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DUAL NUMBERS AND TOPOLOGICAL HJELMSLEV PLANES

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ABSTRACT. In 1929 J. Hjelmslev introduced a geometry over the dual numbers $\mathbb{R}+t\mathbb{R}$ with $t^2=0$. The dual numbers form a Hjelmslev ring, that is a local ring whose (unique) maximal ideal is equal to the set of 2 sided zero divisors and whose ideals are totally ordered by inclusion. This paper first shows that if we endow the dual numbers with the product topology of $\mathbb{R}^2$, then we obtain the only locally compact connected hausdorff topological Hjelmslev ring of topological dimension two. From this fact we establish that Hjelmslev's original geometry, suitably topologized, is the only locally compact connected hausdorff topological desarguesian projective Hjelmslev plane to topological dimension four.

1. **Introduction.** Projective Hjelmslev planes (PH-planes) are generalizations of ordinary projective planes where two points (lines) may be joined by (may intersect in) more than one line (point). The elements involved in multiple joinings or intersections are neighbours and the induced neighbour relations on points respectively lines are equivalence relations whose quotient spaces define an ordinary projective plane called the canonical image of the Hjelmslev plane. The plane is topological if its point and line sets are topological spaces so that the joining of non-neighbouring points and the intersection of non-neighbouring lines are continuous maps, and the neighbour relations are closed.

A desarguesian projective Hjelmslev plane (DPH-plane) is represented via homogeneous coordinates over a Hjelmslev ring (See [10] and [11]). The desarguesian plane is topological exactly when its coordinate ring is a topological ring whose jacobson radical is closed with a void interior ([13]). If we endow the dual numbers $\mathbb{R}+t\mathbb{R}$, where $\mathbb{R}$ is the reals and $t^2=0$, which were introduced by Study in 1903 ([17]), with the product topology from $\mathbb{R}^2$, then we can consider Hjelmslev's original geometry of 1929 over the dual numbers ([7]) as a topological DPH-plane.

In this paper we characterize topologically Hjelmslev's original geometry as

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the only locally compact connected hausdorff topological DPH-plane of topological dimension four (4.2). We accomplish this by initially proving that the dual numbers are the only locally compact connected hausdorff Hjelmslev ring of topological dimension two (2.1). In the process we also obtain several other characterizations of the geometry over the dual numbers. A PH-plane is uniform if any two neighbouring lines with a common intersection point have a common intersection with the neighbours of that point. Hjelmslev's geometry is then the only uniform topological DPH-plane whose canonical image is topologically isomorphic to the real projective plane.

2. **Topological Hjelmslev rings.** Let $\mathbb{R}$ be the reals. Then, the *real algebra of dual numbers is* $\mathbb{D}_2(\mathbb{R})=\mathbb{R}+\mathbb{R}t$ where t commutes with every real number and $t^2=0$. $\mathbb{D}_2(\mathbb{R})$ has the following algebraic properties.

(HR1) *The lattice of left ideals and the lattice of right ideals both form a chain.*

(HR2) *The zero-divisors, $\mathcal{N}$, form a two-sided ideal and each zero-divisior is a two-sided zero divisor.*

(HR3) *Each nonzero divisor is a unit.*

Any associative ring $\mathcal{H}$, with $0\neq 1$, satisfying the above three conditions is a *Hjelmslev ring* ([18]). $\mathcal{J}$ is the jacobson radical and $\mathcal{J}^{(m)}$ is the product of the ideal $\mathcal{J}$, in $\mathcal{H}$, by itself m-times ([1, page 20]). As in Artmann [2] we can also describe $\mathcal{D}_2(\mathbb{R})$ as the set of 2×2 real matrices of the form $\binom{a\ b}{0\ a}$. Then, from Lorimer [15] section seven, we have that $\mathcal{D}_2(\mathbb{R})$, endowed with the product topology from $\mathbb{R}^2$, is a *topological ring* (i.e. the maps $(A, B)\mapsto A-B$ and $(A, B)\mapsto AB$ are continuous) whose group of units under multiplication forms an open topological group. That is $\mathbb{D}_2(\mathbb{R})$ is a *Gelfand H-ring.*

By *topological dimension* we mean the *small inductive dimension* for regular hausdorff spaces. However, in our situation all spaces will be seen to be separable metric spaces where all dimension theories coincide ([6]).

Clearly, $\mathbb{D}_2(\mathbb{R})$ is a locally compact connected hausdorff Gelfand ring of topological dimension 2. We next show that it is the only such H-ring.

