded in $L^\infty(\mathbb{R}^d)$ by $e^{4|\tau|}\|H\|_{4,\mathrm{dil}}$. The result follows. ■

5 Proofs of the Results

5.1 Proof of Theorem 1.1

Assume that $\mathrm{Im}(z) > 0$. The estimates for $\mathrm{Im}(z) < 0$ are obtained by taking the adjoint. We recall that $F(\lambda) = \arctan(\lambda)$. As in [17], we observe that

$$\begin{aligned} 2\mathrm{Im}\Big(\big(F(hA)-\tfrac{\pi}{2}\big)u, (P-z)u\Big) &= 2\mathrm{Im}(F(hA)u, Pu) - 2\Big(\mathrm{Im}(z)\big(F(hA)-\tfrac{\pi}{2}\big)u, u\Big) \\ &= \big(i[P, F(hA)]u, u\big) - 2\Big(\mathrm{Im}(z)\big(F(hA)-\tfrac{\pi}{2}\big)u, u\Big) \\ &\geq \big(i[P, F(hA)]u, u\big). \end{aligned}$$

By (1.8) and Propositions 3.1, 4.1, and 4.2, we have

$$\begin{aligned} &\big(i[P, F(hA)]u, u\big) \\ &\quad\geq h\big(P_0(hA+i)^{-1}u, (hA+i)^{-1}u\big) - Ch^2\||D|(hA+i)^{-1}u\|^2_{L^2} \\ &\quad\geq \tfrac{\varepsilon}{2}h\big\||D|(hA+i)^{-1}u\big\|^2_{L^2}, \end{aligned} \tag{5.1}$$

by taking h small enough so that $Ch \leq \varepsilon/2$. Notice that the constant C in (5.1) is of order $1+\|H\|_{4,\mathrm{dil}}$, so that we may choose h^{-1} of order $(1+\|H\|_{4,\mathrm{dil}})$.

On the other hand, we may write

$$\Big(\big(F(hA)-\tfrac{\pi}{2}\big)u, (P-z)u\Big) = \Big(|D|\big(F(hA)-\tfrac{\pi}{2}\big)(hA+i)^{-1}u, |D|^{-1}(hA-i)(P-z)u\Big).$$

Thus, once we have proved Proposition 5.1, we shall get the estimate

$$\big\||D|(hA+i)^{-1}u\big\|_{L^2} \leq \tfrac{C}{h}\big\||D|^{-1}(hA-i)(P-z)u\big\|_{L^2}$$

which gives (1.9).

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Proposition 5.1 *For all* $0 < h_0 < 1$, *there exists* $C > 0$ *such that*

$$\big\| \, |D| F(hA)(hA+i)^{-1} u \big\|_{L^2} \leq C \big\| \, |D| (hA+i)^{-1} u \big\|_{L^2},$$

for all $u \in \mathcal{S}(\mathbb{R}^d)$ *and* $0 < h \leq h_0$.

Proof Since we have

$$\big\| F(hA)|D|(hA+i)^{-1} u \big\|_{L^2} \leq \|F\|_\infty \big\| \, |D|(hA+i)^{-1} u \big\|_{L^2},$$

the result is clearly equivalent to an estimate on the commutator $[|D|, F(hA)]$. The latter can be computed explicitly using the same argument as in Proposition 3.1. We obtain

$$(5.2)\quad \big(i \big[|D|, F(hA) \big] u_1, u_2 \big) = \frac{h}{2} \int_{\mathbb{R}} e^{-|t|} \Big(\frac{1}{t} \int_0^t e^{sh} (e^{ithA} |D| u_1, u_2) ds \Big) dt, \qquad u_1, u_2 \in \mathcal{S}(\mathbb{R}^d),$$

since $e^{-ishA} i[|D|, A] e^{ishA} = e^{sh}|D|$. This implies that

$$\big| ([|D|, F(hA)] u_1, u_2) \big| \leq \frac{h}{2} \int e^{-(1-h)|t|} dt \big\| \, |D| u_1 \big\|_{L^2} \|u_2\|_{L^2},$$

i.e., that $\|[|D|, F(hA)] u_1\|_{L^2} \lesssim (1-h)^{-1} \big\| \, |D| u_1 \big\|_{L^2}$. The result then follows clearly. ■

5.2 Proof of Corollary 1.2

Using the homogeneous Sobolev imbedding (1.11), we have, for any $f \in L^2$

$$(5.3)\qquad \big\| (hA+i)^{-1} (P-z)^{-1} f \big\|_{L^{2^*}} \leq C \big\| \, |D| (hA+i)^{-1} (P-z)^{-1} f \big\|_{L^2}.$$

Then, by choosing $f = (hA-i)^{-1} g$ with $g \in L^2 \cap L^{2_*}$, we have

$$\begin{aligned}
&\big\| \, |D| (hA+i)^{-1} (P-z)^{-1} f \big\|_{L^2} \\
&\quad = \sup_{\|u\|_{L^2}=1} \big| \big(|D| (hA+i)^{-1} (P-z)^{-1} f, u \big) \big| \\
&\quad = \sup_{\|u\|_{L^2}=1} \big| \big(g, (hA+i)^{-1} (P-\bar{z})^{-1} (hA-i)^{-1} |D| u \big) \big| \\
&\quad \leq \sup_{\|u\|_{L^2}=1} \|g\|_{L^{2_*}} \big\| (hA+i)^{-1} (P-\bar{z})^{-1} (hA-i)^{-1} |D| u \big\|_{L^{2^*}} \\
&\quad \leq C \big\| \, |D| (hA+i)^{-1} (P-\bar{z})^{-1} (hA-i)^{-1} |D| \big\|_{L^2 \to L^2} \|g\|_{L^{2_*}},
\end{aligned}$$

which combined with (5.3) completes the proof. ■

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5.3 Proof of Corollary 1.3

By Hölder's inequality,

$$\|\langle x\rangle^{-1-\epsilon}u\|_{L^2} \lesssim \|u\|_{L^{2^*}}, \qquad \|\langle x\rangle^{-1-\epsilon}v\|_{L^{2_*}} \lesssim \|v\|_{L^2}. \tag{5.4}$$

Next choose $\chi \in C_0^\infty(\mathbb{R})$, which is equal to 1 near $[0,1]$. It is classical that

$$(1-\chi^2)(P)(P-z)^{-1} \colon L^{2_*} \to L^{2^*}$$

by Sobolev embeddings, with norm uniformly bounded for $|z| \leq 1$. This follows for instance from the fact that the L^2 bounded operator $(1-\chi^2)(P)(P-z)^{-1}$ is a pseudo-differential operator of order -2. It is therefore sufficient to show that

$$\langle x\rangle^{-1}\chi(P)(P-z)^{-1}\chi(P)\langle x\rangle^{-1} \colon L^{2_*} \to L^{2^*}$$

is bounded uniformly with respect to $|z| < 1$, $z \notin \mathbb{R}$. To get the latter, we simply write

$$\chi(P)\langle x\rangle^{-1} = (hA-i)^{-1}(hA-i)\chi(P)\langle x\rangle^{-1}$$

and use the fact that $\chi(P)\langle x\rangle^{-1}$ and $A\chi(P)\langle x\rangle^{-1}$ are bounded on L^p for all p, which follows from the fact that these operators are pseudo-differential operators of order $-\infty$ (see, for instance, [6] for more details on such properties). ■

