European Journal of Applied Mathematics (2024), 1–29
doi:10.1017/S0956792524000810

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PAPER

Discrete-to-continuum limits of optimal transport with linear growth on periodic graphs

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Received: 30 November 2023; **Revised:** 11 September 2024; **Accepted:** 30 October 2024

Keywords: optimal transport; discrete-to-continuum; homogenisation; linear growth; gamma-convergence

Abstract

We prove discrete-to-continuum convergence for dynamical optimal transport on $\mathbb{Z}^d$-periodic graphs with cost functional having linear growth at infinity. This result provides an answer to a problem left open by Gladbach, Kopfer, Maas, and Portinale (Calc Var Partial Differential Equations 62(5), 2023), where the convergence behaviour of discrete boundary-value dynamical transport problems is proved under the stronger assumption of superlinear growth. Our result extends the known literature to some important classes of examples, such as scaling limits of 1-Wasserstein transport problems. Similarly to what happens in the quadratic case, the geometry of the graph plays a crucial role in the structure of the limit cost function, as we discuss in the final part of this work, which includes some visual representations.

1. Introduction

In the Euclidean setting, the Benamou–Brenier [5] formulation of the distance on the space $\mathscr{P}_2(\mathbb{R}^d)$ known as *2-Wasserstein* or *Kantorovich–Rubinstein* distance is given by the minimisation problem

$$\mathbb{W}_2(\mu_0,\mu_1)^2 = \inf\left\{ \int_0^1 \int_{\mathbb{R}^d} \frac{|\nu_t|^2}{\mu_t}\,\mathrm{d}x\,\mathrm{d}t : \partial_t\mu_t + \nabla\cdot\nu_t = 0, \quad \mu_{t=0}=\mu_0, \quad \mu_{t=1}=\mu_1 \right\}, \tag{1.1}$$

for every $\mu_0,\mu_1 \in \mathscr{P}_2(\mathbb{R}^d)$. The Partial Differential Equation (PDE) constraint is called *continuity equation* (we write $(\boldsymbol{\mu},\boldsymbol{\nu}) \in \mathsf{CE}$ when $(\boldsymbol{\mu},\boldsymbol{\nu})$ is a solution). Over the years, the Benamou–Brenier formula (1.1) has revealed significant connections between the theory of optimal transport and different fields of mathematics, including partial differential equations [29], functional inequalities [35], and the novel notion of Lott–Sturm–Villani's synthetic Ricci curvature bounds for metric measure spaces [30, 31, 36, 37]. Inspired by the dynamical formulation (1.1), in independent works, Maas [32] (in the setting of Markov chains) and Mielke [33] (in the context of reaction-diffusion systems) introduced a notion of optimal transport in discrete settings structured as a dynamical formulation *à la* Benamou–Brenier as in (1.1). One of the features of this discretisation procedure is the replacement of the continuity equation with a discrete counterpart: when working on a (finite) graph $(\mathcal{X},\mathcal{E})$ (resp. vertices and edges), the discrete continuity equation reads

$$\partial_t m_t(x) + \sum_{y\sim x} J_t(x,y) = 0, \quad \forall x\in\mathcal{X}, \quad \big(\text{we write } (\boldsymbol{m},\boldsymbol{J}) \in \mathcal{CE}_{\mathcal{X}}\big)$$

where (m_t, J_t) corresponds to discrete masses and fluxes (s.t. $J_t(x,y) = -J_t(y,x)$). Maas' proposed distance $\mathcal{W}$ [32] is obtained by minimising, under the above constraint, a discrete analogue of the Benamou–Brenier action functional with reference measure $\pi \in \mathscr{P}(\mathcal{X})$ and weight function $\omega \in \mathbb{R}_+^{\mathcal{E}}$,

https://doi.org/10.1017/S0956792524000810 Published online by Cambridge University Press

of the form

$$\int_0^1 \frac{1}{2} \sum_{(x,y)\in\mathcal{E}} \frac{|J_t(x,y)|^2}{\widehat{r}_t(x,y)} \omega(x,y)\,\mathrm{d}t, \quad \text{where} \quad \widehat{r}_t(x,y) := \theta_{\log}(r_t(x), r_t(y)), \quad r_t(x) := \frac{m_t(x)}{\pi(x)},$$

and where $\theta_{\log}(a,b) := \int_0^1 a^s b^{1-s}\,\mathrm{d}s$ denotes the 1-homogeneous, positive mean called *logarithmic mean*. With this particular choice of the mean, it was proved [32], [33] (see also [6]) that the discrete heat flow coincides with the gradient flow of the relative entropy with respect to the discrete distance $\mathcal{W}$. In discrete settings, the equivalence between static and dynamical optimal transport breaks down, and the latter stands out in applications to evolution equations, discrete Ricci curvature and functional inequalities [9, 15, 34, 16, 11, 18, 14]. Subsequently, several contributions have been devoted to the study of the scaling behaviour of discrete transport problems, in the setting of discrete-to-continuum approximation problems. The first convergence results were obtained in [21] for symmetric grids on a d-dimensional torus, and by [20] in a stochastic setting. In both cases, the authors obtained convergence of the discrete distances towards $\mathbb{W}_2$ in the limit of the discretisation getting finer and finer.

Nonetheless, it turned out that the geometry of the graph plays a crucial role in the game. A general result was obtained in [24], where it is proved that the convergence of discrete distances associated with finite-volume partitions with vanishing size to the 2-Wasserstein space is substantially equivalent to an *asymptotic isotropy condition* on the mesh. The first complete characterisation of limits of transport costs on periodic graphs (Figure 1) in arbitrary dimension for general action functionals (not necessarily quadratic) was established in [22, 23]: in this setting, the limit action functional (more precisely, the energy density) can be explicitly characterised in terms of a cell formula, which is a finite-dimensional constrained minimisation problem depending on the initial graph and the cost function at the discrete level. The *action functionals* considered in [23] are of the form

$$(\boldsymbol{\mu},\boldsymbol{\nu}) \in \mathsf{CE} \quad \mapsto \quad \mathbb{A}(\boldsymbol{\mu},\boldsymbol{\nu}) := \int_{(0,1)\times\mathbb{T}^d} f\big(\rho, j\big)\,\mathrm{d}\mathscr{L}^{d+1} + \int_{(0,1)\times\mathbb{T}^d} f^\infty\big(\rho^\perp, j^\perp\big)\,\mathrm{d}\boldsymbol{\sigma}, \tag{1.2}$$

where we used the Lebesgue decomposition

$$\boldsymbol{\mu} = \rho\mathscr{L}^{d+1} + \boldsymbol{\mu}^\perp, \quad \boldsymbol{\nu} = j\mathscr{L}^{d+1} + \boldsymbol{\nu}^\perp, \quad \text{and} \quad \boldsymbol{\mu}^\perp = \rho^\perp\boldsymbol{\sigma}, \quad \boldsymbol{\nu}^\perp = j^\perp\boldsymbol{\sigma}, \quad \big(\boldsymbol{\sigma} \perp \mathscr{L}^{d+1}\big)$$

and where the *energy density* $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is some given convex, lower semicontinuous function with *at least linear growth*, i.e. satisfying

$$f(\rho, j) \geq c|j| - C(\rho+1), \qquad \forall \rho \in \mathbb{R}_+ \quad \text{and} \quad j \in \mathbb{R}^d, \tag{1.3}$$

whereas f^∞ denotes its *recession function* (see (2.2) for the precise definition). The choice $f(\rho,j) := |j|^2/\rho$ corresponds to the $\mathbb{W}_2$ distance. At the discrete level, on a locally finite connected graph $(\mathcal{X},\mathcal{E})$ embedded in $\mathbb{R}^d$, the natural counterpart is represented by action functionals of the form

$$(\boldsymbol{m},\boldsymbol{J}) \in \mathcal{CE}_{\mathcal{X}} \quad \mapsto \quad \mathcal{A}(\boldsymbol{m},\boldsymbol{J}) := \int_0^1 F(m,J)\,\mathrm{d}t, \tag{1.4}$$

for a given lower semicontinuous, convex, and local *cost function* F, which also has at least linear growth with respect to the second variable (see (2.8) for the precise definition).

The main result in [23] is the Γ-convergence for constrained functionals as in (1.4), after a suitable rescaling of the graph $\mathcal{X}_\varepsilon := \varepsilon\mathcal{X}$, $\mathcal{E}_\varepsilon := \varepsilon\mathcal{E}$, and of the cost F_ε (and associated action $\mathcal{A}_\varepsilon$), in the framework of $\mathbb{Z}^d$-periodic graphs. In particular, the limit action is of the form (1.2), where the energy density $f = f_{\mathrm{hom}}$ is given in terms of a *cell formula*, explicitly reading

$$f_{\mathrm{hom}}(\rho, j) := \inf\left\{F(m,J) : (m,J) \in \mathsf{Rep}(\rho,j)\right\}, \qquad \rho \in \mathbb{R}_+, \ \ j \in \mathbb{R}^d,$$

where $\mathsf{Rep}(\rho,j)$ denotes the set of discrete *representatives* of ρ and j, given by all $\mathbb{Z}^d$-periodic functions $m : \mathcal{X} \to \mathbb{R}_+$ with

$$\sum_{x\in\mathcal{X}\cap[0,1)^d} m(x) = \rho$$

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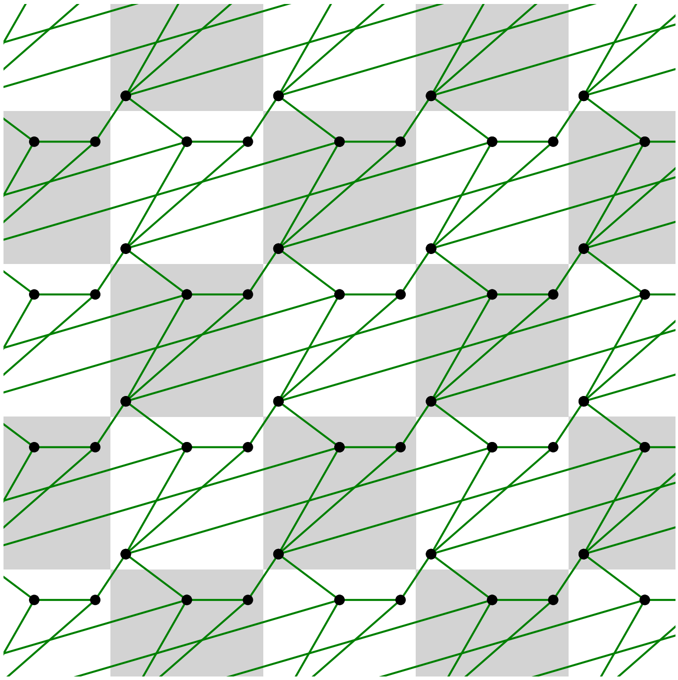

Figure 1. *Example of $\mathbb{Z}^d$-periodic graph embedded in $\mathbb{R}^d$.*

and all $\mathbb{Z}^d$-periodic anti-symmetric discrete vector fields $J : \mathcal{E} \to \mathbb{R}$ with zero *discrete divergence* and with *effective flux* equal to j, i.e.,

$$\operatorname{div} J(x) := \sum_{y \sim x} J(x,y) = 0 \quad \forall x \in \mathcal{X} \quad \text{and} \quad \mathsf{Eff}(J) := \frac{1}{2} \sum_{\substack{(x,y)\in\mathcal{E} \\ x \in [0,1)^d}} J(x,y)(y-x) = j. \tag{1.5}$$

The result covers several examples, both for what concerns the geometric properties of the graph (such as isotropic meshes of $\mathbb{T}^d$, or the simple nearest-neighbour interaction on the symmetric grid) as well as the choice of the cost functionals (including discretisation of p-Wasserstein distances in arbitrary dimension and flow-based models, i.e. when F – or f – does not depend on the first variable).