2.1. THEOREM. *Let $\mathcal{H}$ be a topological hausdorff H-ring whose radical $\mathcal{J}$ has a void interior and $\mathcal{J}\neq(0)$. Then, $\mathcal{H}$ is topologically and algebraically isomorphic to the real algebra of dual numbers if and only if $\mathcal{H}$ is a locally compact connected H-ring of topological dimension two.*

Proof. The necessity is clear from our previous comments.

Now, assume $\mathcal{H}$ is a locally compact connected $\mathcal{H}$-ring of topological dimension 2. By [15; 5.19.1] $\mathcal{H}=\mathbb{R}^n$ is a finite dimensional topological algebra over the reals. Moreover, by [15; 5.20], the radical $\mathcal{J}$ is nilpotent, say $\mathcal{J}^{(m)}=(0)$. Since $\mathcal{H}$ has topological dimension two and $\mathcal{J}\neq(0)$, we have $1<m\leq n=2$ or $m=n=2$. By [15; 5.16] $\mathcal{H}/\mathcal{J}$ is either the reals, the complex numbers or the quaternions. Now by [16; page 239] $\dim(\mathcal{H}/J)=\dim(\mathcal{H})-\dim(\mathcal{J})$; and in

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addition [15; 5.15] yields $\dim(\mathcal{J})=\dim(\mathcal{H}/\mathcal{J})$. Since $\dim(\mathcal{H})=2$, we conclude that $\dim(\mathcal{H}/\mathcal{J})=1$ or equivalently $\mathcal{H}/\mathcal{J}$ is the reals, $\mathbb{R}$. Let $v:\mathcal{H}\to\mathcal{H}/\mathcal{J}$ be the usual quotient map. Now $\mathcal{H}$ is a finite dimensional algebra and $\mathcal{H}/\mathcal{J}=\mathbb{R}$ is a separable algebra ([1; page 239]). By the Wedderburn Principal Theorem ([1; page 303] and [9; page 45]) $\mathcal{H}=\mathcal{S}+\mathcal{J}$ (vector space direct sum) where $v \mid \mathcal{S}:\mathcal{S}\to\mathcal{H}/\mathcal{J}=\mathbb{R}$ is an algebra isomorphism. Moreover, since $v:\mathcal{H}\to\mathcal{H}/\mathcal{J}$ is open and continuous ([8]) $v \mid \mathcal{S}$ is a homeomorphism. Thus, $\mathcal{H}=\mathbb{R}+\mathcal{J}$. *By* [15; 5.11], $\mathcal{J}=\mathcal{H}a=a\mathcal{H}$ where $a^2=0$, since $\mathcal{J}^{(2)}=(0)$. Hence, $\mathcal{J}=\mathbb{R}a$ and so $\mathcal{H}=\mathbb{R}+\mathbb{R}a$. Moreover, $\mathbb{R}a=\mathcal{J}=a\mathbb{R}$ and $\{1, a\}$ is a basis for $\mathcal{H}$ as a left and right vector space. This defines an automorphism $\sigma:\mathbb{R}\to\mathbb{R}$ where $ra=ar^\sigma$ for all reals r (see [4; page 288]). But, the reals have no non-trivial automorphisms and so $ra=ar$ for all reals r. $\mathcal{H}$ is thus the real algebra of dual numbers.

From the proof we also have,

2.2 Corollary. *If $\mathcal{H}$ is a locally compact connected hausdorff $\mathcal{H}$-ring of topological dimension* 2 *and the radical $\mathcal{J}$ has void interior then $\mathcal{H}$ is commutative and $\mathcal{J}^{(2)}=(0)$.*

3. **Topological Hjelmslev planes.** *Incidence structures and their homomorphisms* are defined as in Dembowski's "Finite Geometries". Blocks are called lines here. For any incidence structure $\langle\mathbb{P},\mathcal{L},I\rangle$: points are denoted by $P, Q, R,\ldots$ and lines $l, m, n, \ldots$.

An incidence structure $H=\langle\mathbb{P},\mathcal{L},E\rangle$ is a *projective Hjelmslev plane* (PH-plane) if it satisfies the following axioms ([10]):

(PH1) *Any two points lie on at least one line.*

We say two ponts P, Q are *neighbours* ($P\sim Q$) if there exist two distinct lines both containing P and Q.

(PH2) *The dual of* (PH1).

