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COMPACTNESS OF INVARIANT DENSITIES FOR FAMILIES OF EXPANDING, PIECEWISE MONOTONIC TRANSFORMATIONS

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1. Introduction. Let $I = [0, 1]$ and let $\mathcal{L}_1 \equiv \mathcal{L}_1(I, m)$ be the space of all integrable functions on I, where m denotes Lebesque measure on I. Let $\| \|_1$ be the $\mathcal{L}_1$-norm and let $\tau: I \longrightarrow I$ be a measurable, nonsingular transformation on I. Let

$$\mathcal{D} = \{f \in \mathcal{L}_1 : \|f\|_1 = 1, f(x) \geqq 0\}$$

denote the space of densities. The probability measure μ is invariant under τ if for all measurable sets A, $\mu(A) = \mu(\tau^{-1}A)$. The measure μ is absolutely continuous if there exists an $f^* \in \mathcal{D}$ such that for any measurable set A

$$\mu(A) = \int_A f^*(x)m(dx).$$

We refer to f^* as the invariant density of τ (with respect to m). It is well-known that f^* is a fixed point of the Frobenius-Perron operator $P_\tau : \mathcal{L}_1 \longrightarrow \mathcal{L}_1$ defined by

$$P_\tau f(x) = \frac{d}{dx} \int_{\tau^{-1}[0,x]} f(s)\tilde{m}(ds).$$

The operator P_τ has a simple physical interpretation. If x is a random variable with density f, then the random variable $\tau(x)$ has $P_\tau f$ as its density. Thus the orbit $\{P_\tau^n f\}$ describes the evolution of the density f in time.

For τ piecewise C^2 and expanding, it is shown in [**11**] that P_τ has a fixed point. (Other existence results are established in [**15**]. The proof in [**3**, Chapter **7** , Section 4] appears to have an error.) Since the invariant density f^* describes the asymptotic properties of orbits $\{\tau^i(X)\}$, the computation of f^* is an important problem. Unfortunately, solving the functional equation $P_\tau f = f$ explicitly is possible only in the very simplest cases. Practical procedures for approximating the solution of the functional equation directly do not seem to be known.

From the proof in [**11**], we obtain that any invariant density f^* is approximated by the sequence

$$\left\{ \frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k \right\},$$

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where $f \in \mathcal{L}_1$. Since

$$P_\tau f(x) = \sum_{i=1}^{N} f(\tau_i^{-1}(x))|(\tau_i^{-1})'x)|\chi_{\tau(I_i)}(x),$$

where χ_A is the characteristic function of the set A, it is a very complex procedure to use the sequence of partial sums. Even if P_τ is a constrictive operator [**10**] and

$$||P_\tau^k f - f^*||_1 \longrightarrow 0 \quad \text{as } k \longrightarrow \infty,$$

the iteration of P_τ is prohibitively complex.

In [**12**], Li carried out a suggestion of Ulam and proved a method of approximating f^* by using the eigenvectors of certain matrices. Drawbacks of the results in [**12**] are the requirement that $|\tau'(x)| > 2$, and the computation of the matrices, which are not the Frobenius-Perron operators of piecewise linear Markov maps, can be difficult. This idea is carried out for maps on the real line in [**2**].

The key inequality in [**11**] is:

$$(1) \qquad \bigvee_0^1 P_\tau f \leqq \gamma ||f||_1 + \beta \bigvee_0^1 f,$$

where

$$\beta = \frac{2}{\lambda} < 1, \quad \gamma = K + 2\left(\min_i |I_i|\right)^{-1},$$

$\{I_i\}$ being the intervals of smoothness of τ. The fact that γ depends on the partition makes it impossible to use (1) for families of maps whose defining partitions become finer and finer. It is this difficulty which prohibits the use of the stability results in [**7**, **9**], where it is essential that the number of intervals in the partitions of the approximating transformations be bounded. For special Renyi maps τ, it is shown in [**6**] that there exist piecewise linear Markov maps τ_n, $\tau_n \longrightarrow \tau$, such that the associated invariant densities f_n converge uniformly to f, the invariant density. The method relies heavily on bounded variation arguments and is lengthy. A more general procedure is outlined in [**8**], but there are major gaps in the proof and there appear to be errors. The main objective of this note is to give a correct version of the results stated in [**8**].

In Section 2 the main compactness result is proved. In Section 3, this result is used to prove the existence of sequences of piecewise linear Markov maps, whose densities converge to the invariant density of the map.