As a consequence of this Γ-convergence (in time-space) and a compactness result for curves of measures with bounded action [[23], Theorem 5.9], one obtains as a corollary [[23], Theorem 5.10] that, under the stronger assumption of *superlinear growth* on F, also the corresponding discrete boundary-value problems (i.e. the associated squared distances, in the case of the quadratic Wasserstein problems) Γ-converge to the corresponding continuous one, namely $\mathcal{MA}_\varepsilon \xrightarrow{\Gamma} \mathbb{MA}_{\mathrm{hom}}$ (with respect to the weak topology), where

$$\mathcal{MA}_\varepsilon(m_0, m_1) := \inf\left\{\mathcal{A}_\varepsilon(\boldsymbol{m}, \boldsymbol{J}) : (\boldsymbol{m}, \boldsymbol{J}) \in \mathcal{CE}_{\mathcal{X}_\varepsilon} \quad \text{and} \quad \boldsymbol{m}_{t=0} = m_0, \ \boldsymbol{m}_{t=1} = m_1\right\},$$
$$\mathbb{MA}_{\mathrm{hom}}(\mu_0, \mu_1) := \inf\left\{\mathbb{A}_{\mathrm{hom}}(\boldsymbol{\mu}, \boldsymbol{\nu}) : (\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathsf{CE} \quad \text{and} \quad \boldsymbol{\mu}_{t=0} = \mu_0, \ \boldsymbol{\mu}_{t=1} = \mu_1\right\}$$

are the *minimal* discrete and homogenised action functionals, respectively. The superlinear growth condition, at the continuous level, is a reinforcement of the condition (1.3) and assumes the existence of a function $\theta : [0,\infty) \to [0,\infty)$ with $\lim_{t\to\infty} \frac{\theta(t)}{t} = \infty$ and a constant $C \in \mathbb{R}$ such that

$$f(\rho, j) \ge (\rho+1)\theta\left(\frac{|j|}{\rho+1}\right) - C(\rho+1), \qquad \forall \rho \in \mathbb{R}_+, \ \ j \in \mathbb{R}^d.$$

In particular, this forces every $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathsf{CE}$ with finite action to satisfy $\boldsymbol{\nu} \ll \boldsymbol{\mu} + \mathscr{L}^{d+1}$ [[23], Remark 6.1], and it ensures compactness in $\mathcal{C}_{\mathrm{KR}}\big([0,1]; \mathcal{M}_+(\mathbb{T}^d)\big)$ [[23], Theorem 5.9], i.e. with respect to the

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time-uniform convergence in the Kantorovich–Rubinstein norm (recall that the KR norm metrises weak convergence on $\mathcal{M}_+(\mathbb{T}^d)$, see [[23], Appendix A]). This compactness property makes the proof of the convergence $\mathcal{MA}_\varepsilon \xrightarrow{\Gamma} \mathbb{MA}_{\text{hom}}$ an easy corollary of the convergence of the time-space energies.

Without the assumption of superlinear growth the situation is much more subtle: in particular, the lower semicontinuity of $\mathbb{MA}$ obtained minimising the functional $\mathbb{A}$ associated to a function f satisfying only (1.3) is not trivial. This is due to the fact that, in this framework, being a solution to CE with bounded action only ensures bounds for $\boldsymbol{\mu} \in \mathrm{BV}_{\mathrm{KR}}\big((0,1);\mathcal{M}_+(\mathbb{T}^d)\big)$, which does not suffice to pass to the limit in the constraint given by the boundary conditions: *jumps* may occur at $t \in \{0,1\}$ in the limit. Therefore, when the cost F grows linearly (linear bounds from both below and above), the scaling behaviour of the discrete boundary-value problems $\mathcal{MA}_\varepsilon$, as well as the lower semicontinuity of $\mathbb{MA}$, cannot be understood with the techniques utilised in [23]. The main goal of this work is, thus, to provide discrete-to-continuum results for $\mathcal{MA}_\varepsilon$ for cost functionals with linear growth, as well as for every flow-based type of cost, i.e. $F(m,J) = F(J)$. With similar arguments, we can also show the lower semicontinuity of $\mathbb{MA}$ for a general energy density f under the same assumptions, see Section 3.3.

Theorem 1.1 (Main result). *Assume that either F satisfies the linear growth condition, i.e.*

$$F(m,J) \le C\left(1 + \sum_{\substack{(x,y)\in\mathcal{E}\\ |x|\le R}} |J(x,y)| + \sum_{\substack{x\in\mathcal{X}\\ |x|\le R}} m(x)\right)$$

for some constant $C < \infty$ and some $R > 0$, or that F does not depend on the m-variable (flow-based type). Then, as $\varepsilon \to 0$, the discrete functionals $\mathcal{MA}_\varepsilon$ Γ-converge to the continuous functional $\mathbb{MA}_{\text{hom}}$ with respect to weak convergence.

The contribution of this paper is twofold. On one side, thanks to our main result, we can now include important examples, such as the $\mathbb{W}_1$ distance and related approximations, see in particular Section 4 for some explicit computations of the cell formula, including the equivalence between static and dynamical formulations (4.3), as well as some simulations. As typical in this discrete-to-continuum framework, also for $\mathbb{W}_1$-type problems, the geometry of the graph plays an important role in the homogenised norm obtained in the limit, giving rise to a whole class of *crystalline norms*, see Proposition 4.4 as well as Figure 2. On the other hand, this work provides ideas and techniques on how to handle the presence of singularities/jumps in the framework of curves of measures which are only of bounded variation, which is of independent interest.

Related literature. In their seminal work [29], Jordan, Kinderlehrer, and Otto showed that the heat flow in $\mathbb{R}^d$ can be seen as the gradient flow of the relative entropy with respect to the 2-Wasserstein distance. In the same spirit, a discrete counterpart was proved in [32] and [33], independently, for the discrete heat flow and discrete relative entropy on Markov chains. In [19], the authors proved the evolutionary Γ-convergence of the discrete gradient-flow structures associated with finite-volume partitions and discrete Fokker–Planck equations, generalising a previous result obtained in [8] in the setting of isotropic, one-dimensional meshes. Similar results were later obtained in [26], [27] for the study of the limiting behaviour of random walks on tessellations in the diffusive limit. Generalised gradient-flow structures associated to jump processes and approximation results of nonlocal- and local-interaction equations have been studied in a series of works [12], [13], [10]. Recently, [17] considered the more general setting where the graph also depends on time.

2. General framework: continuous and discrete transport problems

In this section, we first introduce the general class of problems at the continuous level we are interested in, discussing main properties and known results. We then move to the discrete, periodic framework in the spirit of [23], summarise the known convergence results and discuss the open problems we want to treat in this work. In contrast with [23], for the sake of the exposition we restrict our analysis to the

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time interval $\mathcal{I} := (0, 1)$. Nonetheless, our main results easily extend to a general bounded, open interval $\mathcal{I} \subset \mathbb{R}$.

2.1. The continuous setting: transport problems on the torus

We start by recalling the definition of solutions to the continuity equation on $\mathbb{T}^d$.

Definition 2.1 *(Continuity equation)*. A pair of measures $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathcal{M}_+\big((0, 1) \times \mathbb{T}^d\big) \times \mathcal{M}^d\big((0, 1) \times \mathbb{T}^d\big)$ is said to be a solution to the continuity equation

$$\partial_t \boldsymbol{\mu} + \nabla \cdot \boldsymbol{\nu} = 0$$

if, for all functions $\varphi \in \mathcal{C}_c^1\big((0, 1) \times \mathbb{T}^d\big)$, the identity

$$\int_{(0,1)\times\mathbb{T}^d} \partial_t \varphi \, \mathrm{d}\boldsymbol{\mu} + \int_{(0,1)\times\mathbb{T}^d} \nabla\varphi \cdot \, \mathrm{d}\boldsymbol{\nu} = 0$$

holds. We use the notation $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathsf{CE}$.

Throughout the whole paper, we consider energy densities f with the following properties.

Assumption 2.2. *Let $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous and convex function, whose domain $\mathsf{D}(f)$ has nonempty interior. We assume that there exist constants $c > 0$ and $C < \infty$ such that the (at least) linear growth condition*

$$f(\rho, j) \ge c|j| - C(\rho + 1) \tag{2.1}$$

holds for all $\rho \in \mathbb{R}_+$ and $j \in \mathbb{R}^d$.

The corresponding *recession function* $f^\infty : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$f^\infty(\rho, j) := \lim_{t\to+\infty} \frac{f(\rho_0 + t\rho, j_0 + tj)}{t}, \tag{2.2}$$

for every $(\rho_0, j_0) \in \mathsf{D}(f)$. It is well established that the function f^∞ is lower semicontinuous, convex, and it satisfies the inequality

$$f^\infty(\rho, j) \ge c|j| - C\rho, \qquad \rho \in \mathbb{R}_+, \ j \in \mathbb{R}^d, \tag{2.3}$$

see [1, Section 2.6].

Let $\mathscr{L}^{d+1}$ denote the Lebesgue measure on $(0, 1) \times \mathbb{T}^d$. For $\boldsymbol{\mu} \in \mathcal{M}_+\big((0, 1) \times \mathbb{T}^d\big)$ and $\boldsymbol{\nu} \in \mathcal{M}^d\big((0, 1) \times \mathbb{T}^d\big)$, we write their Lebesgue decompositions as

$$\boldsymbol{\mu} = \rho \mathscr{L}^{d+1} + \boldsymbol{\mu}^\perp, \qquad \boldsymbol{\nu} = j \mathscr{L}^{d+1} + \boldsymbol{\nu}^\perp,$$

for some $\rho \in L^1_+\big((0, 1) \times \mathbb{T}^d\big)$ and $j \in L^1\big((0, 1) \times \mathbb{T}^d; \mathbb{R}^d\big)$. Given these decompositions, there always exists a measure $\boldsymbol{\sigma} \in \mathcal{M}_+\big((0, 1) \times \mathbb{T}^d\big)$ such that

$$\boldsymbol{\mu}^\perp = \rho^\perp \boldsymbol{\sigma}, \qquad \boldsymbol{\nu}^\perp = j^\perp \boldsymbol{\sigma},$$

for some $\rho^\perp \in L^1_+(\boldsymbol{\sigma})$ and $j^\perp \in L^1(\boldsymbol{\sigma}; \mathbb{R}^d)$ (take for example $\boldsymbol{\sigma} := |\boldsymbol{\mu}^\perp| + |\boldsymbol{\nu}^\perp|$).

Definition 2.3 *(Action functionals)*. We define the action functionals by

$$\begin{aligned}
&\mathbb{A} : \mathcal{M}_+\big((0, 1) \times \mathbb{T}^d\big) \times \mathcal{M}^d\big((0, 1) \times \mathbb{T}^d\big) \to \mathbb{R} \cup \{+\infty\}, \\
&\mathbb{A}(\boldsymbol{\mu}, \boldsymbol{\nu}) := \int_{(0,1)\times\mathbb{T}^d} f\big(\rho, j\big) \, \mathrm{d}\mathscr{L}^{d+1} + \int_{(0,1)\times\mathbb{T}^d} f^\infty\big(\rho^\perp, j^\perp\big) \, \mathrm{d}\boldsymbol{\sigma}, \\
&\mathbb{A}(\boldsymbol{\mu}) := \inf_{\boldsymbol{\nu}} \big\{ \mathbb{A}(\boldsymbol{\mu}, \boldsymbol{\nu}) : (\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathsf{CE} \big\}.
\end{aligned}$$

Remark 2.4. This definition does not depend on the choice of $\boldsymbol{\sigma}$, due to the 1-homogeneity of f^∞. As $f(\rho, j) \ge -C(1 + \rho)$ and $f^\infty(\rho, j) \ge -C\rho$ from (2.1) and (2.3), the fact that $\boldsymbol{\mu}\big((0, 1) \times \mathbb{T}^d\big) < \infty$ ensures that $\mathbb{A}(\boldsymbol{\mu}, \boldsymbol{\nu})$ is well defined in $\mathbb{R} \cup \{+\infty\}$.