Dually, we speak of *neighbouring lines.*

(PH3) *There exists an ordinary projective plane H^* and an epimorphism $\phi:H\to H^*$ so that*

$$\phi(P)=\phi(Q)\Leftrightarrow P\sim Q$$

$$\phi(l)=\phi(m)\Leftrightarrow l\sim m.$$

$\bar{H}=\langle\mathbb{P}/\sim, \mathcal{L}/\sim, I\rangle$ is the canonical image of H, with equivalence classes $\bar{P}$ and $\bar{l}$, and $\pi:H\to\bar{H}$ is the canonical projection ([18]). Then, $\bar{H}$ is isomorphic to H^*.

If P and Q are non-neighbours ($P\not\sim Q$), then $P\vee Q$ *is the unique line through* P and Q. Dually, $l\wedge m$ *is the unique point incident with the non-neighbouring lines* l and m.

$H=\langle\mathbb{P},\mathcal{L},E\rangle$ is a *topological PH-plane* or TPH-plane

for short if its satisfies the following additional axioms ([13]).

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(TPH1) *$\mathbb{P}$ and $\mathcal{L}$ are topological spaces.*
(TPH2) *The maps $v:\mathbb{P}\times\mathbb{P}\setminus\sim\to\mathcal{L}$ and $\wedge:\mathcal{L}\times\mathcal{L}\setminus\sim\to\mathbb{P}$ are continuous.*
(TPH3) *$\sim$ is a closed set of $\mathbb{P}^2$ and $\mathcal{L}^2$.*

We say H *has a topological property* $(*)$ *if* $\mathbb{P}$ does. Finally, two TPH-planes are *topologically isomorphic* if they are related by a continuous and open collineation.

Let H be a TPH-plane. From [13] and [15] we have the following results.

3.1 *H is hausdorff if and only if all lines are closed subsets of $\mathbb{P}$.*

3.2 *$\bar{H}$, endowed with the quotient topologies from the neighbour relations, is a (hausdorff) topological projective plane; and not discrete if the interior of one neighbour class, $\bar{P}$, is void. Moreover, $\pi: H\to\bar{H}$ is a continuous-open map.*

4. **Topological Desarguesian PH-planes.** Let $\mathcal{H}$ be an H-ring (see Section 1). The *PH-plane over* $\mathcal{H}$, $H(\mathcal{H})$, defined via homogeneous coordinates ([11]) with some notation from [13, §8] is constructed as follows:

Let $\approx_l$ and $\approx_r$ be the equivalence relations on the set $\mathcal{H}\times\mathcal{H}\times\mathcal{H}\setminus\mathcal{J}\times\mathcal{J}\times\mathcal{J}=\mathcal{H}_3^*$ with equivalence classes $\langle xyz\rangle=\{h(x,y,z)\mid h\in\mathcal{H}\setminus\mathcal{J}\}$ and $[uvw]=\{(u,v,w)h\mid h\in\mathcal{H}\setminus\mathcal{J}\}$ respectively. Then $H(\mathcal{H})=\langle\mathcal{H}_3^*/\approx_l,\ \mathcal{H}_3^*/\approx_r, I\rangle$ with incidence defined by $\langle xyz\rangle I[uvw]\Leftrightarrow xu+yv+zw=0$ is a PH-plane. We say a PH-plane is *desarguesian* if it is isomorphic to some $H(\mathcal{H})$.

If $\mathcal{H}$ is a topological H-ring, then $H(\mathcal{H})$ inherits the quotient topologies from $\approx_l$ and $\approx_r$. Subsequently, we have

4.1. THEOREM. *Let $H(\mathcal{H})$ be a desarguesian* TPH-*plane with one neighbour class $\bar{P}$ having a void interior. Then $H(\mathcal{H})$ is a locally compact connected hausdorff* TPH-*plane of topological dimension $2n$ if and only if $\mathcal{H}$ is a locally compact connected hausdorff topological H-ring of topological dimension n. Moreover, $n=r2^s$ $(s=0,1,2;\ r=2,3,4\ldots)$.*

Proof. The theorem, without the dimension assertions, follows from [15; 6.2, 6.13]. If $H(\mathcal{H})$ is locally compact hausdorff, then by [15; 6.4], $H(\mathcal{H})$ is a topological manifold. The affine H-plane with [001] as its line at infinity ([10]) is topologically isomorphic to $\mathcal{A}(\mathcal{H})=\langle\mathcal{H}\times\mathcal{H},\ \mathbb{L}_1\cup\mathbb{L}_2, \|, \varepsilon\rangle$ (as defined in [12]) by [13; 8.3]. Since $\mathcal{A}(\mathcal{H})$ is an open-dense substructure of $H(\mathcal{H})$ ([14; 6.7]) the result now follows immediately. The last assertion is just [15; 5.18].