2. Strong compactness of invariant densities. The transformation $\tau : [0,1] \longrightarrow [0,1]$ is called piecewise expanding if there exist $0 = a_0 < a_1 < \ldots < a_N = 1$ and a constant $\lambda > 1$, such that for any $i = 0, 1, \ldots, N-1$

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(i) $\tau|_{(a_i,a_{i+1})}$ is of class C^1 and the limits $\tau'(a_i^+)$, $\tau'(a_{i+1}^-)$ exist (or are infinite)

(ii) $|\tau'(x)| \geqq \lambda > 1$ for $x \in (a_i, a_{i+1})$

(iii) $\left|\frac{1}{\tau'}\right|$ is a function of bounded variation

We shall denote by Q the set $\{a_0, a_1, \ldots, a_N\}$ and by $\mathcal{I}$ the partition of $[0, 1]$ into closed intervals with endpoints belonging to Q : $I_1 = [a_0, a_1], \ldots, I_N = [a_{N-1}, a_N]$.

In this section we shall prove the compactness of the set of invariant densities for any family of piecewise expanding maps which satisfy conditions (i) and (ii) uniformly. We do not assume that they have a common defining partition $\mathcal{I}$, although we need a weak supplementary condition on their defining partitions.

THEOREM 1. *Let $\{\tau_\alpha\}_{\alpha\in A}$ be a family of piecewise expanding transformations satisfying the following conditions:*

(1) *There exists a constant $\lambda > 1$ such that*

$$|\tau'_\alpha(x)| \geqq \lambda,$$

whenever the derivative exists for any $\alpha \in \mathcal{A}$;

(2) *There exists a constant $W > 0$ such that for any $\alpha \in \mathcal{A}$*

$$\operatorname{Var}\left|\frac{1}{\tau'_\alpha}\right| \leqq W;$$

(3) *There exists a constant $\delta > 0$ such that for any $\alpha \in \mathcal{A}$, there exists a finite partition $\mathcal{K}_\alpha$ such that for $I \in \mathcal{K}_\alpha, \tau_\alpha|_I$ is one-to-one, $\tau_\alpha(I)$ is an interval, and*

$$\min_{I\in\mathcal{K}_\alpha} \operatorname{diam}(I) > \delta.$$

(4) *For any $m \geqq 1$, there exists $\delta_m > 0$ such that if*

$$\mathcal{K}_\alpha^{(m)} = \bigvee_{j=0}^{m-1} \tau_\alpha^{-j}(\mathcal{K}_\alpha),$$

then

$$\min_{I\in\mathcal{K}_\alpha^{(m)}} \operatorname{diam}(I) \geqq \delta_m > 0.$$

Then, for any density f of bounded variation, there exists a constant V such that any $\alpha \in \mathcal{A}$, and any $k = 1, 2, 3, \ldots$

$$\operatorname{Var} P_{\tau_\alpha}^k f \leqq V.$$

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This implies that any $\tau_\alpha, \alpha \in A$, admits an invariant density f_α and that the set $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ is of uniformly bounded variation and hence is precompact in $\mathcal{L}_1$.

Remarks. (I) In a personal communication, Gerhard Keller pointed out to us that the proof of Theorem 1 does not work without assumption (4). Keller's example is given at the end of this section.

(II) The assumption (4) is a consequence of assumption (3) for families of transformations considered in Section 3, i.e., for families of Markov transformations associated with a piecewise expanding transformation τ.

(III) Although condition (3) may appear to be strong, it in fact holds for all the approximating maps of interest. Let τ be a fixed piecewise expanding transformation on a finite partition

$$\mathcal{K} = \{I_i\}_{i=1}^N.$$

Let τ_n be a tranformation which approximates $\tau : \tau_n$ is monotonic on each I_i, but $\tau_{n|I_i}$ is allowed to have discontinuities in its derivative, as shown for example in Figure 1. Then condition (3) is satisfied for

$$\mathcal{K}_n = \mathcal{K} = \{(0,.5),(.5,1)\}$$

and the family $\{\tau_n\}$. This idea is developed in detail in Section 3.

LEMMA 1. *If the family $\{\tau_\alpha\}_{\alpha\in A}$ satisfies the conditions of Theorem 1, then for any $m = 1, 2, \ldots$, the family $\{\tau_\alpha^m\}_{\alpha\in A}$ satisfies analogous conditions. Moreover, the new partitions*

$$\mathcal{K}_\alpha^{(m)} = \bigvee_{j=0}^{m-1} \tau_\alpha^{-1}(\mathcal{K}_\alpha)$$

satisfy

$$\max_{I\in\mathcal{K}_\alpha^{(m)}} \operatorname*{Var}_I \left|\frac{1}{(\tau_\alpha^m)'}\right| \leqq m\left(\frac{1}{\lambda}\right)^{m-1} W.$$