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The natural setting to work in is the space $\mathrm{BV_{KR}}\big((0,1);\mathcal{M}_+(\mathbb{T}^d)\big)$ of the curves of measures $\boldsymbol{\mu}:(0,1)\to\mathcal{M}_+(\mathbb{T}^d)$ such that the BV-seminorm $\|\boldsymbol{\mu}\| = \|\boldsymbol{\mu}\|_{\mathrm{BV_{KR}}\left((0,1);\mathcal{M}_+(\mathbb{T}^d)\right)}$ defined by

$$\|\boldsymbol{\mu}\| := \sup\left\{\int_{(0,1)}\int_{\mathbb{T}^d}\partial_t\varphi_t\,\mathrm{d}\mu_t\,\mathrm{d}t : \varphi\in\mathcal{C}_c^1\big((0,1);\mathcal{C}^1(\mathbb{T}^d)\big),\ \max_{t\in(0,1)}\|\varphi_t\|_{\mathcal{C}^1(\mathbb{T}^d)}\le 1\right\}$$

is finite. Note that, by the trace theorem in BV, curves of measures in $\mathrm{BV_{KR}}\big((0,1);\mathcal{M}_+(\mathbb{T}^d)\big)$ have a well-defined trace at $t=0$ and $t=1$. As shown in [[23], Lemma 3.13], any solution $(\boldsymbol{\mu},\boldsymbol{\nu})\in\mathsf{CE}$ can be disintegrated as $\mathrm{d}\boldsymbol{\mu}(t,x) = \mathrm{d}\mu_t(x)\,\mathrm{d}t$ for some measurable curve $t\mapsto\mu_t\in\mathcal{M}_+(\mathbb{T}^d)$ with finite constant mass. If $\mathbb{A}(\boldsymbol{\mu})<\infty$, then this curve belongs to $\mathrm{BV_{KR}}\big((0,1);\mathcal{M}_+(\mathbb{T}^d)\big)$ and

$$\|\boldsymbol{\mu}\|_{\mathrm{BV_{KR}}\left((0,1);\mathcal{M}_+(\mathbb{T}^d)\right)} \le |\boldsymbol{\nu}|\big((0,1)\times\mathbb{T}^d\big).$$

2.2. *Boundary conditions and lower semicontinuity*

Define the minimal homogenised action for $\mu_0,\mu_1\in\mathcal{M}_+(\mathbb{T}^d)$ with $\mu_0(\mathbb{T}^d)=\mu_1(\mathbb{T}^d)$ as

$$\mathbb{MA}(\mu_0,\mu_1) := \inf_{\boldsymbol{\mu}\in\mathrm{BV_{KR}}\left((0,1);\mathcal{M}_+(\mathbb{T}^d)\right)}\left\{\mathbb{A}(\boldsymbol{\mu}) : \boldsymbol{\mu}_{t=0}=\mu_0,\ \boldsymbol{\mu}_{t=1}=\mu_1\right\}. \tag{2.4}$$

Note that, in general, $\mathbb{MA}$ may be infinite (although the measures have equal masses). Despite the lower semicontinuity property of $\mathbb{A}$ (cfr. [[23], Lemma 3.14]), the lower semicontinuity of $\mathbb{MA}$ with respect to the natural weak topology of $\mathcal{M}_+(\mathbb{T}^d)\times\mathcal{M}_+(\mathbb{T}^d)$ is, in general, nontrivial. More precisely, it is a challenging question to prove (or disprove) that for any two sequences $\mu_0^n,\ \mu_1^n\in\mathcal{M}_+(\mathbb{T}^d)$, such that $\mu_i^n\to\mu_i$ weakly in $\mathcal{M}_+(\mathbb{T}^d)$ as $n\to\infty$ for $i=0,1$, the inequality

$$\liminf_{n\to\infty}\mathbb{MA}(\mu_0^n,\mu_1^n)\ge\mathbb{MA}(\mu_0,\mu_1) \tag{2.5}$$

holds. In this work, we provide a positive answer in the case when f has linear growth or it is flow-based (i.e. it does not depend on the first variable), see Remark 3.14 and Proposition 3.15 below. First, we discuss the main challenges and the setup where the lower semicontinuity is already known to hold.

Remark 2.5 *(Lack of compatible compactness).* We know from [[23], Lemma 3.14] that $(\boldsymbol{\mu},\boldsymbol{\nu})\mapsto\mathbb{A}(\boldsymbol{\mu},\boldsymbol{\nu})$ and $\boldsymbol{\mu}\mapsto\mathbb{A}(\boldsymbol{\mu})$ are lower semicontinuous w.r.t. the weak topology. Moreover, if $\boldsymbol{\mu}^n$ is a sequence with

$$\sup_n\mathbb{A}(\boldsymbol{\mu}^n)<\infty\quad\text{and}\quad\sup_n\boldsymbol{\mu}^n\big((0,1)\times\mathbb{T}^d\big)<\infty$$

then $\boldsymbol{\mu}^n$ is weakly compact and any limit $\boldsymbol{\mu}$ belongs to $\mathrm{BV_{KR}}\big((0,1);\mathcal{M}_+(\mathbb{T}^d)\big)$. This can be proved as in [[23], Theorem 5.4]. Nonetheless, this property does not ensure the lower semicontinuity of $\mathbb{MA}$ because weak convergence does not preserve the boundary conditions (at time $t=0$ and $t=1$). For similar issues in the setting of functionals of $\mathbb{R}^d$-valued curves with bounded variations and their minimisation, see e.g. [2].

Remark 2.6 *(Superlinear growth).* Under the strengthened assumption of superlinear growth on f (with respect to the momentum variable), it is possible to prove the lower semicontinuity property (2.5), in the same way as in the proof of the discrete-to-continuum Γ-convergence of boundary-value problems of [[23], Theorem 5.10]. More precisely, we say that f is of superlinear growth if there exists a function $\theta:[0,\infty)\to[0,\infty)$ with $\lim_{t\to\infty}\frac{\theta(t)}{t}=\infty$ and a constant $C\in\mathbb{R}$ such that

$$f(\rho,j)\ge(\rho+1)\theta\left(\frac{|j|}{\rho+1}\right)-C(\rho+1),\qquad\forall\rho\in\mathbb{R}_+,\ \ j\in\mathbb{R}^d.$$

Arguing as in [[23], Remark 5.6], one shows that any function of superlinear growth must satisfy the growth condition given by Assumption 2.2. Moreover, in this case, the recession function satisfies

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$f^\infty(0,j) = +\infty$, for every $j \neq 0$. See [[23], Examples 5.7 & 5.8] for some examples belonging to this class. By arguing similarly as in the proof of [[23], Theorem 5.9], assuming superlinear growth one can show that if $\boldsymbol{\mu}^n$ is a sequence with bounded action $\mathbb{A}(\boldsymbol{\mu}^n)$ and bounded total mass $\boldsymbol{\mu}^n\big((0,1)\times\mathbb{T}^d\big)$, then, up to a (non-relabelled) subsequence, we have $\boldsymbol{\mu}^n \to \boldsymbol{\mu}$ in $\mathcal{M}_+((0,1)\times\mathbb{T}^d)$ and $\mu_t^n \to \mu_t$ in KR norm uniformly in $t \in (0,1)$, with limit curve $\boldsymbol{\mu} \in W^{1,1}_{\mathrm{KR}}((0,1);\mathcal{M}_+(\mathbb{T}^d))$. Using this fact, it is clear that the problem of "jumps" in the limit explained in Remark 2.5 does not occur, and the lower semicontinuity (2.5) directly follows from the lower semicontinuity of $\mathbb{A}$.

Remark 2.7. *(Nonnegativity)* Without loss of generality, we can assume that $f \geq 0$. Indeed, thanks to the linear growth assumption 2.2, we can define a new function

$$\widetilde{f}(\rho,j) := f(\rho,j) + C(\rho+1) \geq c|j| \geq 0 \tag{2.6}$$

which is now nonnegative and with (at least) linear growth. Furthermore, we can compute the recess $\widetilde{f}^\infty$ and from the definition we see that

$$\widetilde{f}^\infty(\rho,j) = f^\infty(\rho,j) + C\rho. \tag{2.7}$$

Denote by $\widetilde{\mathbb{A}}$ the action functional obtained by replacing f with $\widetilde{f}$. Thanks to (2.6), (2.7), we have that

$$\begin{aligned}\widetilde{\mathbb{A}}(\boldsymbol{\mu}) &:= \inf_{\boldsymbol{\nu}} \big\{\widetilde{\mathbb{A}}(\boldsymbol{\mu},\boldsymbol{\nu}) : (\boldsymbol{\mu},\boldsymbol{\nu}) \in \mathsf{CE}\big\} \\ &= \inf_{\boldsymbol{\nu}} \big\{\mathbb{A}(\boldsymbol{\mu},\boldsymbol{\nu}) : (\boldsymbol{\mu},\boldsymbol{\nu}) \in \mathsf{CE}\big\} + C\big(\boldsymbol{\mu}\big((0,1)\times\mathbb{T}^d\big) + 1\big).\end{aligned}$$

It follows that the corresponding boundary value problems are given by

$$\widetilde{\mathbb{MA}}(\mu_0,\mu_1) = \mathbb{MA}(\mu_0,\mu_1) + C(\mu_0(\mathbb{T}^d)+1), \quad \text{if } \mu_0(\mathbb{T}^d) = \mu_1(\mathbb{T}^d).$$

Therefore, the (weak) lower semicontinuity for $\widetilde{\mathbb{MA}}$ is equivalent to that of $\mathbb{MA}$.

2.3. The discrete framework: transport problems on periodic graphs

We recall the framework of [23]: let $(\mathcal{X},\mathcal{E})$ be a locally finite and $\mathbb{Z}^d$-periodic connected graph of bounded degree. We encode the set of vertices as $\mathcal{X} = \mathbb{Z}^d \times V$, where V is a finite set, and we use coordinates $x = (x_{\mathsf{z}}, x_{\mathsf{v}}) \in \mathcal{X}$. The set of edges $\mathcal{E} \subseteq \mathcal{X}\times\mathcal{X}$ is symmetric and $\mathbb{Z}^d$-periodic, and we use the notation $x \sim y$ whenever $(x,y) \in \mathcal{E}$. Let $R_0 := \max_{(x,y)\in\mathcal{E}} |x_{\mathsf{z}} - y_{\mathsf{z}}|_\infty$ be the maximal edge length in the supremum norm $|\cdot|_\infty$ on $\mathbb{R}^d$. We use the notation $\mathcal{X}^Q := \{x \in \mathcal{X} : x_{\mathsf{z}} = 0\}$ and $\mathcal{E}^Q := \big\{(x,y) \in \mathcal{E} : x_{\mathsf{z}} = 0\big\}$. For a discussion concerning abstract and embedded graphs, see [[23], Remark 2.2].

In what follows, we denote by $\mathbb{R}_+^{\mathcal{X}}$ the set of functions $m \colon \mathcal{X} \to \mathbb{R}_+$ and by $\mathbb{R}_a^{\mathcal{E}}$ the set of *anti-symmetric* functions $J \colon \mathcal{E} \to \mathbb{R}$, that is, such that $J(x,y) = -J(y,x)$. The elements of $\mathbb{R}_a^{\mathcal{E}}$ will often be called *(discrete) vector fields*.

Assumption 2.8 (Admissible cost function). *The function $F \colon \mathbb{R}_+^{\mathcal{X}} \times \mathbb{R}_a^{\mathcal{E}} \to \mathbb{R}\cup\{+\infty\}$ is assumed to have the following properties:*

(a) F is convex and lower semicontinuous.