5.4 Local Compactness of the Riesz Transform

In this subsection we prove a property of the Riesz transform that we shall use in the proof of Theorem 1.4. We first recall the definition of the Riesz transform. Since $P \geq 0$ is self-adjoint, the Spectral Theorem and (1.2) give

$$(Pu,u) = \|P^{1/2}u\|_{L^2}^2 \approx \|\nabla u\|_{L^2}^2 \approx \big\| |D|u \big\|_{L^2}^2, \tag{5.5}$$

where $\approx$ stands for the equivalence of norms. Formally replacing u by $P^{-1/2}v$, this implies that the operators

$$R(j) = \partial_{x_j} P^{-1/2} \tag{5.6}$$

are bounded on $L^2(\mathbb{R}^d)$ for all j. They are the components of the well-known Riesz transform $\nabla P^{-1/2}$. To define $R(j)$ more explicitly, we can use the following integral representation (see, for instance, [3]). For each $n \geq 1$, we consider

$$R_n(j) := \pi^{-1/2}\partial_{x_j}\int_{1/n}^{n} e^{-tP}\frac{dt}{\sqrt{t}},$$

where the integral converges in the strong sense. It is not hard to check that $R_n(j)$ is bounded using that e^{-tP} maps L^2 in $\cap_s H^s$ for all $t > 0$. Let us briefly recall why $R_n(j)$ converges strongly as $n \to \infty$ (for this purpose we could actually consider

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lower and upper bounds in the integral defining $R_n(j)$ going independently to 0 and ∞ respectively, but this is irrelevant for our purpose). Using (5.5), we see that

$$\|R_n(j)u\|_{L^2} \leq C\Big\| \sqrt{P}\int_{1/n}^{n} e^{-tP}\frac{dt}{\sqrt{t}}u\Big\|_{L^2} = C\|f_n(P)u\|_{L^2}, \tag{5.7}$$

with

$$f_n(\lambda) = \int_{1/n}^{n} \lambda^{1/2}e^{-t\lambda}\frac{dt}{\sqrt{t}} = \int_{\lambda/n}^{\lambda n} e^{-\tau}\frac{d\tau}{\sqrt{\tau}}.$$

Since f_n is uniformly bounded with respect to $n \geq 1$ and $\lambda \geq 0$, (5.7) and the Spectral Theorem show that $\|R_n(j)\|_{L^2\to L^2} \leq C$ for all n. Therefore, it is sufficient to prove the strong convergence of $R_n(j)$ on a dense subset. For the latter, we observe that, since 0 is not an eigenvalue of P, the spectral theorem shows that for all $u \in L^2$,

$$\chi_{[\epsilon,\epsilon^{-1}]}(P)u \to u, \qquad \epsilon \to 0, \tag{5.8}$$

$\chi_{[\epsilon,\epsilon^{-1}]}$ denoting the characteristic function of $[\epsilon,\epsilon^{-1}]$. It is then easy to check that $R_n(j)\chi_{[\epsilon,\epsilon^{-1}]}(P)$ converges in the strong sense as $n \to \infty$ for each $\epsilon > 0$, since the spectral projection on $[\epsilon,\epsilon^{-1}]$ guarantees the exponential decay of e^{-tP} as well as the boundedness of $\partial_{x_j}\chi_{[\epsilon,\epsilon^{-1}]}(P)$. By (5.8), functions of the form $\chi_{[\epsilon,\epsilon^{-1}]}(P)u$ are dense in L^2, so this completes the proof of the strong convergence of $R_n(j)$. We may thus define

$$R(j) = \pi^{-1/2}\partial_{x_j}\int_0^{\infty} e^{-tP}\frac{dt}{\sqrt{t}} := \mathrm{s}-\lim_{n\to\infty} R_n(j),$$

which is a reasonable definition for $\partial_{x_j}P^{-1/2}$, since one checks that

$$R(j)P^{1/2}u = \partial_{x_j}u \tag{5.9}$$

for all $u \in D(P)$. This is an elementary consequence of the Spectral Theorem and the Lebesgue theorem since, for all $\lambda > 0$

$$\pi^{-1/2}f_n(\lambda) \to 1, \qquad n \to \infty,$$

and since $\{\lambda = 0\}$ is negligible with respect to the spectral measure, because 0 is not an eigenvalue of P. This completes our definition of $R(j)$.

The main purpose of this subsection is to prove the following result.

Proposition 5.2 *Assume that $d \geq 3$. Then for all $\chi \in C_0^\infty(\mathbb{R}^d)$ and all $\varphi \in C_0^\infty(\mathbb{R})$, $\chi(x)R(j)\varphi(P)$ is a compact operator on $L^2(\mathbb{R}^d)$, for all $j = 1,\dots,d$.*

Proof We split $\pi^{1/2}R(j)$ into $\partial_{x_j}\int_0^2 e^{-tP}dt/t^{1/2} + \partial_{x_j}\int_2^\infty e^{-tP}dt/t^{1/2}$. It is clear that

$$\chi(x)\partial_{x_j}\int_0^2 e^{-tP}\frac{dt}{\sqrt{t}}\chi(P) = \big(\chi(x)\partial_{x_j}\chi(P)\big)\int_0^2 e^{-tP}\frac{dt}{\sqrt{t}}$$

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is compact since the bracket is compact and the integral defines a bounded operator on L^2. We then write the contribution of the second term as

$$\left(\chi(x)\partial_{x_j}e^{-P}\langle x\rangle^N\right)\int_2^\infty \langle x\rangle^{-N}e^{-(t-1)P}\frac{dt}{\sqrt{t}},$$

with $N>0$ to be chosen below. Again the bracket is a compact operator (since e^{-P} is a smoothing operator that preserves polynomial decay). To see that the integral is bounded on L^2, we use the classical Gaussian upper bounds for the kernel $K(t,x,y)$ of e^{-tP} (see, for instance, [2, 13]). For some $C,c>0$ we have,

$$|K(t,x,y)|\le \frac{C}{t^{d/2}}\exp\left(\frac{-c|x-y|^2}{t}\right),\qquad x,y\in\mathbb{R}^d,\ t>0,$$

and thus $\|e^{-tP}\|_{L^2\to L^\infty}\lesssim t^{-d/4}$. Therefore, if $N>d/2$,

$$\|t^{-1/2}\langle x\rangle^{-N}e^{-(t-1)P}u\|_{L^2}\lesssim t^{-\frac12-\frac d4}\|u\|_{L^2},$$

which is integrable on $[2,\infty)$, since $\frac12+\frac d4>1$. This completes the proof. ∎

5.5 Proof of Theorem 1.4

We start by observing that it is sufficient to show that, for some $\lambda>0$ and $h>0$ small enough, we have the bound

$$\left\|\,|D|(hA+i)^{-1}(P-z)^{-1}(hA-i)^{-1}|D|\right\|_{L^2\to L^2}\le C,\qquad |\mathrm{Re}(z)|<\lambda. \tag{5.10}$$

We will then obtain (1.10) exactly as in Corollary 1.3. We may even replace $(P-z)^{-1}$ in this estimate by $(P-z)^{-1}\varphi_0(P/\lambda)$, with $\varphi_0\in C_0^\infty(\mathbb{R})$ such that $\varphi_0\equiv 1$ near $[-1,1]$, since the operator

$$|D|(hA+i)^{-1}(P-z)^{-1}(1-\varphi_0)(P/\lambda)(hA-i)^{-1}|D|$$

is easily seen to be bounded on L^2, uniformly with respect to z such that $|\mathrm{Re}(z)|<\lambda$. It is therefore enough to consider u of the form $u=(P-z)^{-1}\varphi_0(P/\lambda)f$ with $f\in\mathcal{S}(\mathbb{R}^d)$ so that

$$u=\varphi(P/\lambda)u, \tag{5.11}$$

for some $\varphi\equiv 1$ near $\mathrm{supp}(\varphi_0)$.