Proof. It is obvious that the conditions (1) and (2) will be satisfied for different constants. The condition (3) is satisfied by assumption (4). To prove the supplementary inequality, we let

$$W_k = \max_{I\in\mathcal{K}_\alpha^k} \operatorname*{Var}_I \left|\frac{1}{(\tau_\alpha^k)'}\right|$$

and prove

$$W_k \leqq k\left(\frac{1}{\lambda}\right)^{k-1} W$$

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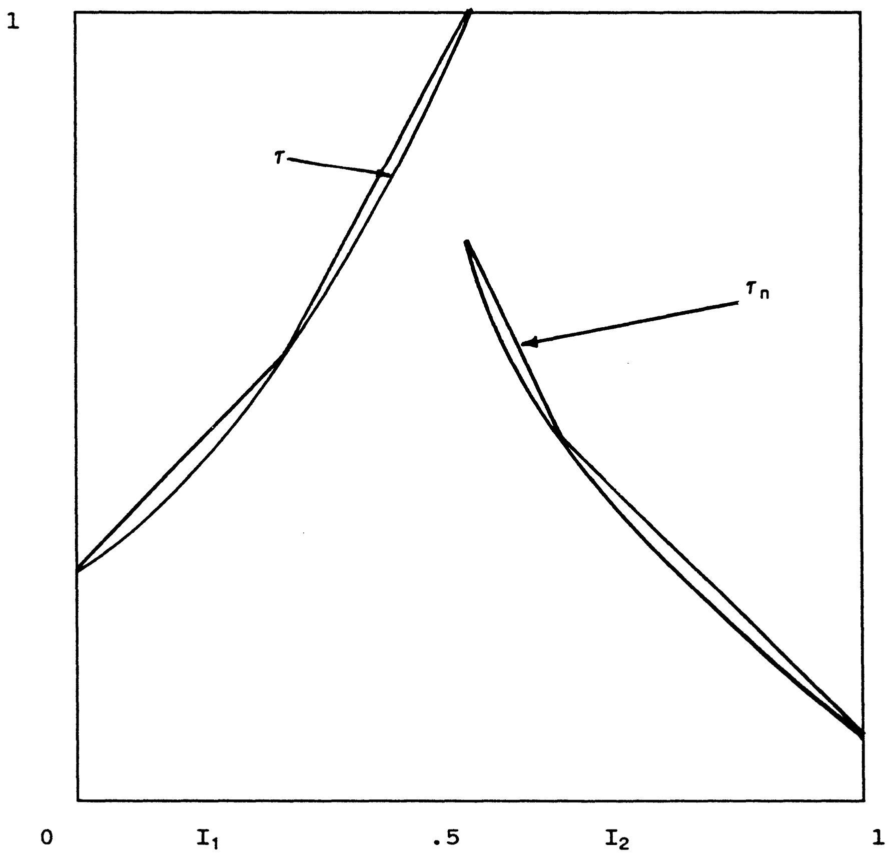

Figure 1.

by induction. Let $t_0 < t_1 < \ldots < t_l$ be any sequence of points in $I \in \mathcal{K}_\alpha^{(k+1)}$. Then

$$\sum_{j=0}^{l-1} \left| \frac{1}{(\tau_\alpha^{k+1})'(t_{j+1})} - \frac{1}{(\tau_\alpha^{k+1})'(t_j)} \right|$$

$$= \sum_{j=0}^{l-1} \left| \frac{1}{(\tau_\alpha^{k})'(\tau_\alpha t_{j+1}) \cdot \tau_\alpha'(t_{j+1})} - \frac{1}{(\tau_\alpha^{k})'(\tau_\alpha t_j) \cdot \tau_\alpha'(t_j)} \right|$$

$$\leqq \sum_{j=0}^{l-1} \left| \frac{1}{(\tau_\alpha^{k})'(\tau_\alpha t_{j+1}) \cdot \tau_\alpha'(t_{j+1})} - \frac{1}{(\tau_\alpha^{k})'(\tau_\alpha t_j) \cdot \tau_\alpha'(t_{j+1})} \right|$$

$$+ \left| \frac{1}{(\tau_\alpha^{k})'(\tau_\alpha t_j) \cdot \tau_\alpha'(t_{j+1})} - \frac{1}{(\tau_\alpha^{k})'(\tau_\alpha t_j) \cdot \tau_\alpha'(t_j)} \right|$$

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$$\leqq \frac{1}{\lambda}\sum_{j=0}^{l-1}\left|\frac{1}{(\tau_\alpha^k)'(\tau_\alpha t_{j+1})}-\frac{1}{(\tau_\alpha^k)'(\tau_\alpha t_j)}\right|$$

$$+\frac{1}{\lambda^k}\sum_{j=0}^{l-1}\left|\frac{1}{\tau'_\alpha(t_{j+1})}-\frac{1}{\tau'_\alpha(t_j)}\right|$$

$$\leqq \frac{1}{\lambda}W_k+\frac{1}{\lambda^k}W \leqq \frac{1}{\lambda}\frac{k}{\lambda^{k-1}}W+\frac{1}{\lambda^k}W.$$

Hence

$$W_{k+1} \leqq (k+1)\frac{1}{\lambda^k}W,$$

which completes the proof.