(b) F is local, meaning that, for some number $R_1 < \infty$, we have $F(m,J) = F(m',J')$ whenever $m, m' \in \mathbb{R}_+^{\mathcal{X}}$ and $J, J' \in \mathbb{R}_a^{\mathcal{E}}$ agree within a ball of radius R_1, i.e.

$$\begin{aligned} m(x) &= m'(x) && \text{for all } x \in \mathcal{X} \text{ with } |x_{\mathsf{z}}|_\infty \leq R_1, \quad \text{and} \\ J(x,y) &= J'(x,y) && \text{for all } (x,y) \in \mathcal{E} \text{ with } |x_{\mathsf{z}}|_\infty, |y_{\mathsf{z}}|_\infty \leq R_1. \end{aligned}$$

(c) F is of at least linear growth, i.e. there exist $c > 0$ and $C < \infty$ such that

$$F(m,J) \geq c \sum_{(x,y)\in\mathcal{E}^Q} |J(x,y)| - C\Bigg(1 + \sum_{\substack{x\in\mathcal{X} \\ |x_{\mathsf{z}}|_\infty \leq R_{\max}}} m(x)\Bigg) \tag{2.8}$$

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for any $m \in \mathbb{R}_+^{\mathcal{X}}$ and $J \in \mathbb{R}_a^{\mathcal{E}}$. Here, $R_{\max} := \max\{R_0, R_1\}$.

(d) *There exist a $\mathbb{Z}^d$-periodic function $m^\circ \in \mathbb{R}_+^{\mathcal{X}}$ and a $\mathbb{Z}^d$-periodic and divergence-free vector field $J^\circ \in \mathbb{R}_a^{\mathcal{E}}$ such that*

$$(m^\circ, J^\circ) \in \mathsf{D}(F)^\circ.$$

Remark 2.9. Important examples that satisfy the growth condition (2.8) are of the form

$$F(m,J) = \frac{1}{2} \sum_{(x,y)\in\mathcal{E}^Q} \frac{|J(x,y)|^p}{\Lambda\big(q_{xy}m(x), q_{yx}m(y)\big)^{p-1}},$$

where $1 \le p < \infty$, the constants $q_{xy}, q_{yx} > 0$ are fixed parameters defined for $(x,y) \in \mathcal{E}^Q$, and Λ is a suitable mean. Functions of this type naturally appear in discretisations of Wasserstein gradient-flow structures [32], [33], [6], see also [[23], Remark 2.6].

The rescaled graph. Let $\mathbb{T}_\varepsilon^d = (\varepsilon\mathbb{Z}/\mathbb{Z})^d$ be the discrete torus of mesh size $\varepsilon \in 1/\mathbb{N}$. We denote by $[\varepsilon z]$ for $z \in \mathbb{Z}^d$ the corresponding equivalence classes. Equivalently, $\mathbb{T}_\varepsilon^d = \varepsilon\mathbb{Z}_\varepsilon^d$ where $\mathbb{Z}_\varepsilon^d = \left(\mathbb{Z}/\frac{1}{\varepsilon}\mathbb{Z}\right)^d$. The rescaled graph $(\mathcal{X}_\varepsilon, \mathcal{E}_\varepsilon)$ is defined as

$$\mathcal{X}_\varepsilon := \mathbb{T}_\varepsilon^d \times V \quad \text{and} \quad \mathcal{E}_\varepsilon := \left\{ \big(T_\varepsilon^0(x), T_\varepsilon^0(y)\big) : (x,y) \in \mathcal{E} \right\}$$

where for $\bar z \in \mathbb{Z}_\varepsilon^d$,

$$T_\varepsilon^{\bar z}: \mathcal{X} \to \mathcal{X}_\varepsilon, \qquad (z,v) \mapsto \big([\varepsilon(\bar z + z)], v\big).$$

For $x = \big([\varepsilon z], v\big) \in \mathcal{X}_\varepsilon$, we write

$$x_{\mathsf{z}} := z \in \mathbb{Z}_\varepsilon^d, \qquad x_{\mathsf{v}} := v \in V.$$

The equivalence relation $\sim$ on $\mathcal{X}$ is equivalently defined on $\mathcal{X}_\varepsilon$ by means of $\mathcal{E}_\varepsilon$. Hereafter, we always assume that $\varepsilon R_0 < \frac{1}{2}$.

The rescaled energies. Let $F: \mathbb{R}_+^{\mathcal{X}} \times \mathbb{R}_a^{\mathcal{E}} \to \mathbb{R} \cup \{+\infty\}$ be a cost function satisfying Assumption 2.8. For $\varepsilon > 0$ satisfying the conditions above, we can define a corresponding energy functional $\mathcal{F}_\varepsilon$ in the rescaled periodic setting: following [23], for $\bar z \in \mathbb{Z}_\varepsilon^d$, each function $\psi: \mathcal{X}_\varepsilon \to \mathbb{R}$ induces a $\frac{1}{\varepsilon}\mathbb{Z}^d$-periodic function

$$\tau_\varepsilon^{\bar z}\psi: \mathcal{X} \to \mathbb{R}, \qquad \big(\tau_\varepsilon^{\bar z}\psi\big)(x) := \psi\big(T_\varepsilon^{\bar z}(x)\big) \quad \text{for } x \in \mathcal{X}.$$

Similarly, each function $J: \mathcal{E}_\varepsilon \to \mathbb{R}$ induces a $\frac{1}{\varepsilon}\mathbb{Z}^d$-periodic function

$$\tau_\varepsilon^{\bar z}J: \mathcal{E} \to \mathbb{R}, \qquad \big(\tau_\varepsilon^{\bar z}J\big)(x,y) := J\big(T_\varepsilon^{\bar z}(x), T_\varepsilon^{\bar z}(y)\big) \quad \text{for } (x,y) \in \mathcal{E}.$$

Definition 2.10 *(Discrete energy functional).* The rescaled energy is defined by

$$\mathcal{F}_\varepsilon: \mathbb{R}_+^{\mathcal{X}_\varepsilon} \times \mathbb{R}_a^{\mathcal{E}_\varepsilon} \to \mathbb{R} \cup \{+\infty\}, \qquad (m,J) \overset{\mathcal{F}_\varepsilon}{\longmapsto} \sum_{z\in\mathbb{Z}_\varepsilon^d} \varepsilon^d F\left(\frac{\tau_\varepsilon^z m}{\varepsilon^d}, \frac{\tau_\varepsilon^z J}{\varepsilon^{d-1}}\right).$$

Remark 2.11. As observed in [[23], Remark 2.8], the value $\mathcal{F}_\varepsilon(m,J)$ is well defined as an element in $\mathbb{R} \cup \{+\infty\}$. Indeed, the condition (2.8) yields

$$\begin{aligned}\mathcal{F}_\varepsilon(m,J) = \sum_{z\in\mathbb{Z}_\varepsilon^d} \varepsilon^d F\left(\frac{\tau_\varepsilon^z m}{\varepsilon^d}, \frac{\tau_\varepsilon^z J}{\varepsilon^{d-1}}\right) &\ge -C \sum_{z\in\mathbb{Z}_\varepsilon^d} \varepsilon^d \left(1 + \sum_{\substack{x\in\mathcal{X} \\ |x_{\mathsf{z}}|_\infty \le R_{\max}}} \frac{\tau_\varepsilon^z m(x)}{\varepsilon^d}\right) \\ &\ge -C\left(1 + (2R_{\max}+1)^d \sum_{x\in\mathcal{X}_\varepsilon} m(x)\right) > -\infty.\end{aligned}$$

Definition 2.12 *(Discrete continuity equation).* A pair $(\boldsymbol{m}, \boldsymbol{J})$ is said to be a solution to the discrete continuity equation if $\boldsymbol{m}: (0,1) \to \mathbb{R}_+^{\mathcal{X}_\varepsilon}$ is continuous, $\boldsymbol{J}: (0,1) \to \mathbb{R}_a^{\mathcal{E}_\varepsilon}$ is Borel measurable, and

$$\partial_t m_t(x) + \sum_{y\sim x} J_t(x,y) = 0 \tag{2.9}$$

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holds for all $x \in \mathcal{X}_\varepsilon$ in the sense of distributions. We use the notation

$$(\boldsymbol{m}, \boldsymbol{J}) \in \mathcal{CE}_\varepsilon.$$

Remark 2.13. We may write (2.9) as $\partial_t m_t + \mathsf{div} J_t = 0$ using the discrete divergence operator, given by

$$\mathsf{div} J \in \mathbb{R}^{\mathcal{X}_\varepsilon}, \qquad \mathsf{div} J(x) := \sum_{y \sim x} J(x, y), \qquad \forall J \in \mathbb{R}_a^{\mathcal{E}_\varepsilon}.$$

The proof of the following lemma can be found in [23].

Lemma 2.14 (Mass preservation). *Let $(\boldsymbol{m}, \boldsymbol{J}) \in \mathcal{CE}_\varepsilon$. Then we have $m_s(\mathcal{X}_\varepsilon) = m_t(\mathcal{X}_\varepsilon)$ for all $s, t \in (0, 1)$.*

We are now ready to define one of the main objects in this paper.

Definition 2.15 *(Discrete action functional).* For any continuous function $\boldsymbol{m}: (0, 1) \to \mathbb{R}_+^{\mathcal{X}_\varepsilon}$ such that $t \mapsto \sum_{x \in \mathcal{X}_\varepsilon} m_t(x) \in L^1\big((0, 1)\big)$ and any Borel measurable function $\boldsymbol{J}: (0, 1) \to \mathbb{R}_a^{\mathcal{E}_\varepsilon}$, we define

$$\mathcal{A}_\varepsilon(\boldsymbol{m}, \boldsymbol{J}) := \int_0^1 \mathcal{F}_\varepsilon(m_t, J_t)\, \mathrm{d}t \in \mathbb{R} \cup \{+\infty\}.$$

Furthermore, we set

$$\mathcal{A}_\varepsilon(\boldsymbol{m}) := \inf_{\boldsymbol{J}} \Big\{ \mathcal{A}_\varepsilon(\boldsymbol{m}, \boldsymbol{J}) : (\boldsymbol{m}, \boldsymbol{J}) \in \mathcal{CE}_\varepsilon \Big\}.$$

Arguing as in Remark 2.11, one can show [[23], Remark 2.13] that $\mathcal{A}_\varepsilon(\boldsymbol{m}, \boldsymbol{J})$ is well defined as an element in $\mathbb{R} \cup \{+\infty\}$, as a consequence of the growth condition (2.8).

Definition 2.16 *(Minimal discrete action functional).* For any pair of boundary data $m_0, m_1 \in \mathbb{R}_+^{\mathcal{X}_\varepsilon}$, we define the associated discrete boundary value problem as

$$\mathcal{MA}_\varepsilon(m_0, m_1) := \inf \left\{ \mathcal{A}_\varepsilon(\boldsymbol{m}) : \boldsymbol{m}: (0, 1) \to \mathbb{R}_+^{\mathcal{X}_\varepsilon}, \quad \boldsymbol{m}_{t=0} = m_0 \quad \text{and} \quad \boldsymbol{m}_{t=1} = m_1 \right\}.$$

The aim of this work is to study the asymptotic behaviour of the energies $\mathcal{MA}_\varepsilon$ as $\varepsilon \to 0$ under the Assumption 2.8.

3. Statement and proof of the main result

In this paper, we extend the Γ-convergence result for the functionals $\mathcal{MA}_\varepsilon$ towards $\mathbb{MA}_{\mathrm{hom}}$, proved in [23] for superlinear cost functionals, to two cases: under the assumption of linear growth (bound both from below and above) and when the function F does not depend on ρ.

Assumption 3.1 (Linear growth). *We say that a function $F: \mathbb{R}_+^{\mathcal{X}} \times \mathbb{R}_a^{\mathcal{E}} \to \mathbb{R} \cup \{+\infty\}$ has linear growth if it satisfies*

$$F(m, J) \leq C \left(1 + \sum_{\substack{(x,y) \in \mathcal{E} \\ |x_{\mathsf{z}}|_\infty \leq R}} |J(x, y)| + \sum_{\substack{x \in \mathcal{X} \\ |x_{\mathsf{z}}|_\infty \leq R}} m(x) \right)$$

for some constant $C < \infty$ and some $R > 0$.