Independently, observe that, as in the proof of Theorem 1.1, we have, for all $u\in\mathcal{S}(\mathbb{R}^d)$,

$$\big(i[P,F(hA)]u,u\big)\ge\frac h2\big(P_0(hA+i)^{-1}u,(hA+i)^{-1}u\big)-Ch^2\big\|\,|D|(hA+i)^{-1}u\big\|_{L^2}^2, \tag{5.12}$$

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but the difference is now that P_0 is not necessarily elliptic in a compact set. It is however elliptic outside a large enough compact set, since P_0 is close to P, or equivalently to $-\Delta$, at infinity and we may thus write $P_0 = \widetilde{P}_0 + P_c$ with a uniformly elliptic part

$$\widetilde{P}_0 = \sum_{jk} D_j \widetilde{G}_0(x) D_k$$

for some matrix $\widetilde{G}_0$ satisfying (1.2), and a compactly supported part

$$P_c = \sum_{j,k=1}^{d} D_j \big(b_{jk}(x) D_k \big), \qquad b_{jk} \in C_0^\infty(\mathbb{R}^d).$$

We shall absorb the contribution of $(P_c(hA+i)^{-1}u, (hA+i)^{-1}u)$ as in the original proof of Mourre [27] by considering u that are spectrally localized very close to 0. Using (5.12) and the uniform ellipticity of $\widetilde{P}_0$ there exists $c > 0$ such that, for all u satisfying (5.11), we have

$$\text{(5.13)} \qquad \big(i[P, F(hA)]u, u \big) \geq ch \big\| \nabla (hA+i)^{-1} u \big\|_{L^2}^2 - Ch^2 \big\| |D| (hA+i)^{-1} u \big\|_{L^2}^2 + \tfrac{h}{2} \big(P_c (hA+i)^{-1} \varphi(P/\lambda) u, (hA+i)^{-1} u \big).$$

Using (5.6), we now introduce

$$R_c = -\sum_{jk} R(j)^* b_{jk}(x) R(k),$$

i.e., $R_c = P^{-1/2} P_c P^{-1/2}$ formally. Actually, by (5.9), we have $P^{1/2} R_c P^{1/2} = P_c$ at least in the form sense, and this allows us to rewrite the last term of (5.13) as $h/2$ times the sum of the following two terms

$$\text{(5.14)} \qquad \big(R_c \varphi(P/\lambda) P^{1/2} (hA+i)^{-1} u, P^{1/2} (hA+i)^{-1} u \big),$$
$$\Big(P_c \big[(hA+i)^{-1}, \varphi(P/\lambda) \big] u, (hA+i)^{-1} u \Big).$$

The local compactness of the Riesz transform is crucial for the following result.

Proposition 5.3 *As $\lambda \downarrow 0$, we have $\|R_c \varphi(P/\lambda)\|_{L^2 \to L^2} \to 0$.*

Proof The operator $R_c \varphi(P/\lambda)$ can be written for λ small enough, $(R_c \varphi(P)) \varphi(P/\lambda)$ since $\varphi \equiv 1$ near 0. The bracket is compact by Proposition 5.2, and $\varphi(P/\lambda)$ goes strongly to 0 as $\lambda \downarrow 0$, by the Spectral Theorem, since 0 is not an eigenvalue of P. Since $R_c \varphi(P)$ is compact, $(R_c \varphi(P)) \varphi(P/\lambda)$ goes to 0 in operator norm. ∎

By Proposition 5.3 and by choosing $\lambda > 0$ small enough, we can make (5.14) small so that, using (5.5), we get the existence of $c' > 0$ such that

$$\text{(5.15)} \qquad (i[P, F(hA)]u, u) \geq c'h \big\| P^{1/2} (hA+i)^{-1} u \big\|_{L^2}^2 - Ch^2 \big\| |D| (hA+i)^{-1} u \big\|_{L^2}^2 - \frac{h}{2} \big| \big(P_c [(hA+i)^{-1}, \varphi(P/\lambda)] u, (hA+i)^{-1} u \big) \big|$$

for all $0 < h < 1/4$ and all u satisfying (5.11). It remains to deal with the last term of (5.15). This is the purpose of the following proposition.

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Proposition 5.4 *For all $\lambda > 0$, there exists $C_\lambda > 0$ such that for all $v \in \mathcal{S}(\mathbb{R}^d)$ and all h*

$$\big\| |D|[(hA+i)^{-1}, \varphi(P/\lambda)]v \big\|_{L^2} \le C_\lambda h \big\| P^{1/2}(hA+i)^{-1}v \big\|_{L^2}.$$

Proof The proof relies on a standard combination of the resolvent identity

$$|D|\big[(hA+i)^{-1}, \varphi(P/\lambda)\big] = -h|D|(hA+i)^{-1}\big[A, \varphi(P/\lambda)\big](hA+i)^{-1}, \tag{5.16}$$

and, for instance, the following Helffer–Sjöstrand formula (see [15])

$$\varphi(P/\lambda) = \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\widetilde{\varphi}_\lambda(z)(P-z)^{-1}L(dz),$$

where $L(dz)$ is the Lebesgue measure on $\mathbb{C} \simeq \mathbb{R}^2$, and $\widetilde{\varphi}_\lambda \in C_0^\infty(\mathbb{C})$ is an almost analytic extension of $\varphi_\lambda := \varphi(\frac{\cdot}{\lambda})$, *i.e.*, which coincides with φ_λ on $\mathbb{R}$ and such that $\bar{\partial}\widetilde{\varphi}_\lambda = \mathcal{O}(|\mathrm{Im}(z)|^\infty)$. We have

$$\big[A, \varphi(P/\lambda)\big] = -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\widetilde{\varphi}_\lambda(z)(P-z)^{-1}[A,P](P-z)^{-1}L(dz), \tag{5.17}$$

hence, using (2.4) we can rewrite (5.16) as

$$\frac{h}{\pi}\Big(\int_{\mathbb{C}} \bar{\partial}\widetilde{\varphi}_\lambda(z)(hA+i+ih)^{-1}|D|P^{-1/2} \times \big\{ (P-z)^{-1}P^{1/2}[A,P]P^{-1/2}(P-z)^{-1}\big\} L(dz)\Big) P^{1/2}(hA+i)^{-1}$$

where it is not hard to check that the operator $\{\cdot\}$ is bounded on L^2 with norm of polynomial growth with respect to $|\mathrm{Im}(z)|^{-1}$ (for z in the support of $\widetilde{\varphi}_\lambda$).