The following lemma is similar to a result in [**17**], and to Lemma 5 in [**14**].

LEMMA 2. *Let τ be a transformation satisfying the assumptions of Theorem* 1. *Let*

$$\eta = \max_{I\in\mathcal{K}} \operatorname*{Var}_I \left|\frac{1}{\tau'}\right|.$$

Then for any density f of bounded variation

$$\operatorname{Var} P_\tau f \leqq \left(\frac{3}{\lambda}+\eta\right)\operatorname{Var} f+\frac{2+n}{\delta}\|f\|_1.$$

Proof. For any $I \in \mathcal{K}$ we define $\phi_I = (\tau|_I)^{-1}$, and $A_I = \tau(I)$. Let f be a density of bounded variation. We will estimate the variation of $P_\tau f$.

Let $0 = t_0 < t_1 < \ldots < t_r = 1$. We have:

$$\sum_{j=1}^{r}|P_\tau f(t_j)-P_\tau f(t_{j-1})|$$

$$=\sum_{j=1}^{r}\left|\sum_{I\in\mathcal{K}} f(\phi_I(t_j))|\phi'_I(t_j)|\chi_{A_I}(t_j)\right.$$

$$\left.-\sum_{I\in\mathcal{K}} f(\phi_I(t_{j-1}))|\phi'_I(t_{j-1})|\chi_{A_I}(t_{j-1})\right|$$

$$\leqq \sum_{j=1}^{r}{}'\sum_{i\in\mathcal{K}}{}'\,|f(\phi_I(t_j))|\phi'_I(t_j)|-f(\phi_I(t_{j-1}))|\phi'_I(t_{j-1})||$$

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$$+\sum_{j=1}^{r}{}'' \sum_{I\in\mathcal{K}}{}'' |f(\phi_I(t_j))|\phi_I'(t_j)||$$

$$+\sum_{j=1}^{r}{}''' \sum_{I\in\mathcal{K}}{}''' |f(\phi_I(t_{j-1}))|\phi_I'(t_{j-1})||$$

where

$\sum'$ is taken over $1 \leqq j \leqq r$ and $I \in \mathcal{K}$ such that $t_j, t_{j-1} \in A_I$;
$\sum''$ is taken over $1 \leqq j \leqq r$ and $I \in \mathcal{K}$ such that $t_j \in A_I, t_{j-1} \notin A_I$;
$\sum'''$ is taken over $1 \leqq j \leqq r$ and $I \in \mathcal{K}$ such that $t_j \notin A_I, t_{j-1} \in A_I$.
The first sum can be estimated by:

$$\sum_{j=1}^{r}{}' \sum_{I\in\mathcal{K}}{}' |[f(\phi_I(t_j)) - f(\phi_I(t_{j-1})]\phi_I'(t_{j-1})|$$

$$+\sum_{j=1}^{r}{}' \sum_{I\in\mathcal{K}}{}' |f(\phi_I(t_{j-1}))|[\phi_I'(t_j) - \phi_I'(t_{j-1})]|$$

$$\leqq \frac{1}{\lambda}\operatorname{Var} f + \left(\operatorname{Var} f + \frac{1}{\delta}\int_0^1 |f| dx\right)\cdot \eta.$$

We have used the inequalities:

$$|f(\phi_I(t_{j-1}))| \leqq \inf_f |f| + \operatorname*{Var}_I f$$

$$\leqq \frac{1}{|I|}\int_I |f| dx + \operatorname*{Var}_I f$$

and

$$\sum_{j=1}^{r}{}' [\phi_I'(t_j) - \phi_I'(t_{j-1})] \leqq \operatorname*{Var}_I |\phi_I'| \leqq \eta.$$

Let $j(I)$ be the smallest j such that $t_j \in I$ and $j'(I)$ be the biggest j such that $t_j \in I$. The remaining two sums can then be estimated by:

$$\frac{1}{\lambda}\sum_{I\in\mathcal{K}}(|f(\phi_I(t_{j(I)}))| + |(f(\phi_I(t_{j'(I)}))|)$$

$$\leqq \frac{1}{\lambda}\left(2\operatorname{Var} f + \frac{2}{\delta}\int_0^1 |f| dx\right)$$

since, if $x, y \in I$, then

$$|f(x)| + |f(y)| \leqq 2\operatorname*{Var}_I f + 2\inf_I |f|.$$

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Consequently we have:

$$\operatorname{Var} P_\tau f \leqq \left(\frac{3}{\lambda} + \eta\right) \operatorname{Var} f + \frac{2+\eta}{\delta} \|f\|_1 .$$

Remark. Under the assumptions of Lemma 2, a little more careful reasoning gives us the following better estimate:

$$\operatorname{Var} P_\tau f \leqq \left(\frac{2}{\lambda} + \eta\right) \operatorname{Var} f + \frac{2+\eta}{\delta} \|f\|_1 .$$

See [**17**].