Assumption 3.2 (Flow-based). *We say that a function $F: \mathbb{R}_+^{\mathcal{X}} \times \mathbb{R}_a^{\mathcal{E}} \to \mathbb{R} \cup \{+\infty\}$ is of flow-based type if it depends only on the the second variable, i.e. (with a slight abuse of notation) $F(m, J) = F(J)$, for some $F: \mathbb{R}_a^{\mathcal{E}} \to \mathbb{R} \cup \{+\infty\}$.*

Similarly, we say that $f:\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is of *flow-based type* if it does not depend on the ρ variable, i.e. $f(\rho, j) = f(j)$. In this case, the problem simplifies significantly, and the dynamical variational problem described in (2.4) admits an equivalent, static formulation (see (3.23)).

Remark 3.3 *(Linear growth vs Lipschitz).* While working with convex functions, to assume a linear growth condition (from above) is essentially equivalent to require Lipschitz continuity with respect to the second variable.

Lemma 3.4 (Lipschitz continuity). *Let $f:\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ be a function, convex in the second variable. Let $C > 0$. Then the following are equivalent:*

1. *For every $\rho \in \mathbb{R}_+$ and $j \in \mathbb{R}^d$ the inequality $f(\rho, j) \le C(1 + \rho + |j|)$ holds.*
2. *For every $\rho \in \mathbb{R}_+$, the function $f(\rho, \cdot):\mathbb{R}^d \to \mathbb{R}_+$ is Lipschitz continuous (uniformly in ρ) with constant C, and the inequality $f(\rho, 0) \le C(1 + \rho)$ holds.*

In the very same spirit, one can show the analogous result at the discrete level.

Lemma 3.5 (Lipschitz continuity II). *Let $F:\mathbb{R}_+^{\mathcal{X}} \times \mathbb{R}_a^{\mathcal{E}} \to \mathbb{R} \cup \{+\infty\}$ be convex in the second variable. Let $C, R > 0$. Then the following are equivalent:*

1. *F is of linear growth, in the sense of Assumption 3.1, with the same constants C and R.*
2. *For every $m \in \mathbb{R}_+^{\mathcal{X}}$, we have that*

$$F(m, 0) \le C\Bigg(1 + \sum_{\substack{x \in \mathcal{X} \\ |x_{\mathsf{z}}|_\infty \le R}} m(x)\Bigg),$$

as well as that F is Lipschitz continuous with constant C in the second variable, in the sense that

$$|F(m, J_1) - F(m, J_2)| \le C \sum_{\substack{(x,y) \in \mathcal{E} \\ |x_{\mathsf{z}}|_\infty \le R}} |J_1(x, y) - J_2(x, y)|, \tag{3.1}$$

for every $J_1, J_2 \in \mathbb{R}_a^{\mathcal{E}}$.

Proof of Lemma 3.4. Let us assume the first condition and fix $\rho \in \mathbb{R}_+$ as well as $j_1, j_2 \in \mathbb{R}^d$. It follows from the convexity in the second variable that the function

$$\mathbb{R} \ni t \mapsto f(\rho, j_1 + t(j_2 - j_1))$$

is convex. In particular, the inequalities

$$f(\rho, j_2) - f(\rho, j_1) \le \frac{f(\rho, j_1 + t(j_2 - j_1)) - f(\rho, j_1)}{t} \le \frac{C\big(1 + \rho + \big|j_1 + t(j_2 - j_1)\big|\big) - f(\rho, j_1)}{t}$$

hold for every $t \ge 1$. Letting $t \to \infty$, we thus find

$$f(\rho, j_2) - f(\rho, j_1) \le C|j_2 - j_1|$$

and, by arbitrariness of the arguments, the claimed Lipschitz continuity. The fact that $f(\rho, 0) \le C(1 + \rho)$ trivially follows from the first condition.

Conversely, if the second condition holds, we necessarily have

$$f(\rho, j) \le C|j| + f(\rho, 0) \le C(1 + \rho + |j|),$$

for every $\rho \in \mathbb{R}_+$ and $j \in \mathbb{R}^d$, which is precisely the first condition in the statement. □

Let us recall the homogenised energy density f_{hom}, which describes the limit energy and is given by a cell formula. For given $\rho \ge 0$ and $j \in \mathbb{R}^d$, $f_{\text{hom}}(\rho, j)$ is obtained by minimising over the unit cube the cost among functions m and vector fields J *representing* ρ and j. More precisely, the function $f_{\text{hom}}:\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ is given by

$$f_{\text{hom}}(\rho, j) := \inf_{m,J}\big\{F(m, J) : (m, J) \in \mathsf{Rep}(\rho, j)\big\},$$

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where the set of *representatives* $\mathsf{Rep}(\rho, j)$ consists of all $\mathbb{Z}^d$-periodic functions $m: \mathcal{X} \to \mathbb{R}_+$ and all $\mathbb{Z}^d$-periodic anti-symmetric discrete vector fields $J: \mathcal{E} \to \mathbb{R}$ satisfying

$$\sum_{x \in \mathcal{X}^Q} m(x) = \rho, \quad \mathsf{div} J = 0, \quad \text{and} \quad \mathsf{Eff}(J) := \frac{1}{2} \sum_{(x,y) \in \mathcal{E}^Q} J(x,y)(y_{\mathsf{z}} - x_{\mathsf{z}}) = j. \tag{3.2}$$

The set of representatives is nonempty for every choice of ρ and j by [[23], Lemma 4.5 (iv)]. In the case of embedded graphs, the definition of effective flux coincides with the one provided in the introduction (cfr. (3.2)), see [[23], Proposition 9.1].

Remark 3.6. It is not hard to show that if F is of linear growth, then f_{hom} is also of linear growth (and therefore, in view of Lemma 3.4, it is Lipschitz in the second variable uniformly w.r.t. the first one), see e.g. [25].

We denote by $\mathbb{A}_{\mathrm{hom}}$ and $\mathbb{MA}_{\mathrm{hom}}$ the action functionals corresponding to the choice $f = f_{\mathrm{hom}}$. In order to talk about Γ-convergence, we need to specificy which type of discrete-to-continuum topology/convergence we adopt (in the same spirit of [23]).

Definition 3.7 *(Embedding)*. For $\varepsilon > 0$ and $z \in \mathbb{R}^d$, let $Q_\varepsilon^z := \varepsilon z + [0, \varepsilon)^d \subseteq \mathbb{T}^d$ denote the projection of the cube with side length ε based at εz to the quotient $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$. For $m \in \mathbb{R}_+^{\mathcal{X}_\varepsilon}$ and $J \in \mathbb{R}_a^{\mathcal{E}_\varepsilon}$, we define $\iota_\varepsilon m \in \mathcal{M}_+(\mathbb{T}^d)$ and $\iota_\varepsilon J \in \mathcal{M}^d(\mathbb{T}^d)$ by

$$\iota_\varepsilon m := \varepsilon^{-d} \sum_{x \in \mathcal{X}_\varepsilon} m(x) \mathscr{L}^d|_{Q_\varepsilon^{x_{\mathsf{z}}}},$$

$$\iota_\varepsilon J := \varepsilon^{-d+1} \sum_{(x,y) \in \mathcal{E}_\varepsilon} \frac{J(x,y)}{2} \left(\int_0^1 \mathscr{L}^d|_{Q_\varepsilon^{(1-s)x_{\mathsf{z}} + s y_{\mathsf{z}}}} \, \mathrm{d}s \right) (y_{\mathsf{z}} - x_{\mathsf{z}}).$$

With a slight abuse of notation, given $\boldsymbol{m}: (0,1) \to \mathbb{R}_+^{\mathcal{X}_\varepsilon}$ we also write $\iota_\varepsilon \boldsymbol{m} \in \mathcal{M}_+\big((0,1) \times \mathbb{T}^d\big)$ for the continuous space-time measure with time disintegration given by $t \mapsto \iota_\varepsilon m_t$. Moreover, for a given sequence of nonnegative discrete measures $m^\varepsilon \in \mathbb{R}_+^{\mathcal{X}_\varepsilon}$, we write

$$m_\varepsilon \to \mu \in \mathcal{M}_+(\mathbb{T}^d) \quad \text{weakly} \qquad \text{iff} \qquad \iota_\varepsilon m^\varepsilon \to \mu \quad \text{weakly in } \mathcal{M}_+(\mathbb{T}^d).$$

Similarly, for $\boldsymbol{m}^\varepsilon: (0,1) \to \mathbb{R}_+^{\mathcal{X}_\varepsilon}$ we write $\boldsymbol{m}^\varepsilon \to \boldsymbol{\mu} \in \mathcal{M}_+\big((0,1) \times \mathbb{T}^d\big)$ with an analogous meaning. Similar notation is used for (Borel, possibly discontinuous) curves of fluxes $\boldsymbol{J}: (0,1) \to \mathbb{R}_a^{\mathcal{E}_\varepsilon}$ and convergent sequences of (curves of) fluxes.

Remark 3.8 (Preservation of the continuity equation). *The definition of this embedding for masses and fluxes ensures that solutions to the discrete continuity equation are mapped to solutions of* CE, *cfr. [[23], Lemma 4.9].*

We are ready to state our main result.

Theorem 3.9 (Main result). *Let $(\mathcal{X}, \mathcal{E}, F)$ be as described in Section 2.2 and Assumption 2.8. Assume that F is either of flow-based type (Assumption 3.2) or with linear growth (Assumption 3.1). Then, in either case, the functionals $\mathcal{MA}_\varepsilon$ Γ-convergence to $\mathbb{MA}_{\mathrm{hom}}$ as $\varepsilon \to 0$ with respect to the weak topology of $\mathcal{M}_+(\mathbb{T}^d) \times \mathcal{M}_+(\mathbb{T}^d)$. More precisely, we have*

(1) ***Liminf inequality****: For any sequences $m_0^\varepsilon, m_1^\varepsilon \in \mathcal{M}_+(\mathcal{X}_\varepsilon)$ such that $m_i^\varepsilon \to \mu_i$ weakly in $\mathcal{M}_+(\mathbb{T}^d)$ for $i = 0, 1$, we have that*

$$\liminf_{\varepsilon \to 0} \mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) \geq \mathbb{MA}_{\mathrm{hom}}(\mu_0, \mu_1).$$

(2) ***Limsup inequality****: For any $\mu_0, \mu_1 \in \mathcal{M}_+(\mathbb{T}^d)$, there exist sequences $m_0^\varepsilon, m_1^\varepsilon \in \mathcal{M}_+(\mathcal{X}_\varepsilon)$ such that $m_i^\varepsilon \to \mu_i$ weakly in $\mathcal{M}_+(\mathbb{T}^d)$ for $i = 0, 1$, and*

$$\limsup_{\varepsilon \to 0} \mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) \leq \mathbb{MA}_{\mathrm{hom}}(\mu_0, \mu_1).$$

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Remark 3.10 (Convergence of the actions and superlinear regime). *The Γ-convergence of the energies $\mathcal{A}_\varepsilon$ towards $\mathbb{A}_{\text{hom}}$ under Assumption 2.8 is the main result of [[23], Theorem 5.1]. Related to it, similarly as discussed in Remark 2.6, the superlinear case [[23], Assumption 5.5], not included in the statement, has already been proved in [23], and it follows directly from the aforementioned convergence $\mathcal{A}_\varepsilon \xrightarrow{\Gamma} \mathbb{A}_{\text{hom}}$ and a strong compactness result which holds in such a framework, see in particular [[23], Theorems 5.9 & 5.10]. Without the superlinear growth assumption, the proof is much more involved and requires extra work and new ideas, which are the main contribution of this paper.*

Remark 3.11 (Compactness under linear growth from below). *Just assuming Assumption 2.8, the following compactness result for sequences of bounded action was proved in [[23], Theorem 5.4]: if $\boldsymbol{m}^\varepsilon:(0,1) \to \mathbb{R}_+^{\mathcal{X}_\varepsilon}$ is such that*

$$\sup_{\varepsilon>0} \mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon) < \infty \quad \text{and} \quad \sup_{\varepsilon>0} \boldsymbol{m}^\varepsilon\big((0,1)\times\mathcal{X}_\varepsilon\big) < \infty,$$

then there exists a curve $\boldsymbol{\mu} = \mu_t(\,\mathrm{d}x)\,\mathrm{d}t \in \mathrm{BV}_{\mathrm{KR}}\big((0,1);\mathcal{M}_+(\mathbb{T}^d)\big)$ such that, up to a (non-relabelled) subsequence, we have

$$\boldsymbol{m}^\varepsilon \to \boldsymbol{\mu} \quad \text{weakly in } \mathcal{M}_+\big((0,1)\times\mathbb{T}^d\big) \quad \text{and} \quad m_t^\varepsilon \to \mu_t \quad \text{weakly in } \mathcal{M}_+(\mathbb{T}^d),$$

for a.e. $t \in (0,1)$. This is going to be an important tool in the proof of our main result.