The result follows. ∎

Completion of the Proof of Theorem 1.4 Since P_c is of div-grad type, the last term of (5.15) is bounded by $-C_\lambda h^2\|P^{1/2}(hA+i)u\|_{L^2}^2$ from below. Thus, by choosing h and $c_\lambda > 0$ both small enough, we finally get

$$\big(i[P, F(hA)]u, u\big) \ge c_\lambda h \big\| P^{1/2}(hA+i)^{-1}u \big\|_{L^2}^2$$

for all u satisfying (5.11). We then obtain (5.10) as in the proof of Theorem 1.1. This completes the proof. ∎

6 Obstacle Perturbations

In this section, we show how to extend our results to the more general context of a long range perturbation of div-grad type outside a compact obstacle. We will show, as is well known in the case of positive frequencies, that our relatively simple and explicit approach on $\mathbb{R}^d$ can be modified to handle topological perturbations. Using

Dirichlet boundary conditions, we will furthermore see that we have a control on the resolvent at low frequency even in dimension 2.

We denote by K the closure of a non-empty bounded open subset of $\mathbb{R}^d$ with smooth boundary, and let $\Omega = \mathbb{R}^d \setminus K$ be the exterior of this obstacle. We assume that $d \geq 2$. We denote by P^D the self-adjoint realization of (1.1) on Ω with Dirichlet boundary conditions, that is, with domain

$$\mathrm{Dom}(P^D) = \{u \in H^2(\Omega) \mid u_{|\partial\Omega} = 0\},$$

which coincides with $H^2(\Omega) \cap H_0^1(\Omega)$. Notice in particular that on this domain

$$(P^D u, u) = \|(P^D)^{1/2} u\|^2_{L^2(\Omega)} \approx \|\nabla u\|^2_{L^2(\Omega)}, \tag{6.1}$$

where $\approx$ stands for the equivalence of norms.

We will prove the following result.

Theorem 6.1 *Assume that $d \geq 2$. There exists a self-adjoint differential operator A_Ω that coincides with the generator of dilations outside a compact set such that, for some $h > 0$ and $\lambda_0 > 0$ small enough, we have*

$$\big\| (P^D)^{1/2}(hA_\Omega + i)^{-1}(P^D - z)^{-1}(hA_\Omega - i)^{-1}(P^D)^{1/2} \big\|_{L^2 \to L^2} \lesssim 1, \ |\mathrm{Re}(z)| < \lambda_0, \ z \notin \mathbb{R}.$$

Furthermore, for all $\epsilon > 0$, we have

$$\|\langle x\rangle^{-2-\epsilon}(P^D - z)^{-1}\langle x\rangle^{-2-\epsilon}\| \lesssim 1, \quad \mathrm{Re}(z) \to 0.$$

In dimensions 3 and higher, estimates on gradients (or on $P^{1/2}$) lead to weighted estimates via

$$\|\langle x\rangle^{-1-\epsilon} v\|_{L^2} \lesssim \|\nabla v\|_{L^2}, \tag{6.2}$$

which is a consequence of the Sobolev embedding (1.11) and the Hölder inequality (5.4). Using the Poincaré inequality, this also holds in dimension 2 for Dirichlet boundary conditions.

Proposition 6.2 (Poincaré inequality) *Assume that $d = 2$. Then for all $\epsilon > 0$, there exists $C > 0$ such that $\|\langle x\rangle^{-1-\epsilon} v\|_{L^2} \leq C\|\nabla v\|_{L^2}$, for all $v \in C_0^\infty(\Omega)$.*

Proof By possibly translating the obstacle, we may assume that 0 belongs to the interior of K and that a Euclidean ball $B(0, r_0)$ is contained in K. Define ψ by

$$\psi(r, \omega) = v(x), \qquad r = |x|, \ \omega = x/|x|.$$

Then if v is supported outside K, hence outside $B(0, r_0)$, we have

$$\psi(r_1, \omega) = \int_{r_0}^{r_1} \partial_r \psi(r, \omega) dr = \int_{r_0}^{r_1} r^{-\frac{1}{2}} \partial_r \psi(r, \omega) r^{\frac{1}{2}} dr,$$

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so, by the Cauchy–Schwarz inequality, we have

$$|\psi(r_1,\omega)|^2 \le \ln(r_1/r_0)\left(\int_{r_0}^{r_1} |\partial_r\psi(r,\omega)|^2 r dr\right).$$

Integrating with respect to ω and using that $\partial_r\psi = |x|^{-1}x\cdot\nabla_x v$ yields

$$\int_{\mathbb{S}^1} |\psi(r_1,\omega)|^2 d\omega \le C\ln(r_1/r_0)\|\nabla v\|_{L^2}^2,$$

and thus

$$\int_{r_0}^{\infty}\int_{\mathbb{S}^1} r_1^{-2-2\epsilon}|\psi(r_1,\omega)|^2 r_1 dr_1 d\omega \le C\int_{r_0}^{\infty}\langle \ln(r_1)\rangle r_1^{-1-2\epsilon}dr_1\|\nabla v\|_{L^2}^2$$

leads to the result. ■

We next study the properties of a suitable conjugate operator, basically obtained by cutting off the generator of dilation (1.5) near the obstacle. Let $R>0$ be such that $K\subset B(0,R/2)$ and let $\rho\in C^\infty(\mathbb{R},\mathbb{R})$ be such that

$$\rho(r) = \begin{cases} 0, & \text{if } r\le R/2,\\ 1, & \text{if } r\ge R.\end{cases}$$

We consider the vector field, defined on $\mathbb{R}^d$, $V(x)=\rho(|x|)x$. Since $\partial_j V$ is bounded for all j, the flow of V, *i.e.*, the solution $\phi^t(x_0) := x_t$ to

$$\dot{x}_t = V(x_t), \qquad x_{t=0}=x_0\in\mathbb{R}^d,$$

is defined for $t\in\mathbb{R}$. In the next proposition, we record some elementary properties of this flow.

Proposition 6.3 (i) *For $t\in\mathbb{R}$ and $|x|\le R/2$, we have $\phi^t(x)=x$. If $|x|>R/2$, then $|\phi^t(x)|>R/2$ for all $t\in\mathbb{R}$.*

(ii) *For $|x|\ge R$ and $t\ge 0$, we have*

$$\phi^t(x) = e^t x \begin{cases} \text{if } |x|\ge R \text{ and } t\ge 0,\\ \text{if } |x|\ge e^{-t}R \text{ and } t\le 0.\end{cases} \tag{6.3}$$

In particular, $\phi^t(x)=e^t x$ for $t\in\mathbb{R}$ and $|x|\ge e^{|t|}R$.

(iii) *There exists $C>0$ such that*

$$e^{-C|t|}|x| \le |\phi^t(x)| \le e^{C|t|}|x|, \qquad t\in\mathbb{R},\ x\in\mathbb{R}^d.$$

(iv) *For all multi-index $\alpha\ne 0$, there exists $C_\alpha>0$ such that*

$$|\partial_x^\alpha\phi^t(x)| \le C_\alpha e^{C_\alpha|t|}, \qquad t\in\mathbb{R},\ x\in\mathbb{R}^d.$$

Proof The first part of (i) is easily verified. Then if $|x| > R/2$ and $\phi^{t_0}(x) \leq R/2$ for some t_0, we have

$$R/2 < |x| = |\phi^{-t_0} \circ \phi^{t_0}(x)| = |\phi^{t_0}(x)| \leq R/2,$$

which yields a contradiction. The statement (ii) is easy to check for $t \geq 0$. If $t \leq 0$, one simply uses that $y = \phi^t \circ \phi^{-t}(y) = \phi^t(e^{-t}y)$ for $|y| \geq R$, and then applies this identity to $y = e^t x$. The upper bound in (iii) follows from the Gronwall inequality (one may take $C = \|\rho\|_\infty$). The lower bound follows from the upper bound and the identity $|\phi^{-t} \circ \phi^t(x)| = |x|$. The estimates in (iv) are obtained by induction on $|\alpha|$ by applying ∂^α to the equation $\dot{\phi}^t = V(\phi^t)$ and using the Gronwall inequality. ■

In the applications below, Proposition 6.3(iii) will be used very often in connection with the following.