LEMMA 3. *If the conclusions of Lemma* 2 *is true for the family* $\{\tau_n^m\}_{n \geqq 1}$, *m a fixed positive integer, then it is true for the family* $\{\tau_n\}_{n \geqq 1}$ *itself.*

Proof. Let $\tau \in \{\tau_n\}_{n \geqq 1}$. It is enough to prove that if

$$\operatorname{Var} P_{\tau^m}^n f < V_1$$

for any n, and any f of bounded variation, then

$$\operatorname{Var} P_\tau^n f < V_2$$

for any n and some V_2. Let $n = k \cdot m + j \quad 0 \leqq j \leqq m-1$. We have

$$\begin{aligned} \operatorname{Var} P_\tau^n f &= \operatorname{Var} P_\tau^j P_{\tau^m}^k f \\ &\leqq \left(\frac{3}{\lambda} + \eta\right) \operatorname{Var}\left(P_\tau^{j-1} P_{\tau^m}^k f\right) + \frac{3+\eta}{\delta} \|f\|_1 \leqq \dots \\ &\leqq \left(\frac{3}{\lambda} + \eta\right)^j V_1 + \left(\frac{3}{\lambda} + \eta\right)^{j-1} \left(\frac{2+\eta}{\delta}\right) \|f\|_1 + \dots + \left(\frac{2+\eta}{\delta}\right) \|f\|_1 . \end{aligned}$$

It is easy to see that we can find an appropriate V_2.

Proof of Theorem 1. Let us fix a positive integer m such that

$$(*) \qquad \left(\frac{3}{\lambda^m} + m \frac{1}{\lambda^{m-1}} W\right) < 1.$$

The family $\{\tau_\alpha^m\}_{\alpha \in A}$ satisfies the conditions of Theorem 1 with

$$\tilde{\lambda} \leqq \frac{1}{\lambda^m}, \quad \tilde{\eta} \leqq m \frac{1}{\lambda^{m-1}} . W, \quad \text{and } \tilde{\delta} \geqq \delta_m$$

(Lemma 1). By Lemma 2, we have that for any $\alpha \in \mathcal{A}$ and any density of bounded variation:

$$\operatorname{Var} P_{\tau_\alpha^m} f \leqq r \operatorname{Var} f + D,$$

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where

$$r = \left(\frac{3}{\lambda^m} + m\frac{1}{\lambda^{m-1}}W\right) < 1 \quad \text{and} \quad D = \frac{2 + m\lambda^{-m+1} \cdot W}{\delta_m}.$$

This implies that there exists a constant V_1 such that:

$$\operatorname{Var} P^k_{\tau^m_\alpha} f \leqq V_1$$

for any $k = 1, 2, \ldots$, and any $\alpha \in \mathcal{A}$, and hence the conclusion of Theorem 1 for the family $\{\tau^m_\alpha\}_{\alpha \in A}$. Applying Lemma 3 completes the proof.

COROLLARY 1. *The conclusion of Theorem* 1 *is valid for families of* $\{\tau_\alpha\}_{\alpha \in \mathcal{A}}$ *such that for some fixed positive integer n, the family* $\{\tau^n_\alpha\}_{\alpha \in \mathcal{A}}$ *satisfies the conditions of the theorem.*

Example. (Gerhard Keller) Let

$$\tau_\alpha(x) = \begin{cases} b - \alpha + x(1 - b + \alpha)/b, & 0 \leqq x < b \\ (x - b/(1 - b) & , \quad x \leqq b \leqq 1 \end{cases}$$

for $\alpha \in \mathcal{A} = (0, b)$. It is easy to see that $\{\tau_\alpha\}_{\alpha \in \mathcal{A}}$ satisfies assumptions (1), (2), (3) of Theorem 1. However, it does not satisfy assumption (4) since

$$\min_{I \in \mathcal{K}^{(2)}_\alpha} \operatorname{diam}(I) = \alpha$$

and therefore tends to 0 as α goes to 0. This example, therefore, shows that our proof of Theorem 1 does not work without assumption (4).

3. Approximation by Markov maps. The piecewise expanding transformation τ is called Markov with respect to the partition $\mathcal{I}$ if it transforms the set Q of endpoints of intervals of $\mathcal{I}$ into itself, i.e., $\tau(Q) \subset Q$. This condition implies that if

$$\operatorname{int}(\tau(I_i) \cap I_j) \neq \phi,$$

then $\tau(I_i) \supset I_j$ for $I_i, I_j \in \mathcal{I}$. For the Markov transformation τ, we define a transition matrix $M = (m_{ij})$, $1 \leqq i, j \leqq N$ as follows:

$$m_{ij} = \begin{cases} 0 & \text{if int } (\tau(I_i) \cap I_j) = \phi \\ 1 & \text{if int } (\tau(I_i) \cap I_j \neq \phi. \end{cases}$$

The matrix M is called primitive of these exists a positive integer m such that $M^m > 0$. If $M^m > 0$, then $\tau^m(I_i) = [0, 1]$ for $i = 1, \ldots, N$.