3.1. Proof of the limsup inequality

In this section, we prove the limsup inequality in Theorem 3.9. This proof does not require Assumption 3.1 or Assumption 3.2, but rather a weaker hypothesis, which is satisfied under either of the two assumptions.

Proposition 3.12 *(Γ-limsup). Let μ_0, μ_1 be nonnegative measures on $\mathbb{T}^d$. Assume that there exists a $\mathbb{Z}^d$-periodic and divergence-free vector field $\bar{J} \in \mathbb{R}_a^{\mathcal{X}}$ such that*

$$F(m,\bar{J}) \le C\left(1 + \sum_{\substack{x\in\mathcal{X}\\ |x|_\infty \le R}} m(x)\right), \qquad m \in \mathbb{R}_+^{\mathcal{X}}, \tag{3.3}$$

for some finite constants C and R. Then there exist two sequences $(m_0^\varepsilon)_{\varepsilon>0}$ and $(m_1^\varepsilon)_{\varepsilon>0}$ in $\mathbb{R}_+^{\mathcal{X}_\varepsilon}$ such that $m_i^\varepsilon \to \mu_i$ weakly in $\mathcal{M}_+(\mathbb{T}^d)$ for $i = 0, 1$ and

$$\limsup_{\varepsilon\to0} \mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) \le \mathbb{MA}_{\text{hom}}(\mu_0, \mu_1). \tag{3.4}$$

Proof. We may and will assume that $\mathbb{MA}_{\text{hom}}(\mu_0,\mu_1) < \infty$. We also claim that it suffices to prove the statement with $\mathbb{MA}(\mu_0,\mu_1) + 1/k$ in place of the right-hand side of (3.4) for every $k \in \mathbb{N}_1$. Indeed, assume that we know of the existence of sequences $(m_i^{\varepsilon,k})_\varepsilon$ such that $m_i^{\varepsilon,k} \to \mu_i$ and

$$\limsup_{\varepsilon\to0} \mathcal{MA}_\varepsilon(m_0^{\varepsilon,k}, m_1^{\varepsilon,k}) \le \mathbb{MA}(\mu_0,\mu_1) + 1/k,$$

for every $k \in \mathbb{N}_1$. Since $\mathbb{T}^d$ is compact, the weak convergence is equivalent to convergence in the Kantorovich–Rubinstein norm. Hence, for every k, we can find ε_k such that when $\varepsilon \le \varepsilon_k$,

$$\mathcal{MA}_\varepsilon(m_0^{\varepsilon,k}, m_1^{\varepsilon,k}) \le \mathbb{MA}_{\text{hom}}(\mu_0,\mu_1) + 2/k \quad \text{and} \quad \max_{i=0,1} \|\iota_\varepsilon m_i^{\varepsilon,k} - \mu_i\|_{\mathrm{KR}} \le 1/k.$$

We can also assume that $\varepsilon_{k+1} \le \frac{\varepsilon_k}{2}$, for every k. It now suffices to set

$$k_\varepsilon := \max\{k \in \mathbb{N}_1 : \varepsilon_k \ge \varepsilon\} \quad \text{and} \quad m_i^\varepsilon := m_i^{\varepsilon,k_\varepsilon},$$

for every ε and $i = 0, 1$ to get

$$\limsup_{\varepsilon\to0} \mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) \le \mathbb{MA}_{\text{hom}}(\mu_0,\mu_1) + \limsup_{\varepsilon\to0} \frac{2}{k_\varepsilon}$$

as well as

$$\limsup_{\varepsilon\to 0} \max_{i=0,1} \|\iota_\varepsilon m_i^\varepsilon - \mu_i\|_{\mathrm{KR}} \le \limsup_{\varepsilon\to 0} \frac{1}{k_\varepsilon}.$$

The claim is proved, since $k_\varepsilon \to_\varepsilon \infty$, as can be readily verified.

Thus, let us now choose k and keep it fixed. By definition of $\mathbb{MA}_{\mathrm{hom}}$, there exists $\boldsymbol{\mu} = \mu_t(\,\mathrm{d}x)\,\mathrm{d}t \in \mathrm{BV}_{\mathrm{KR}}\big((0,1);\mathcal{M}_+(\mathbb{T}^d)\big)$ with $\boldsymbol{\mu}_{t=0} = \mu_0$, $\boldsymbol{\mu}_{t=1} = \mu_1$ and such that

$$\mathbb{A}_{\mathrm{hom}}(\boldsymbol{\mu}) \le \mathbb{MA}_{\mathrm{hom}}(\mu_0, \mu_1) + 1/k.$$

Recall from Remark 3.10 that $\mathcal{A}_\varepsilon \xrightarrow{\Gamma} \mathbb{A}_{\mathrm{hom}}$; in particular, there exists a recovery sequence $(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) \in \mathcal{CE}_\varepsilon$ such that $\boldsymbol{m}^\varepsilon \to \boldsymbol{\mu}$ weakly and

$$\limsup_{\varepsilon\to 0} \mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) \le \mathbb{A}_{\mathrm{hom}}(\boldsymbol{\mu}).$$

We shall prove that $\|\iota_\varepsilon m_t^\varepsilon - \mu_t\|_{\mathrm{KR}(\mathbb{T}^d)} \to 0$ in ($\mathscr{L}^1$-)measure or, equivalently, that

$$\lim_{\varepsilon\to 0} \int_0^1 \min\left\{\|\iota_\varepsilon m_t^\varepsilon - \mu_t\|_{\mathrm{KR}(\mathbb{T}^d)}, 1\right\} \,\mathrm{d}t = 0. \tag{3.5}$$

In order to do this, assume by contradiction that there exists a subsequence such that

$$\int_0^1 \min\left\{\|\iota_{\varepsilon_n} m_t^{\varepsilon_n} - \mu_t\|_{\mathrm{KR}(\mathbb{T}^d)}, 1\right\} \,\mathrm{d}t > \delta, \qquad n \in \mathbb{N},$$

for some $\delta > 0$. Up to possibly extracting a further subsequence, it can be easily checked that the hypotheses of [[23], Theorem 5.4] are satisfied (cfr. Remark 3.11); hence, there exists a further (not relabelled) subsequence such that, for almost every $t \in (0,1)$, $m_t^{\varepsilon_n} \to \mu_t$ weakly and thus $\|\iota_{\varepsilon_n} m_t^{\varepsilon_n} - \mu_t\|_{\mathrm{KR}(\mathbb{T}^d)} \to 0$. The dominated convergence theorem yields an absurd.

From (3.5) we deduce that for every $T \in (0, 1/2)$ there exists a sequence of times $(a_\varepsilon^T)_\varepsilon \subseteq (0, T)$ such that

$$\lim_{\varepsilon\to 0} \|\iota_\varepsilon m_{a_\varepsilon^T}^\varepsilon - \mu_{a_\varepsilon^T}\|_{\mathrm{KR}(\mathbb{T}^d)} = 0.$$

With a diagonal argument, we find a sequence $(a_\varepsilon)_\varepsilon \subseteq (0, 1/2)$ such that

$$\lim_{\varepsilon\to 0} a_\varepsilon = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} \|\iota_\varepsilon m_{a_\varepsilon}^\varepsilon - \mu_{a_\varepsilon}\|_{\mathrm{KR}(\mathbb{T}^d)} = \lim_{\varepsilon\to 0} \|\iota_\varepsilon m_{a_\varepsilon}^\varepsilon - \mu_0\|_{\mathrm{KR}(\mathbb{T}^d)} = 0.$$

Similarly, we can find another sequence $(b_\varepsilon)_\varepsilon \subseteq (1/2, 1)$ such that

$$\lim_{\varepsilon\to 0} b_\varepsilon = 1 \quad \text{and} \quad \lim_{\varepsilon\to 0} \|\iota_\varepsilon m_{b_\varepsilon}^\varepsilon - \mu_1\|_{\mathrm{KR}(\mathbb{T}^d)} = 0.$$

We claim the sought recovery sequences is provided by $m_0^\varepsilon := m_{a_\varepsilon}^\varepsilon$ and $m_1^\varepsilon := m_{b_\varepsilon}^\varepsilon$. In order to show this, let us define $\widehat{J}^\varepsilon : \mathcal{E}_\varepsilon \to \mathbb{R}$ via the formula[1] (recall the assumption (3.3))

$$\frac{\tau_\varepsilon^z \widehat{J}^\varepsilon}{\varepsilon^{d-1}} := \bar{J}, \qquad z \in \mathbb{Z}_\varepsilon^d,$$

so that $\widehat{J}^\varepsilon$ is divergence-free. Now define

$$\widetilde{m}_t^\varepsilon := \begin{cases} m_{a_\varepsilon}^\varepsilon & \text{if } t \in [0, a_\varepsilon) \\ m_t^\varepsilon & \text{if } t \in [a_\varepsilon, b_\varepsilon] \\ m_{b_\varepsilon}^\varepsilon & \text{if } t \in (b_\varepsilon, 1] \end{cases} \quad \text{and} \quad \widetilde{J}_t^\varepsilon := \begin{cases} \widehat{J}^\varepsilon & \text{if } t \in [0, a_\varepsilon) \\ J_t^\varepsilon & \text{if } t \in [a_\varepsilon, b_\varepsilon] \\ \widehat{J}^\varepsilon & \text{if } t \in (b_\varepsilon, 1] \end{cases}.$$

[1] The definition is well-posed because εR_0 is assumed to be smaller than $1/2$.