Proposition 6.4 *For all $s \geq 0$, there exists $C_s \geq 0$ such that*

$$\langle e^t x\rangle^{-s} \leq C_s(1 + e^{-st})\langle x\rangle^{-s}, \qquad t \in \mathbb{R},\ x \in \mathbb{R}^d.$$

In particular, $\langle \phi^t(x)\rangle^{-s} \lesssim e^{Cs|t|}\langle x\rangle^{-s}$.

Proof Let $\chi \in C_0^\infty(\mathbb{R}^d)$ be equal to 1 near 0. Write

$$\langle e^t x\rangle^{-s} = \langle e^t x\rangle^{-s}\chi(x) + \langle e^t x\rangle^{-s}(1-\chi)(x),$$

where the first term in the right-hand side is clearly $\mathcal{O}(\langle x\rangle^{-s})$. To estimate the second term, we simply observe that $\langle y\rangle^{-s} \leq |y|^{-s}$, so the second term is dominated by

$$\frac{(1-\chi)(x)}{e^{st}|x|^s} \leq C\langle x\rangle^{-s}e^{-st},$$

and the conclusion follows. The estimate on $\langle \phi^t(x)\rangle^{-s}$ follows from the lower bound in Proposition 6.3(iii). ■

Using the flow $(\phi^t)_{t\in\mathbb{R}}$, it is completely standard to check that one defines a strongly continuous group of unitary operators on $L^2(\mathbb{R}^d)$ by setting $\widetilde{U}(t)v = (\det D_x\phi^t)^{1/2}v \circ \phi^t$, and an easy calculation shows that its generator (*i.e.*, $\tilde{U}(t) = e^{it\tilde{A}}$) is

$$\tilde{A} = \frac{\rho(|x|)x \cdot \nabla + \nabla \cdot \rho(|x|)x}{2i}. \tag{6.4}$$

Let us denote by $e_\Omega \colon L^2(\Omega) \to L^2(\mathbb{R}^d)$ the operator of extension by 0 outside Ω and by $r_\Omega \colon L^2(\mathbb{R}^d) \to L^2(\Omega)$ the operator of restriction to Ω. Note in particular that, if χ_Ω is the (multiplication operator by the) characteristic function of Ω, one has

$$e_\Omega r_\Omega = \chi_\Omega. \tag{6.5}$$

In the sequel, we define a family of operators on $L^2(\Omega)$ by

$$U(t)\varphi = r_\Omega \tilde{U}(t)e_\Omega\varphi, \qquad t \in \mathbb{R},\ \varphi \in L^2(\Omega).$$

Proposition 6.5 *$(U(t))_{t\in\mathbb{R}}$ is a strongly continuous group of unitary operators on $L^2(\Omega)$.*

Proof The unitarity (resp. strong continuity) of $U(t)$ is a direct consequence of the unitarity (resp. strong continuity) of $\tilde{U}(t)$. The non-obvious point is the group property. To prove the latter, it suffices to observe that if $\varphi \in C_0^\infty(\Omega)$, one has

$$\tilde{U}(t)\varphi(x) = \varphi(x), \qquad t \in \mathbb{R},\ |x| \leq R/2, \tag{6.6}$$

by Proposition 6.3(i), which vanishes near K. Indeed, we then have

$$U(t_1 + t_2)\varphi = r_\Omega \tilde{U}(t_1)\tilde{U}(t_2)e_\Omega\varphi = r_\Omega \tilde{U}(t_1)\chi_\Omega \tilde{U}(t_2)e_\Omega\varphi = U(t_1)U(t_2)\varphi,$$

where the second equality is a consequence of (6.6), and the third one follows from (6.5). ■

In the rest of this section we denote by A_Ω the self-adjoint generator of $U(t)$, *i.e.*, $U(t) = e^{itA_\Omega}$. Clearly, A_Ω is a self-adjoint realization of the restriction of (6.4) to Ω.

We next consider the conjugation of differential operators by $U(t)$. If $a \in C_b^\infty(\overline{\Omega})$, then

$$U(-t)a(x)U(t) = a \circ \phi^{-t}(x). \tag{6.7}$$

Note that the right-hand side is well defined (*i.e.*, does not depend on an extension of a to $\mathbb{R}^d$) by Proposition 6.3(i). Regarding derivatives, if $j = 1, \dots, d$, we have

$$U(-t)\partial_j U(t) = \left(\partial_j \phi^t\right)\left(\phi^{-t}(x)\right) \cdot \nabla_x + \frac{1}{2}\frac{\partial_{y_j}\det(D_y\phi^t)}{\det(D_y\phi^t)}\Big|_{y=\phi^{-t}(x)}. \tag{6.8}$$

In particular, using (6.3) we see that $U(-t)\partial_j U(t) = e^t\partial_j$, on the region $|x| \geq e^{|t|}R$. More precisely, (6.8) shows that $U(-t)\partial_j U(t) - e^t\partial_j$ is a differential operator of order 1 with coefficients supported in $|x| \leq e^{|t|}R$ and with coefficients whose derivatives in x and t grow at most exponentially in t by Proposition 6.3(iv). We shall see that such operators fall into the class given in Definition 6.7. Before stating this definition, we record here the following useful result.

Proposition 6.6 *There exists $C > 0$ such that*

$$\|\nabla U(t)u\|_{L^2(\Omega)} \leq Ce^{C|t|}\|\nabla u\|_{L^2(\Omega)}, \tag{6.9}$$

for all $u \in H^1(\Omega)$ and all $t \in \mathbb{R}$. Furthermore, for h_0 small enough and all $h \in (0, h_0]$, $H^1(\Omega)$ is stable by $(hA_\Omega \pm i)^{-1}$ and

$$\|\nabla(hA_\Omega \pm i)^{-1}u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}, \tag{6.10}$$

for all $h \in (0, h_0]$ and all $u \in H^1(\Omega)$.