Let $Q^{(0)} = Q, \mathcal{I}^{(0)} = \mathcal{I}$. We define

$$Q^{(k)} = \bigcup_{j=0}^{k} \tau^{-j}(Q^{(0)}), \quad k = 1, 2, \ldots$$

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$$\mathcal{J}^{(k)} = \bigvee_{j=0}^{k} \tau^{-j}(\mathcal{J}^{(0)}), \quad k = 1, 2, \dots$$

It is easy to see that $Q^{(k)}$ is the set of endpoints of intervals belonging to $\mathcal{J}^{(k)}$.

For a piecewise expanding map τ, we define a sequence of piecewise expanding Markov transformations τ_n (with respect to $\mathcal{J}^{(n)}$, $n = 1, 2, \dots$) associated with τ, as follows:

a) If $I = [a, b] \in \mathcal{J}^{(n)}$ and $I \cap Q^{(0)} = \phi$, then $\tilde{\tau}_n|_I$ is a C^1 monotonic function such that

$$\tilde{\tau}_n(a) = \tau(a), \quad \tilde{\tau}_n(b) = \tau(b),$$

$$\inf_I |\tilde{\tau}'_n(x)| \geqq \inf_I |\tau'(x)|, \quad \text{and}$$

$$\operatorname*{Var}_I \left| \frac{1}{\tilde{\tau}'_n} \right| \leqq \operatorname*{Var}_I \left| \frac{1}{\tau'} \right| .$$

b) If $I = [a_i, q] \in \mathcal{J}^{(n)}$, $a_i \in Q^{(0)}$, then if $\tau|_I$ is increasing, we take $q_{a_i}^{(n)} \in Q^{(n)}$ such that

$$q_{a_i}^{(n)} \leqq \tau(a_i) \quad \text{and} \quad (q_{a_i}^{(n)}, \tau(a_i)) \cap Q^{(n)} = \phi.$$

If $\tau|_I$ is decreasing, we take the point $q_{a_i}^{(n)} \in Q^{(n)}$ such that

$$q_{a_i}^{(n)} \geqq \tau(a_i) \quad \text{and} \quad (\tau(a_i), q_{a_i}^{(n)}) \cap Q^{(n)} = \phi.$$

We define $\tilde{\tau}_n|_I$ as a C^1 monotonic function such that

$$\tilde{\tau}_n(a_i) = q_{a_i}^{(n)}, \quad \tilde{\tau}_n(q) = \tau(q),$$

$$\inf_I |\tilde{\tau}'_n(x)| \geqq \inf_I |\tau'(x)|$$

and

$$\operatorname*{Var}_I \left| \frac{1}{\tilde{\tau}'_n} \right| \leqq \operatorname*{Var}_I \left| \frac{1}{\tau'} \right| .$$

c) If $I = [q, a_i], a_i \in Q^{(0)}$, the definition is analogous to the one given in b). In either case,

$$\tilde{\tau}_n(I) \supseteq \tau(I), \quad \min_I |\tilde{\tau}'_n|_I(x)| \geqq \min_I |\tau'(x)| \quad \text{and}$$

$$\operatorname*{Var}_I \left| \frac{1}{\tilde{\tau}'_n} \right| \leqq \operatorname*{Var}_I \left| \frac{1}{\tau'} \right| ,$$

for any $I \in \mathcal{J}^{(n)}$

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It is easy to see that $\tilde{\tau}_n$ is a piecewise expanding Markov transformation with respect to the partition $\mathcal{J}^{(n)}$. We shall now prove that the family of Markov transformations associated with a piecewise expanding transformation τ satisfies the assumptions of Theorem 1, which implies that the family $\{f_n\}_{n\geqq 1}$ of their invariant densities forms a compact set in $\mathcal{L}_1$

THEOREM 2. *Let τ be a piecewise expanding transformation, and $\{\tilde{\tau}_n\}_{n\geqq 1}$ a family of Markov maps associated with τ. Then any $\tilde{\tau}_n$, $n = 1, 2, \ldots$, admits an invariant density f_n and the set $\{f_n\}_{n\geqq 1}$ is precompact in $\mathcal{L}_1$.*

Proof. It is enough to show that the assumptions of Theorem 1 are satisfied. The conditions (1) and (2) are satisfied by construction. For any $n = 1, 2, \ldots,$ we can choose $K_n = \mathcal{J}$. It is then obvious that condition (3) is satisfied.