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It is readily verified that $(\widetilde{\boldsymbol{m}}^\varepsilon, \widetilde{\boldsymbol{J}}^\varepsilon)$ solves the continuity equation for every ε. Therefore,

$$
\begin{aligned}
\mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) \le \mathcal{A}_\varepsilon(\widetilde{\boldsymbol{m}}^\varepsilon, \widetilde{\boldsymbol{J}}^\varepsilon) &= \int_0^1 \sum_{z\in\mathbb{Z}_\varepsilon^d} \varepsilon^d F\left(\frac{\tau_\varepsilon^z \widetilde{m}_t^\varepsilon}{\varepsilon^d}, \frac{\tau_\varepsilon^z \widetilde{J}_t^\varepsilon}{\varepsilon^{d-1}}\right) \,\mathrm{d}t \\
&= \int_0^{a_\varepsilon} \sum_{z\in\mathbb{Z}_\varepsilon^d} \varepsilon^d F\left(\frac{\tau_\varepsilon^z m_{a_\varepsilon}^\varepsilon}{\varepsilon^d}, \bar{J}\right) \,\mathrm{d}t + \int_{a_\varepsilon}^{b_\varepsilon} \sum_{z\in\mathbb{Z}_\varepsilon^d} \varepsilon^d F\left(\frac{\tau_\varepsilon^z m_t^\varepsilon}{\varepsilon^d}, \frac{\tau_\varepsilon^z J_t^\varepsilon}{\varepsilon^{d-1}}\right) \,\mathrm{d}t \\
&\quad + \int_{b_\varepsilon}^1 \sum_{z\in\mathbb{Z}_\varepsilon^d} \varepsilon^d F\left(\frac{\tau_\varepsilon^z m_{b_\varepsilon}^\varepsilon}{\varepsilon^d}, \bar{J}\right) \,\mathrm{d}t \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{3.6}
$$

The first and last integral can be estimated using the assumption (3.3). Indeed,

$$
\begin{aligned}
I_1 + I_3 &\le C \sum_{z\in\mathbb{Z}_\varepsilon^d} \left((a_\varepsilon + 1 - b_\varepsilon)\varepsilon^d + \sum_{\substack{x\in\mathcal{X} \\ |x_{\mathsf{z}}|_\infty \le R}} \left(a_\varepsilon (\tau_\varepsilon^z m_{a_\varepsilon}^\varepsilon)(x) + (1-b_\varepsilon)(\tau_\varepsilon^z m_{b_\varepsilon}^\varepsilon)(x) \right) \right) \\
&\le C \left((a_\varepsilon + 1 - b_\varepsilon) + (2R+1)^d \sum_{x\in\mathcal{X}_\varepsilon} \left(a_\varepsilon m_{a_\varepsilon}^\varepsilon(x) + (1-b_\varepsilon) m_{b_\varepsilon}^\varepsilon(x) \right) \right) \\
&= C \left((a_\varepsilon + 1 - b_\varepsilon) + (2R+1)^d \left(a_\varepsilon \iota_\varepsilon m_{a_\varepsilon}^\varepsilon(\mathbb{T}^d) + (1-b_\varepsilon)\iota_\varepsilon m_{b_\varepsilon}^\varepsilon(\mathbb{T}^d) \right) \right),
\end{aligned}
$$

and in the limit we find

$$
\limsup_{\varepsilon\to 0} I_1 + I_3 \le C \left(0 + (2R+1)^d (0 \cdot \mu_0(\mathbb{T}^d) + 0 \cdot \mu_1(\mathbb{T}^d)) \right) = 0. \tag{3.7}
$$

As for the second integral, thanks to Assumption 2.8(c) we have that

$$
\begin{aligned}
I_2 - \mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) &= - \int_{(0,a_\varepsilon)\cup(b_\varepsilon,1)} \sum_{z\in\mathbb{Z}_\varepsilon^d} \varepsilon^d F\left(\frac{\tau_\varepsilon^z m_t^\varepsilon}{\varepsilon^d}, \frac{\tau_\varepsilon^z J_t^\varepsilon}{\varepsilon^{d-1}}\right) \,\mathrm{d}t \\
&\le C' \left((a_\varepsilon + 1 - b_\varepsilon) + (2R_{\max}+1)^d \iota_\varepsilon \boldsymbol{m}^\varepsilon \left(((0,a_\varepsilon)\cup(b_\varepsilon,1)) \times \mathbb{T}^d \right) \right).
\end{aligned} \tag{3.8}
$$

Since $(\iota_\varepsilon \boldsymbol{m}^\varepsilon)_\varepsilon$ converges weakly, for every $a, b \in (0,1)$, we have that

$$
\begin{aligned}
\limsup_{\varepsilon\to 0} \iota_\varepsilon \boldsymbol{m}^\varepsilon \left(((0,a_\varepsilon)\cup(b_\varepsilon,1)) \times \mathbb{T}^d \right) &\le \limsup_{\varepsilon\to 0} \iota_\varepsilon \boldsymbol{m}^\varepsilon \left(\left((0,a]\cup[b,1)\right) \times \mathbb{T}^d \right) \\
&\le \boldsymbol{\mu} \left(\left((0,a]\cup[b,1)\right) \times \mathbb{T}^d \right).
\end{aligned}
$$

Using the fact that the previous estimate holds for every $a, b \in (0,1)$, we obtain that

$$
\limsup_{\varepsilon\to 0} \iota_\varepsilon \boldsymbol{m}^\varepsilon \left(((0,a_\varepsilon)\cup(b_\varepsilon,1)) \times \mathbb{T}^d \right) = 0.
$$

This, together with the estimate obtained in (3.8), gives us the inequality

$$
\limsup_{\varepsilon\to 0} I_2 \le \limsup_{\varepsilon\to 0} \mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon). \tag{3.9}
$$

In conclusion, from (3.6), (3.7), and (3.9), we find

$$
\limsup_{\varepsilon\to 0} \mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) \le \limsup_{\varepsilon\to 0} \mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) \le \mathbb{A}(\boldsymbol{\mu}) \le \mathbb{MA}(\mu_0, \mu_1) + 1/k,
$$

which is sought upper bound. □

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3.2. Proof of the liminf inequality

In this section, we provide the proof of the liminf inequality in Theorem 3.9. Let m_0^ε, m_1^ε be sequences of measures weakly converging to μ_0, μ_1, respectively. We want to show that

$$\liminf_{\varepsilon\to 0} \mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) \geq \mathbb{MA}_{\mathrm{hom}}(\mu_0, \mu_1). \tag{3.10}$$

Without loss of generality, we will assume that the limit inferior in the latter is a true finite limit, and that $m_0^\varepsilon(\mathcal{X}_\varepsilon) = m_1^\varepsilon(\mathcal{X}_\varepsilon)$ for every $\varepsilon > 0$.

We split the proof into two parts: first for F with linear growth and then for F of flow-based type, respectively Assumption 3.1 and Assumption 3.2.

3.2.1. Case 1: F with linear growth

Assume that F satisfies Assumption 3.1. Recall that, as a consequence of Lemma 3.5, F is Lipschitz continuous as well, in the sense of (3.1).

Proof of the liminf inequality (linear growth). With a very similar argument as the one provided by Remark 2.7 in the continuous setting, we can with no loss of generality assume that F is nonnegative. Let $(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) \in \mathcal{CE}_\varepsilon$ be approximate optimal solutions associated to $\mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon)$, i.e. such that

$$\lim_{\varepsilon\to 0}\Big(\mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) - \mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon)\Big) = 0. \tag{3.11}$$

As usual, we write $\mathrm{d}\boldsymbol{m}^\varepsilon(t,x) = m_t^\varepsilon(\mathrm{d}x)\,\mathrm{d}t$ for some measurable curve $t \mapsto m_t^\varepsilon \in \mathbb{R}_+^{\mathcal{X}_\varepsilon}$ of constant, finite mass. By compactness (Remark 3.11), we know that up to a further non-relabelled subsequence, $\boldsymbol{m}^\varepsilon \to \boldsymbol{\mu}$ weakly in $\mathcal{M}_+\big((0,1)\times\mathbb{T}^d\big)$ with $\boldsymbol{\mu} \in \mathrm{BV}_{\mathrm{KR}}\big((0,1);\mathcal{M}_+(\mathbb{T}^d)\big)$. Due to the lack of continuity of the trace operators in BV, a priori we cannot conclude that $\boldsymbol{\mu}_{t=0} = \mu_0$ and $\boldsymbol{\mu}_{t=1} = \mu_1$. In other words, there might be a "jump" in the limit as $\varepsilon \to 0$ at the boundary of $(0, 1)$. In order to take care of this problem, we rescale our measures $\boldsymbol{m}^\varepsilon$ in time, so as to be able to "see" the jump in the interior of $(0, 1)$.

To this purpose, for $\delta \in (0, 1/2)$, we define $\mathcal{I}_\delta := (\delta, 1-\delta)$ and $\boldsymbol{m}^{\varepsilon,\delta} \in \mathrm{BV}_{\mathrm{KR}}\big((0,1);\mathcal{M}_+(\mathbb{T}^d)\big)$ as

$$m_t^{\varepsilon,\delta} := \begin{cases} m_0^\varepsilon & \text{if } t \in (0,\delta] \\ m^\varepsilon_{\frac{t-\delta}{1-2\delta}} & \text{if } t \in \mathcal{I}_\delta \\ m_1^\varepsilon & \text{if } t \in [1-\delta, 1) \end{cases}, \qquad \mathrm{d}\boldsymbol{m}^{\varepsilon,\delta}(t,x) := m_t^{\varepsilon,\delta}(\mathrm{d}x)\,\mathrm{d}t. \tag{3.12}$$

By construction, the convergence of the boundary data, and the fact that, by assumption, $\boldsymbol{m}^\varepsilon \to \boldsymbol{\mu}$ weakly, it is straightforward to see that $\boldsymbol{m}^{\varepsilon,\delta} \to \widehat{\boldsymbol{\mu}}^\delta$ weakly, where

$$\widehat{\mu}_t^\delta := \begin{cases} \mu_0 & \text{if } t \in (0,\delta] \\ \mu_{\frac{t-\delta}{1-2\delta}} & \text{if } t \in \mathcal{I}_\delta \\ \mu_1 & \text{if } t \in [1-\delta, 1) \end{cases}, \qquad \mathrm{d}\widehat{\boldsymbol{\mu}}^\delta(t,x) := \widehat{\mu}_t^\delta(\mathrm{d}x)\,\mathrm{d}t. \tag{3.13}$$

Note that the rescaled curve $t \mapsto \widehat{\mu}_t^\delta$ might have discontinuities at $t = \delta$ and $t = 1-\delta$, which correspond to the possible jumps in the limit as $\varepsilon \to 0$ for $\boldsymbol{m}^\varepsilon$ at $\{0, 1\}$. Nevertheless, $\widehat{\boldsymbol{\mu}}^\delta$ is a competitor for $\mathbb{MA}(\mu_0, \mu_1)$, which, by the Γ-convergence of $\mathcal{A}_\varepsilon$ towards $\mathbb{A}_{\mathrm{hom}}$ (Remark 3.10), ensures that

$$\liminf_{\varepsilon\to 0} \mathcal{A}_\varepsilon(\boldsymbol{m}^{\varepsilon,\delta}) \geq \mathbb{A}_{\mathrm{hom}}(\widehat{\boldsymbol{\mu}}^\delta) \geq \mathbb{MA}_{\mathrm{hom}}(\mu_0, \mu_1). \tag{3.14}$$

We are left with estimating from above the left-hand side of the latter displayed equation. To do so, we seek a suitable curve of discrete vector fields $\boldsymbol{J}^{\varepsilon,\delta}$ with $(\boldsymbol{m}^{\varepsilon,\delta}, \boldsymbol{J}^{\varepsilon,\delta}) \in \mathcal{CE}_\varepsilon$ and having an action $\mathcal{A}_\varepsilon(\boldsymbol{m}^{\varepsilon,\delta}, \boldsymbol{J}^{\varepsilon,\delta})$ comparable with $\mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon)$ for small $\delta > 0$. We set

$$J_t^{\varepsilon,\delta} := \begin{cases} 0 & \text{if } t \in (0,\delta] \\ \frac{1}{1-2\delta} J^\varepsilon_{\frac{t-\delta}{1-2\delta}} & \text{if } t \in \mathcal{I}_\delta \\ 0 & \text{if } t \in [1-\delta, 1) \end{cases}, \qquad \mathrm{d}\boldsymbol{J}^{\varepsilon,\delta}(t,x) := J_t^{\varepsilon,\delta}(\mathrm{d}x)\,\mathrm{d}t.$$

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We claim that $(\boldsymbol{m}^{\varepsilon,\delta},\boldsymbol{J}^{\varepsilon,\delta})\in\mathcal{CE}_\varepsilon$ and

$$\mathcal{A}_\varepsilon(\boldsymbol{m}^{\varepsilon,\delta},\boldsymbol{J}^{\varepsilon,\delta})\le\big(1+C(F)\delta\big)\mathcal{A}_\varepsilon(\boldsymbol{m}^{\varepsilon},\boldsymbol{J}^{\varepsilon})+C(F)\delta\Big(1+\iota_\varepsilon\boldsymbol{m}^\varepsilon((0,1)\times\mathbb{T}^d)\Big),\tag{3.15}$$

where $C(F)\in\mathbb{R}_+$ only depends on F (specifically on the constants in Assumption 2.8 and Assumption 3.1). This would suffice to conclude the proof of the sought liminf inequality. Indeed, from (3.11) and (3.15), we infer

$$\begin{aligned}\liminf_{\varepsilon\to0}\mathcal{MA}_\varepsilon(m_0^\varepsilon,m_1^\varepsilon)&=\liminf_{\varepsilon\to0}\mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{J}^\varepsilon)\\&\ge\frac{1}{1+C(F)\delta}\liminf_{\varepsilon\to0}\mathcal{A}_\varepsilon(\boldsymbol{m}^{\varepsilon,\delta},\boldsymbol{J}^{\varepsilon,\delta})-\frac{C(F)\delta}{1+C(F)\delta}\Big(1+\iota_\varepsilon\boldsymbol{m}^\varepsilon((0,1)\times\mathbb{T}^d)\Big)\end{aligned}$$

which, combined with (3.14), yields

$$\liminf_{\varepsilon\to0}\mathcal{MA}_\varepsilon(m_0^\varepsilon,m_1^\varepsilon)\ge\frac{\mathbb{MA}_{\mathrm{hom}}(\mu_0,\mu_1)}{1+C(F)\delta}-\frac{C(F)\delta}{1+C(F)\delta}\Big(1+\mu_0(\mathbb{T}^d)\Big)$$

for any $\delta\in(0,1/2)$. We conclude by letting $\delta\to0$.