A PH-plane is *uniform* if *PIl, m* and $l\sim m$ implies $\bar{P}\cap l=\bar{P}\cap m$. In the Desarguesian situation, $H(\mathcal{H})$ is uniform if $\mathcal{J}^{(2)}=(0)$, where $\mathcal{J}$ is the jacobson radical of our H-ring $\mathcal{H}$. Finally the PH-plane $H(\mathcal{H})$ is *Pappian* if $\mathcal{H}$ is commutative. Now we may state our

4.2. MAIN THEOREM. *Let H be a hausdorff* TPH-*plane with the interior of one*

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neighbour class void. The following statements are equivalent.

(a) *H is a locally compact connected Desarguesian* PH-*plane of topological dimension four.*
(b) *H is topologically isomorphic to the* TPH-*plane over the dual numbers.*
(c) *H is a locally compact connected Pappian uniform* TPH-*plane of topological dimension four.*
(d) *H is a desarguesian* TPH-*plane and each $\bar{P}$ is locally compact connected of topological dimension two.*
(e) *H is a uniform desarguesian* TPH-*plane and $\bar{\mathcal{H}}$ is topologically isomorphic to the real projective plane.*

Proof. (a) $\Rightarrow$ (b). This follows from 4.1 and 2.1.

(b) $\Rightarrow$ (c). This is clear since $\mathbb{D}_2(\mathbb{R})$ is commutative and $\mathcal{J}^{(2)} = (0)$.

(c) $\Rightarrow$ (d). From 4.1, $H = H(\mathcal{H})$ where $\mathcal{H}$ is a locally compact connected H-ring of topological dimension two. Now $\overline{\langle 001\rangle} = \{\langle xyl\rangle \mid (x, y) \in \mathcal{J} \times \mathcal{J}\}$ is homeomorphic to $\mathcal{J} \times \mathcal{J}$ ([13; 8.3); and since $\mathcal{J}^{(2)} = (0)$, [15; 5.15] says that $\mathcal{J}$ is homeomorphic to $\mathcal{H}/\mathcal{J}$. Since the quotient map $v: \mathcal{H} \to \mathcal{H}/\mathcal{J}$ is open-continuous, $\mathcal{H}/\mathcal{J}$, and hence $\mathcal{J}$, are locally compact and connected. Consequently $\overline{\langle 001\rangle}$ is also locally compact and connected. Moreover, by 4.1 and [16; page 239] $\dim(\mathcal{H}/\mathcal{J}) = \dim(\mathcal{H}) - \dim(\mathcal{J}) = 2 - \dim(\mathcal{J})$. Because $\mathcal{J}$ is homeomorphic to $\mathcal{H}/\mathcal{J}$, we have that $\dim(\mathcal{H}/\mathcal{J}) = \dim(\mathcal{J}) = 1$ and so $\overline{\langle 001\rangle}$ has dimension two. By a change in coordinates we observe that any $\bar{P}$ has the form $\overline{\langle 001\rangle}$ and we are done.

(d) $\Rightarrow$ (e). From the above argument $\mathcal{J}$ is locally compact connected of topological dimension 1, and hence so is $\mathcal{H}/\mathcal{J}$. By [15; 5.16] $\mathcal{H}/\mathcal{J}$ is the real numbers and so $\bar{\mathcal{H}}$ is topologically isomorphic to $H(\mathcal{H}/\mathcal{J})$, the real projective plane.

(e) $\Rightarrow$ (a). Since $\mathcal{H}/\mathcal{J} = \mathbb{R}$, $\dim(\mathcal{H}/\mathcal{J}) = \dim(\mathcal{H}) - \dim(\mathcal{J})$ and $\mathcal{J}$ is again homeomorphic to $\mathcal{H}/\mathcal{J}$, it follows from [8; 5.25, 7.14] that $\mathcal{H}$ is locally compact connected of topological dimension 2. The result now follows from 4.1.

Finally, we emphasize that we have shown,

4.3. Corollary. *Every locally compact connected hausdorff desarguesian* TPH-*plane of topological dimension* 4 *is uniform and Pappian.*

4.4. Remark. In Cronheim [4] all finite Hjelmslev rings with jacobson radical nilpotent of degree 2—the *uniform H-rings*—and hence all finite uniform desarguesian PH-planes, were determined. Using the techniques of this paper, we may classify completely all locally compact connected hausdorff uniform H-rings and hence the corresponding TPH-planes. Indeed, our techniques allow one to determine all locally compact connected hausdorff desarguesian TPH-planes. We will present these results elsewhere.

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