We now define two specific families of Markov transformations associated with a piecewise expanding transformation τ.

Kosyakin-Sandler approximations of τ. For any $n = 1, 2, \ldots, \tau_n$ is defined as a Markov map with respect to $\mathcal{J}^{(n)}$, associated with τ, satisfying the following conditions:

(i) if $I \in \mathcal{J}^n$ and $I \cap Q^0 = \phi$, then $\tau_n|_I = \tau|_I$;

(ii) if $I \in \mathcal{J}^{(n)}$ and $I \cap Q^{(0)} \neq \phi$, then $\tau_n|_I$ is a linear function.

Piecewise-linear Markov approximation of τ. For any $n = 1, 2, \ldots, \tau_n$ is defined as a Markov transformation (with respect to $\mathcal{J}^{(n)}$) associated with τ which is linear on any $I \in \mathcal{J}^{(n)}$.

It is easy to see that both families are really families of Markov maps associated with τ.

We now prove a result which will be used in the sequel.

LEMMA 4. *Let τ be a piecewise expanding transformation, and $\{\tau_n\}_{n\geqq 1}$ a family of Markov maps associated with τ. Moreover, we assume that $\tau_n \longrightarrow \tau$ uniformly on the set*

$$[0, 1] - \bigcup_{k\geqq O} \tau^{-k} Q^{(0)},$$

and $\tau'_n \longrightarrow \tau'$ in $\mathcal{L}_1$ as $n \longrightarrow \infty$. Then for any $f \in \mathcal{L}_1$,

$$\|P_{\tau_n} f - P_\tau f\|_1 \longrightarrow 0$$

as $n \longrightarrow \infty$.

Proof. It is enough to prove the convergence to 0 for any continuous function. Let f be a continuous function. For any $n = 1, 2, \ldots$, and any $I \in \mathcal{K} = \mathcal{K}_n = \mathcal{J}$ we define

$$\phi_{n,I} = (\tau_{n|_I})^{-1} \quad \text{and} \quad A_{n,I} = \tau_n(I),$$

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as well as

$$\phi_I = (\tau|_I)^{-1} \quad \text{and} \quad A_I = \tau(I).$$

We have

$$\|P_{\tau_n} f - P_\tau f\|_1$$

$$= \int_0^1 |\sum_{I \in \mathcal{I}} f(\phi_{n,I}(x))|\phi'_{n,I}(x)|\chi_{A_{n,I}}(x)$$

$$- \sum_{i \in \mathcal{I}} f(\phi_I(x))|\phi'_I(x)|\chi_{A_I}(x)|dx$$

$$\leqq \int_0^1 \sum_{I \in \mathcal{I}} |f(\phi_{n,I}(x))|\phi'_{n,I}(x)| - f(\phi_I(x))|\phi'_I(x)|| \chi_{A_I}(x)dx$$

$$+ \int_0^1 \sum_{I \in \mathcal{I}} |f(\phi_{n,I}(x)\phi'_{n,I}(x)| \chi_{A_{n,I} \setminus A_I}(x)dx.$$

The first integral is less than:

$$\sum_{I \in \mathcal{I}} \left(\int_0^1 \left| f(\phi_{n,I}(x)) - f(\phi_I(x)) \right) \right| |\phi'_I(x)|\chi_{A_I}(x)dx$$

$$+ \int_0^1 |\phi'_{n,I}(x) - \phi'_I(x)| \, |f(\phi_{n,I}(x))|\chi_{A_I}(x)dx)$$

$$\leqq (\#\mathcal{I}) \left[w_f \left(\sup_{x \in A_I} |\phi_{n,I}(x) - \phi_I(x)| \right) \sup |\phi'_I \right.$$

$$\left. + \sup |f| \int_0^1 |\phi'_{n,I}(x) - \phi'_I(x)|\chi_{A_I}(x)dx \right] \longrightarrow 0$$

as $n \longrightarrow \infty$, where w_f is the modulus of continuity of f and $\#\mathcal{I}$ denotes the cardinality of $\mathcal{I}$. We have used the fact that if $\tau'_n \longrightarrow \tau'$ in $\mathcal{L}_1$ then for any $I \in \mathcal{I}$:

$$\phi'_{n,I}(x)\chi_{A_I}(x) \longrightarrow \phi'_I(x)\chi_{A_I}(x) \quad \text{in } \mathcal{L}_1.$$

The second integral is less than

$$(\#\mathcal{I}) \cdot \sup |f| \sup\{|\phi'_{n,I}| : n = 1, \ldots, I \in \mathcal{I}\} \sup_I \operatorname{diam}(A_{n,I} \setminus A_I).$$

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Since $\sup\{|\phi'_{n,i}| : n = 1, \ldots I \in \mathcal{I}\}$ is bounded (the variation of $|1/\tau'_n|$ is uniformly bounded by construction) and

$$\operatorname{diam}(A_{n,I} \backslash A_I) \leqq 1/\lambda^n \longrightarrow 0,$$

the second integral also tends to 0 as $n \longrightarrow +\infty$.