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Proof Since

$$\|\nabla U(t)u\|_{L^2} = \|U(-t)\nabla U(t)u\|_{L^2} \quad \text{and} \quad U(-t)\partial_j U(t) = e^t\partial_j + \sum_{|\alpha|\leq 1}\chi_\alpha(t,x)\partial^\alpha,$$

with $\chi_\alpha(t,\cdot)$ supported in the ball (centered at 0) of radius $e^{|t|R}$ and whose L^∞ norm is of order $Ce^{C|t|}$ (see the discussion following (6.8)), Proposition 6.4 implies that for all $N > 0$,

$$\|\nabla U(t)u\|_{L^2} \leq C_N e^{C_N|t|}\left(\|\nabla u\|_{L^2} + \|\langle x\rangle^{-N}u\|_{L^2}\right).$$

Choosing $N = 1+\epsilon$ with $\epsilon > 0$ and using (6.2) in dimension $d \geq 3$ or Proposition 6.2 in dimension 2, we obtain (6.9). To prove (6.10), one simply writes

$$\nabla(hA_\Omega \pm i)^{-1} = \frac{1}{i}\int_0^{\pm\infty} e^{-|t|}\nabla U(ht)dt, \tag{6.11}$$

and then uses (6.9) with h small enough such that $1 - Ch \geq 1/2$. ■

Definition 6.7 For $s \geq 0$ real, we denote by S^{-s}_{adm} the set of smooth functions of $(t,x) \in \mathbb{R}\times\mathbb{R}^d$ such that, for all integer $k \geq 0$ and all multi-index α there exists $C_{k\alpha} \geq 0$ such that,

$$|\partial_t^k\partial_x^\alpha b(t,x)| \leq C_{k\alpha}e^{C_{k\alpha}|t|}\langle x\rangle^{-s-|\alpha|}, \qquad t\in\mathbb{R},\ x\in\mathbb{R}^d.$$

A family of differential operators $(B_t)_{t\in\mathbb{R}}$ of order ≤ 2 and symmetric on $\mathrm{Dom}(P^D)$ will be called admissible if it can be written as

$$B_t = c(t)\Delta + \sum_{|\alpha|\leq 2} b_\alpha(t,x)\partial_x^\alpha, \tag{6.12}$$

for some function $c(\cdot)$ such that for all $k \geq 0$,

$$|\partial_t^k c(t)| \leq C_k e^{C_k|t|}, \qquad t\in\mathbb{R},$$

and some coefficients $b_\alpha \in S^{-s-2+|\alpha|}_{\text{adm}}$ with

$$s > 0. \tag{6.13}$$

Notice that $i^{-1}(U(-t)\partial_j U(t) - e^t\partial_j)$ is symmetric on $\mathrm{Dom}(P^D)$ and hence admissible by (6.8) and Propositions 6.3(iv) and 6.4. Note indeed that if $\chi_{e^{|t|}R}$ denotes the characteristic function of the region $|x| \leq e^{|t|}R$, we have for any $s > 0$,

$$\chi_{e^{|t|}R}(x) = \chi_R(e^{-|t|}x) \leq C_{s,R}\langle x\rangle^{-s}e^{s|t|}.$$

Definition 6.7 must be understood as a robust version of Definition 2.2, which was specific to the case of $\mathbb{R}^d$ and to the group of dilations. In particular, in Definition 2.2, we only considered second order operators in div-grad form. Here we need to consider more general operators since the conjugation by $U(t)$ doesn't preserve vector fields (there may be a zero order term as in (6.8)).

We will need the following result.

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Proposition 6.8 *Let $(B_t)_{t\in\mathbb{R}}$ be an admissible family of differential operators. Then there exists $C > 0$ such that*

$$|(B_t u, v)| \le Ce^{C|t|}\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}$$

for all $u, v \in \mathrm{Dom}(P^D)$ and all $t \in \mathbb{R}$.

Proof Observe first that the second order terms in (6.12) may be written as

$$b_\alpha(t,x)\partial^\alpha = \partial^{\alpha_1} b_\alpha(t,x)\partial^{\alpha_2} - (\partial^{\alpha_1} b_\alpha(t,x))\partial^{\alpha_2}$$

with $\alpha_1 + \alpha_2 = \alpha$, $|\alpha_1| = |\alpha_2| = 1$, where one should note that $\partial^{\alpha_1} b_\alpha \in S^{-s-1}_{\mathrm{adm}}$. Thus, using integration by part in second order terms, we have

$$|(B_t u, v)| \le Ce^{C|t|}\left(\|\nabla u\|_{L^2} + \|\langle x\rangle^{-1-s}u\|_{L^2}\right)\left(\|\nabla v\|_{L^2} + \|\langle x\rangle^{-1-s}v\|_{L^2}\right),$$

and we conclude using (6.2) in dimension $d \ge 3$ and Proposition 6.2 in dimension 2 (in both cases the condition (6.13) is crucial). ∎

To prove Theorem 6.1, we need to compute commutators in the spirit of Section 3. For that purpose, we need the following result.

Lemma 6.9 *The space $\mathrm{Dom}(P^D)$ is stable by $U(t)$ and $(U(t))_{t\in\mathbb{R}}$ is strongly continuous on this space (equipped with the $H^2(\Omega)$ norm). Furthermore, if h is small enough, $\mathrm{Dom}(P^D)$ is stable by $(hA_\Omega \pm i)^{-1}$.*

Proof On one hand, if $u \in H^2(\Omega)$ and $|\alpha| \le 2$, we see that $\partial^\alpha U(t)u$ belongs to L^2 since $U(-t)\partial^\alpha U(t)$ is a second order differential operator with bounded coefficients (for fixed t), which follows from Proposition 6.3(iv) and (6.7)–(6.8). On the other hand, one trivially has $U(t)u_{|\partial\Omega} = 0$ since $U(t)u = u$ near $\partial\Omega$ by Proposition 6.3(i). Then the strong continuity follows from the continuity with respect to t of the flow ϕ^t and its spatial derivatives, using standard density arguments.

To see that the domain of P^D is stable by $(hA_\Omega \pm i)^{-1}$ for h small enough, one uses (6.11) and the fact that $\|\partial^\alpha U(ht)u\|_{L^2} \lesssim e^{C_\alpha h|t|}\|u\|_{H^2}$. ∎

Lemma 6.10 *Let B be a differential operator of order ≤ 2 with coefficients in $C_b^\infty(\overline{\Omega})$ that is symmetric on $\mathrm{Dom}(P^D)$. Then for all u, v in this space and all $t \in \mathbb{R}$,*

$$\big(U(t)u, Bv\big) - \big(Bu, U(-t)v\big) = \int_0^t \big(U(s)u, i[B, A_\Omega]U(s-t)v\big)\, ds,$$

where $[B, A_\Omega]$ is the (usual) commutator of B and A_Ω in the sense of differential operators.

Note that this formula makes sense since $[B, A_\Omega]$ is a second order differential operator with coefficients in $C_b^\infty(\overline{\Omega})$, which thus acts boundedly on $H^2(\Omega)$.

Proof It suffices to compute

$$(U(t)u, BU(t)\tilde{v}) - (u, B\tilde{v}) \tag{6.14}$$

with $\tilde{v} \in \mathrm{Dom}(P^D)$ and then to evaluate this expression with $\tilde{v} = U(-t)v$, which still belongs to $\mathrm{Dom}(P^D)$ by Lemma 6.9. Assume further that u and $\tilde{v}$ vanish for $|x|$ large. Such functions are dense in $\mathrm{Dom}(P^D)$, and it is not hard to check that they also belong to $\mathrm{Dom}(A_\Omega)$ (basically $H^1_{\mathrm{comp}}(\Omega) \subset \mathrm{Dom}(A_\Omega)$). Therefore, one can differentiate (6.14) with respect to t, and we get

$$(iA_\Omega U(t)u, BU(t)\tilde{v}) - (BU(t)u, iA_\Omega U(t)\tilde{v}).$$

Since A_Ω vanishes close to $\partial\Omega$, and $U(t)u, U(t)\tilde{v}$ belong to H^2 and are compactly supported, by standard arguments one can integrate by part in this expression, which can thus be written $(U(t)u, i[B, A_\Omega]U(t)\tilde{v})$. By the symmetry of B, (6.14) vanishes at $t = 0$ thus, by integration between 0 and t, we get

$$\big(U(t)u, BU(t)\tilde{v}\big) - (u, B\tilde{v}) = \int_0^t \big(U(s)u, i[B, A_\Omega]U(s)\tilde{v}\big)\, ds.$$

Both sides of this formula still make sense when u and $\tilde{v}$ are not compactly supported, and the result follows. ■

Similarly to (3.1), from now on we define

$$P_t := U(-t)i[P^D, A_\Omega]U(t), \qquad t \in \mathbb{R}.$$

We have an analogue to Proposition 2.3.