Now we will prove that under the assumptions of Lemma 4 any limit points of the family of τ_n-invariant densities $\{f_n\}$ is an invariant density for τ.

THEOREM 3. *Let τ be a piecewise expanding transformation, and $\{\tau_n\}_{n\geqq 1}$ a family of Markov maps associated with τ. Moreover we assume that $\tau_n \longrightarrow \tau$ uniformly on the set*

$$[0, 1] \backslash \bigcup_{k\geqq 0} Q^{(k)}$$

and $\tau'_n \longrightarrow \tau'$ in $\mathcal{L}_1$ as $n \longrightarrow \infty$. By Theorem 2, any τ_n has an invariant density f_n and $\{f_n\}_{n\geqq 1}$ is a precompact set in L_1. We claim that any limit point of $\{f_n\}_{n\geqq 1}$ is an invariant density of τ.

Proof. Let f be a limit point of $\{f_n\}_{n\geqq 1}$. We can assume that $f_n \longrightarrow f$ to avoid complicated notations. We will prove that $P_\tau f = f$. We have

$$\|P_\tau f - f\|_1 \leqq \|P_\tau f - P_{\tau_n} f\|_1 + \|P_{\tau_n} f - P_{\tau_n} f_n\|_1$$

$$+\|P_{\tau_n} f_n - f_n\|_1 + \|f_n - f\|_1.$$

The first summand goes to 0 by Lemma 4. The second one is less then

$$\|P_{\tau_n}\|_1 \|f - f_n\|_1$$

and goes to 0. The third summand is actually equal to 0 and the fourth one goes to 0 by assumption. This completes the proof.

It is easy to see that our special families approximating τ, namely the Kosyakin-Sandler Markov approximations to τ and the piecewise-linear Markov approximations to τ, satisfy the assumptions of Theorem 3. We, therefore, obtain the following two Propositions:

PROPOSITION 1. *Let τ be a piecewise expanding transformation. Let $\{\tau_n\}_{n\geqq 1}$ be the family of Kosyakin-Sandler Markov approximations of τ. Then any τ_n, $n = 1, 2, \ldots,$ admits an invariant density f_n, the set $\{f_n\}_{n\geqq 1}$ is precompact in $\mathcal{L}_1$ and any of its limit points is an invariant density of τ.*

PROPOSITION 2. *Let τ be a piecewise expanding transformation. Let $\{\tau_n\}_{n\geqq 1}$ be a family of piecewise-linear Markov approximations of τ. Then the conclusion of Proposition 1 holds for this family.*

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We give an example which shows that condition (2) in Theorem 1 cannot be ommited (possibly it can be replaced by some weaker condition).

Example. In [**5**] there is given an example of a piecewise C^1 and expanding transformation τ of an interval without absolutely continuous invariant measure. Actually the transformation τ is Markov and transforms any of three intervals of the defining partition $\mathcal{J}$ on the whole interval.

Let us consider a sequence of piecewise linear Markov approximations of τ: $\{\tau_n\}_{n\geqq 1}$. They satisfy assumptions (1) and (3) of Theorem 1. They also satisfy the assumptions of Theorem 3. Both these facts imply that the set $\{f_n\}_{n\geqq 1}$ (where f_n is an invariant density for τ_n, $n = 1, 2, \ldots$) cannot be precompact in $\mathcal{L}_1$. If it were precompact, any limit point of it would be an invariant density of τ, and τ has no invariant density. Actually the set $\{f_n\}_{n\geqq 1}$ cannot even be weakly precompact in $\mathcal{L}_1$, by analogous argument using Lemma I of [**4**].

Final remarks. (1) Propositions 1 and 2 provide a useful method for approximating the Lyapunov exponent. in [**1**], the Kosyakin-Sandler approximations are used to obtain a sequence of matrices whose left eigenvectors, when viewed as function on [0,1], approximate the invariant density of the tranformation, and are used to approximate the Lyapunov exponent.

(2) The ideas of this paper are carried over to non-expanding maps in [**4**].

(3) The idea of the regularity functional which is used in [**13**, p. 37– 38] in place of variation, is not applicable in our case. When f is piecewise C^1 on possibly countably many pieces, the condition

$$\left|\frac{f'}{f}\right| \leqq M$$

does not imply

$$e^{-M} \leqq f(x) \leqq e^M,$$

as erroneously concluded in [**13**]. Hence we cannot obtain a uniform bound on $\{f_\alpha\}_{\alpha\in A}$, the invariant densities of the family $\{\tau_\alpha\}_{\alpha\in A}$.

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