Proposition 6.11 *Let $(B_t)_{t\in\mathbb{R}}$ be an admissible family of differential operators on Ω in the sense of Definition 6.7. Then if $w\colon [0,1] \to \mathbb{C}$ is continuous, the (families of) operators*

$$\frac{d}{dt}B_t, \qquad U(\pm t)B_tU(\mp t), \qquad \text{and} \qquad \int_0^1 w(\sigma)B_{\sigma t}d\sigma,$$

are admissible too. In particular, $(P_t)_{t\in\mathbb{R}}$ is admissible.

Proof It is straightforward to check that dB_t/dt and $\int_0^1 w(\sigma)B_{\sigma t}d\sigma$ are admissible from Definition 6.7. The fact that $U(\pm t)B_tU(\mp t)$ are admissible follows from (6.7), (6.8), Proposition 6.3(iv) and Proposition 6.4. Finally, for P_t one observes first that the t independent operator P^D is admissible, hence so is $P_t = \frac{d}{dt}U(-t)P^DU(t)$. ■

Now, using Lemma 6.10 and Proposition 6.11, one obtains the following result similarly to Proposition 3.1, where we recall that $F(\lambda) = \arctan(\lambda)$.

Proposition 6.12 *For all $h > 0$ small enough and all $u_1, u_2 \in \mathrm{Dom}(P^D)$, we have*

$$\big(i[P^D, F(hA_\Omega)]u_1, u_2\big) = \frac{h}{2}\int_{\mathbb{R}} e^{-|t|}\left(\frac{1}{t}\int_0^t (U(th)P_{sh}u_1, u_2)\, ds\right) dt.$$

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We omit the proof because it is exactly the same as the one in Section 3, up to the minor fact that one has now to choose h small enough rather than $h < 1$, since we have no explicit value for the rate of exponential growth of coefficients of admissible operators with respect to t (in Definition 2.2 and Section 3 this rate was explicit).

Proof of Theorem 6.1 We follow the steps of the proof of Theorem 1.4 and, when necessary, indicate how to use the analysis of this section to allow us to adapt the arguments.

Step 1: Expansion of the commutator. Expanding P_{sh} to the first order by the Taylor formula as in Section 4, we can write $i[P^D, F(hA_\Omega)]$ as the sum of

$$h(hA_\Omega - i)^{-1}P_0(hA_\Omega + i)^{-1} + h\big[P_0, (hA_\Omega - i)^{-1}\big](hA_\Omega + i)^{-1} =: I_h + II_h \tag{6.15}$$

and of a finite number of integrals of the form

$$III_h^\pm = h^2 \int_0^{\pm\infty} e^{-|t|} w^\pm(t)(hA_\Omega - i)^{-1}U(ht)C_{ht}^\pm(hA_\Omega + i)^{-1},$$

with $(C_t^\pm)_{t\in\mathbb{R}}$ admissible operators and $w^\pm$ polynomials. The latter are obtained by the integration by parts trick of Proposition 4.2, which is justified by Proposition 6.11. Using Proposition 6.8 and (6.9), we obtain

$$|(III_h^\pm u_1, u_2)| \lesssim h^2 \|\nabla(hA_\Omega + i)^{-1}u_1\|_{L^2}\|\nabla(hA_\Omega + i)^{-1}u_1\|_{L^2},$$

for all h small enough and $u_1, u_2 \in \mathrm{Dom}(P^D)$. The second term of (6.15) reads

$$II_h = -h^2(hA_\Omega - i)^{-1}[P_0, A_\Omega](h^2A_\Omega^2 + 1)^{-1},$$

where $[P_0, A_\Omega]$ is admissible by Proposition 6.11 (since is of the form $\big(-i\frac{d}{dt}P_t\big)_{|t=0}$). Therefore, by Proposition 6.8 and (6.10), we obtain

$$|(II_h u_1, u_2)| \lesssim h^2 \|\nabla(hA_\Omega + i)^{-1}u_1\|_{L^2}\|\nabla(hA_\Omega + i)^{-1}u_1\|_{L^2}.$$

Conclusion 1. There exist C and h_0 such that

$$\begin{aligned} \big|(i[P, F(hA_\Omega)]u_1, u_2) - (I_h u_1, u_2)\big| \leq \\ Ch^2\|\nabla(hA_\Omega + i)^{-1}u_1\|_{L^2}\|\nabla(hA_\Omega + i)^{-1}u_1\|_{L^2} \end{aligned} \tag{6.16}$$

for all $h \in (0, h_0]$ and $u_1, u_2 \in \mathrm{Dom}(P^D)$.

Step 2: Reduction to a problem in a bounded set. By (6.16) we have to focus on $(I_h u_1, u_2)$. Since P_0 is close to the Laplacian and of div-grad type near infinity, we may write, $P_0 = \widetilde{P}_0 + P_c$. P_c is a second order operator with compactly supported coefficients, hence automatically admissible, and $\widetilde{P}_0$ is a uniformly elliptic operator of div-gad type such that, for some $c > 0$,

$$(\widetilde{P}_0 v, v) \geq c\|\nabla v\|_{L^2}, \qquad v \in \mathrm{Dom}(P^D).$$

Conclusion 2. There exists $c > 0$ such that for all h small enough and all $u \in \mathrm{Dom}(P^D)$,

$$(I_h u, u) \geq ch\|\nabla(hA_\Omega + i)^{-1}u\|_{L^2}^2 + h\big(P_{\mathrm{c}}(hA_\Omega + i)^{-1}u, (hA_\Omega + i)^{-1}u\big). \tag{6.17}$$

We are thus left with the study of the last term of (6.17), which involves an operator with coefficients supported in a bounded subset of Ω.

Step 3: Spectral localization. We will eventually consider $(i[P^D, F(hA_\Omega)]u, u)$ with u such that $(P^D - z)u = f$, *i.e.,* $u = (P^D - z)^{-1} f$, and z close to zero. By considering $\varphi \in C_0^\infty(\mathbb{R})$ that is equal to 1 near 0, we may therefore assume that

$$u = \varphi(P^D/\lambda)u \tag{6.18}$$

after a suitable similar localization on f. Then $(P_{\mathrm{c}}(hA_\Omega + i)^{-1}u, (hA_\Omega + i)^{-1}u)$, *i.e.,* the last term of (6.17), reads

$$\begin{aligned} &\big(P_{\mathrm{c}}\varphi(P^D/\lambda)(hA_\Omega + i)^{-1}u, (hA_\Omega + i)^{-1}u\big) \\ &\qquad + \big(P_{\mathrm{c}}[(hA_\Omega + i)^{-1}, \varphi(P^D/\lambda)]u, (hA_\Omega + i)^{-1}u\big). \end{aligned} \tag{6.19}$$

We now claim that the commutator $[\varphi(P^D/\lambda), (hA_\Omega + i)^{-1}]$ satisfies the same estimate as in Proposition 5.4. More precisely, since P_{c} is admissible and using Proposition 6.8, the second term of (6.19) is bounded in modulus by

$$C\|\nabla[(hA_\Omega + i)^{-1}, \varphi(P^D/\lambda)]u\|_{L^2}\|\nabla(hA_\Omega + i)^{-1}u\|,$$

where the first factor can be treated as in Proposition 5.4 as follows. We use (5.17) with A replaced by A_Ω. We observe on one hand that

$$\begin{aligned} &\|\nabla(hA_\Omega + i)^{-1}(P^D - z)^{-1}(P^D + 1)^{1/2}\|_{L^2 \to L^2} \\ &\quad \lesssim \|\nabla(P^D - z)^{-1}(P^D + 1)^{1/2}\|_{L^2 \to L^2} \\ &\quad \lesssim \|(P^D)^{1/2}(P^D + 1)^{1/2}(P^D - z)^{-1}\|_{L^2 \to L^2} \\ &\quad \lesssim 1 + |\mathrm{Im}(z)|^{-1}, \end{aligned} \tag{6.20}$$

for $\mathrm{Re}(z)$ in any bounded set and $\mathrm{Im}(z) \neq 0$, which follows from the Spectral Theorem, (6.10), and (6.1). On the other hand we also have

$$\|(P^D + 1)^{-1/2}[A_\Omega, P^D](P^D - z)^{-1}v\|_{L^2} \lesssim (1 + |\mathrm{Im}(z)|^{-1})\|\nabla v\|_{L^2} \tag{6.21}$$

by the Spectral Theorem, since

$$\begin{aligned} &|(u_1, (P^D + 1)^{-1/2}[A_\Omega, P^D](P^D - z)^{-1}u_2)| \\ &\quad \lesssim \|\nabla(P^D + 1)^{-1/2}u_1\|_{L^2}\|\nabla(P^D - z)^{-1}u_2\|_{L^2} \\ &\quad \lesssim \|(P^D)^{1/2}(P^D + 1)^{-1/2}u_1\|_{L^2}\|(P^D)^{1/2}(P^D - z)^{-1}u_2\|_{L^2}, \end{aligned}$$

which follows from (6.1) and (6.10) (recall that $i[A_\Omega, P^D]$ is admissible). Using (6.20) and (6.21), one can easily prove the analogue of Proposition 5.4 in this context .

Conclusion 3. For all $\lambda > 0$, there exists C_λ such that, for all u satisfying (6.18) and all h small enough, we have

$$(6.22)\quad (I_h u, u) \geq \\ (ch - C_\lambda h^2)\|\nabla (hA_\Omega + i)^{-1}u\|^2_{L^2} + h((hA_\Omega + i)^{-1}u, P_c\varphi(P^D/\lambda)(hA_\Omega + i)^{-1}u).$$

Step 4: Compactness argument. By Proposition 6.8 and the compact support of the coefficients of P_c, the last term of (6.22) is bounded in modulus by

$$(6.23)\qquad Ch\|\nabla\chi(x)\varphi(P^D/\lambda)(hA_\Omega + i)^{-1}u\|_{L^2}\|\nabla\chi(x)(hA_\Omega + i)^{-1}u\|_{L^2}$$

for some $\chi \in C_0^\infty(\Omega)$ such that $P_c = \chi P_c \chi$. We then observe that if $\zeta \in C_0^\infty(\Omega)$, $|\alpha| \leq 1$ and $\phi \in C_0^\infty(\mathbb{R})$, the operator $\zeta(x)\partial^\alpha\phi(P^D)\langle x\rangle^N$ is compact for all $N > 0$ by standard estimates. We also observe that, for all $N > 1$, the operator $\langle x\rangle^{-N}(P^D)^{-1/2}$ is well defined and bounded on L^2, basically by the same argument as the one prior to Proposition 5.2, using the key estimate

$$\|\langle x\rangle^{-N}v\|_{L^2} \lesssim \|(P^D)^{1/2}v\|_{L^2},$$

which follows from (6.1) and (6.2) in dimension ≥ 3 or Proposition 6.2 in dimension 2. Thus, by choosing ϕ such that $\phi(P^D)\varphi(P^D/\lambda) = \varphi(P^D/\lambda)$ for all λ small enough, we have

$$\begin{aligned}\zeta(x)\partial^\alpha\varphi(P^D/\lambda) &= \big(\zeta(x)\partial^\alpha\phi(P^D)\langle x\rangle^N\big)\big(\langle x\rangle^{-N}(P^D)^{-1/2}\big)\varphi(P^D/\lambda)(P^D)^{1/2}\\ &= B\varphi(P^D/\lambda)(P^D)^{1/2},\end{aligned}$$

where B is compact. Thus $B\varphi(P^D/\lambda) \to 0$ in operator norm as $\lambda \to 0$. Using this property to make the first norm in (6.23) as small as we want, and using the previous steps we obtain the following.

Conclusion 4. There exists $\lambda > 0$ and $c' > 0$ such that for all $u \in \mathrm{Dom}(P^D)$ satisfying (6.18) and all $h > 0$,

$$(6.24)\qquad \big(i[P^D, F(hA_\Omega)]u, u\big) \geq c'h\|\nabla(hA_\Omega + i)^{-1}u\|^2_{L^2}.$$

Final step. To complete the proof, we recall that if $(P^D - z)u = f$, we have

$$(6.25)\quad 2\mathrm{Im}\Big(\big(F(hA_\Omega) \pm \tfrac{\pi}{2}\big)(hA_\Omega + i)^{-1}u, (hA_\Omega - i)f\Big) \geq \big(i[P^D, F(hA_\Omega)]u, u\big),$$

where the sign $\pm$ is chosen so that $\pm\mathrm{Im}(z) < 0$ (see the proof of Theorem 1.1). To be in position to use (6.24), we have to estimate the left-hand side of (6.25) in term of $\|(P^D)^{1/2}(hA_\Omega + i)^{-1}u\|_{L^2}$ or equivalently $\|\nabla(hA_\Omega + i)^{-1}u\|_{L^2}$. This is the analogue of Proposition 5.1. We need to check that

$$\|\nabla F(hA_\Omega)(hA_\Omega + i)^{-1}v\|_{L^2} \lesssim \|\nabla(hA_\Omega + i)^{-1}v\|_{L^2}. \tag{6.26}$$

This is obtained using a formula analogous to (5.2), namely

$$\big(i[i\partial_j, F(hA_\Omega)]u_1, u_2\big) = \frac{h}{2}\int_{\mathbb{R}} e^{-|t|}\left(\frac{1}{t}\int_0^t e^{sh}\big(U(h(t-s))[\partial_j, A_\Omega]U(hs)u_1, u_2\big)\,ds\right)dt,$$

whose right-hand side is bounded for h small enough by $Ch\|\nabla u_1\|_{L^2}\|u_2\|_{L^2}$, using (6.9) and the fact that $[\partial_j, A_\Omega]$ is the sum of a vector field with bounded coefficients and a compactly supported function (for which we can use (6.2) in dimension ≥ 3 or Proposition 6.2). Then this easily implies (6.26), and the proof of Theorem 6.1 is then completed as in Section 5. ■

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