We are left with the proof of $(\boldsymbol{m}^{\varepsilon,\delta},\boldsymbol{J}^{\varepsilon,\delta})\in\mathcal{CE}_\varepsilon$ and of the claim (3.15).

Proof of $(\boldsymbol{m}^{\varepsilon,\delta},\boldsymbol{J}^{\varepsilon,\delta})\in\mathcal{CE}_\varepsilon$. Let us fix $x\in\mathcal{X}_\varepsilon$ and $\varphi\in C_c^1\big((0,1)\big)$. Set $\widetilde{\varphi}:=\varphi\circ r_\delta$, with $r_\delta(s):=(1-2\delta)s+\delta$. We have

$$\begin{aligned}\int_0^1\partial_t\varphi m_t^{\varepsilon,\delta}(x)\,\mathrm{d}t&=\int_0^\delta\partial_t\varphi\,m_0^\varepsilon(x)\,\mathrm{d}t+\int_{1-\delta}^1\partial_t\varphi\,m_1^\varepsilon(x)\,\mathrm{d}t+\int_{\mathcal{I}_\delta}\partial_t\varphi\,m^\varepsilon_{r_\delta^{-1}(t)}(x)\,\mathrm{d}t\\&=\varphi(\delta)\,m_0^\varepsilon(x)-\varphi(1-\delta)\,m_1^\varepsilon(x)+(1-2\delta)\int_0^1(\partial_t\varphi)\circ r_\delta\,m_s^\varepsilon(x)\,\mathrm{d}s\\&=\widetilde{\varphi}(0)\,m_0^\varepsilon(x)-\widetilde{\varphi}(1)\,m_1^\varepsilon(x)+\int_0^1\partial_s\widetilde{\varphi}\,m_s^\varepsilon(x)\,\mathrm{d}s\\&=\int_0^1\widetilde{\varphi}\sum_{y\sim x}J_s^\varepsilon(x,y)\,\mathrm{d}s=\frac{1}{1-2\delta}\int_{\mathcal{I}_\delta}\varphi\sum_{y\sim x}J^\varepsilon_{r_\delta^{-1}(t)}(x,y)\,\mathrm{d}t\\&=\int_0^1\varphi\sum_{y\sim x}J_t^{\varepsilon,\delta}(x,y)\,\mathrm{d}t,\end{aligned}\tag{3.16}$$

where, in the fourth equality, we used that $(\boldsymbol{m}^\varepsilon,\boldsymbol{J}^\varepsilon)\in\mathcal{CE}_\varepsilon$.

Proof of the action estimate. Define $r_\delta(s):=(1-2\delta)s+\delta$. Note that, by construction, for $(t,(x,y))\in\mathcal{I}_\delta\times\mathcal{E}_\varepsilon$,

$$m_t^{\varepsilon,\delta}(x)=m^\varepsilon_{r_\delta^{-1}(t)}(x),\qquad J_t^{\varepsilon,\delta}(x,y)=\frac{1}{1-2\delta}J^\varepsilon_{r_\delta^{-1}(t)}(x,y).$$

On the other hand, for $(t,(x,y))\in\big((0,\delta]\cup[1-\delta,1)\big)\times\mathcal{E}_\varepsilon$, we have that

$$m_t^{\varepsilon,\delta}(x)=\begin{cases}m_0^\varepsilon(x)&\text{if }t\in(0,\delta]\\m_1^\varepsilon(x)&\text{if }t\in[1-\delta,1)\end{cases}\qquad\text{and}\qquad J_t^{\varepsilon,\delta}(x,y)=0.$$

It follows that the action of $(\boldsymbol{m}^{\varepsilon,\delta},\boldsymbol{J}^{\varepsilon,\delta})$ is given by

$$\mathcal{A}_\varepsilon(\boldsymbol{m}^{\varepsilon,\delta},\boldsymbol{J}^{\varepsilon,\delta})=\int_0^1\mathcal{F}_\varepsilon(m_t^{\varepsilon,\delta},J_t^{\varepsilon,\delta})\,\mathrm{d}t=\mathcal{A}_\varepsilon^{\mathcal{I}_\delta}(\boldsymbol{m}^{\varepsilon,\delta},\boldsymbol{J}^{\varepsilon,\delta})+\delta\sum_{i=0,1}\mathcal{F}_\varepsilon(m_i^\varepsilon,0),\tag{3.17}$$

where we used the notation

$$\mathcal{A}_\varepsilon^{\mathcal{I}_\delta}(\boldsymbol{m}^{\varepsilon,\delta},\boldsymbol{J}^{\varepsilon,\delta}):=\int_{\mathcal{I}_\delta}\mathcal{F}_\varepsilon(m_t^{\varepsilon,\delta},J_t^{\varepsilon,\delta})\,\mathrm{d}t=(1-2\delta)\int_0^1\mathcal{F}_\varepsilon\Big(m_t^\varepsilon,\frac{1}{1-2\delta}J_t^\varepsilon\Big)\,\mathrm{d}t.$$

Using Assumption 3.1, we see that, for $i=0,1$,

$$\mathcal{F}_\varepsilon(m_i^\varepsilon,0)\le C(m_i^\varepsilon(\mathcal{X}_\varepsilon)+1)=C\big(\iota_\varepsilon\boldsymbol{m}^\varepsilon\big((0,1)\times\mathbb{T}^d\big)+1\big)$$

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and, by the Lipschitz continuity exhibited in Lemma 3.5, we also infer that

$$\mathcal{A}_\varepsilon^{\mathcal{I}_\delta}(\boldsymbol{m}^{\varepsilon,\delta},\boldsymbol{J}^{\varepsilon,\delta}) \le (1-2\delta)\left(\mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{J}^\varepsilon) + C\Big(\frac{1}{1-2\delta}-1\Big)\sum_{z\in\mathbb{Z}_\varepsilon^d}\varepsilon^d \sum_{\substack{(x,y)\in\mathcal{E} \\ |x_{\mathsf{z}}|_\infty\le R}} \int_0^1 \frac{|\tau_\varepsilon^z J_t^\varepsilon(x,y)|}{\varepsilon^{d-1}}\,\mathrm{d}t\right)$$
$$\le (1-2\delta)\mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{J}^\varepsilon) + 2\delta C(2R+1)^d \sum_{z\in\mathbb{Z}_\varepsilon^d}\varepsilon^d \sum_{(x,y)\in\mathcal{E}^Q} \int_0^1 \frac{|\tau_\varepsilon^z J_t^\varepsilon(x,y)|}{\varepsilon^{d-1}}\,\mathrm{d}t.$$

Since we assumed F to be nonnegative, we can estimate

$$(1-2\delta)\mathcal{A}_\varepsilon(m^\varepsilon,J^\varepsilon) \le \mathcal{A}_\varepsilon(m^\varepsilon,J^\varepsilon)$$

and, using (2.8),

$$\sum_{z\in\mathbb{Z}_\varepsilon^d}\varepsilon^d \sum_{(x,y)\in\mathcal{E}^Q} \int_0^1 \frac{|\tau_\varepsilon^z J_t^\varepsilon(x,y)|}{\varepsilon^{d-1}}\,\mathrm{d}t \le \frac{1}{c}\mathcal{A}_\varepsilon(m^\varepsilon,J^\varepsilon) + \frac{C}{c}\left(1+(1+2R_{\max})^d\|\iota_\varepsilon m^\varepsilon\|_{\mathrm{TV}}\right).$$

Combining these estimates with (3.17), we find (3.15). □

3.2.2. *Case 2: F is flow-based*

In this section, we show (3.10) in the case where F (and hence f_{hom}) is of flow-based type, i.e. it satisfies Assumption 3.2. We start by observing that, in this particular setting, both the discrete and the continuous formulations of the boundary-value problems admit an equivalent, static formulation.

Let $(\boldsymbol{\mu},\boldsymbol{\nu})\in\mathsf{CE}$ and consider the Lebesgue decomposition

$$\boldsymbol{\mu} = \rho\mathscr{L}^{d+1} + \rho^\perp\boldsymbol{\sigma}, \qquad \boldsymbol{\nu} = j\mathscr{L}^{d+1} + j^\perp\boldsymbol{\sigma}.$$

We know that every solution to the continuity equation can be disintegrated in the form $\boldsymbol{\mu}(\mathrm{d}t,\mathrm{d}x) = \mu_t(\mathrm{d}x)\,\mathrm{d}t$ for some measurable curve $t\mapsto\mu_t\in\mathcal{M}_+(\mathbb{T}^d)$ of constant, finite mass. If f is a function as in Assumption 2.2 that further does *not* depend on ρ, then Jensen's inequality yields

$$\int_0^1\int_{\mathbb{T}^d} f(j_t)\,\mathrm{d}x\,\mathrm{d}t \ge \int_{\mathbb{T}^d} f\left(\int_0^1 j_t\,\mathrm{d}t\right)\mathrm{d}x. \tag{3.18}$$

In order to take care of the singular part, consider the disintegration of $\boldsymbol{\sigma}$ with respect to the projection map $\pi:(t,x)\mapsto x$, in the form

$$\boldsymbol{\sigma}(\mathrm{d}t,\mathrm{d}x) = \sigma^x(\mathrm{d}t)(\pi_\#\boldsymbol{\sigma})(\mathrm{d}x),$$

for some measurable $x\mapsto\sigma^x\in\mathscr{P}\big((0,1)\big)$. Due to the convexity of f^∞, by Jensen's inequality we also obtain

$$\int_{(0,1)\times\mathbb{T}^d} f^\infty(j^\perp)\,\mathrm{d}\boldsymbol{\sigma} \ge \int_{\mathbb{T}^d} f^\infty\left(\int j^\perp\,\mathrm{d}\sigma^x\right)\mathrm{d}\pi_\#\boldsymbol{\sigma}(x). \tag{3.19}$$

Now, we define the new space-time measures

$$\widetilde{\boldsymbol{\mu}} := \widetilde{\mu}_t(\mathrm{d}x)\,\mathrm{d}t \quad\text{and}\quad \widetilde{\boldsymbol{\nu}} := \widehat{j}\mathscr{L}^{d+1} + \widehat{j}^\perp\,\mathrm{d}t\otimes\pi_\#\boldsymbol{\sigma}, \quad\text{where}$$
$$\widetilde{\mu}_t := \mu_0 + t(\mu_1-\mu_0), \quad \widehat{j}(x) := \int_0^1 j_t(x)\,\mathrm{d}t, \quad\text{and } \widehat{j}^\perp(x) := \int j^\perp\,\mathrm{d}\sigma^x, \tag{3.20}$$

and note that $(\widetilde{\boldsymbol{\mu}},\widetilde{\boldsymbol{\nu}})\in\mathsf{CE}$. By (3.18) and (3.19), we, therefore, have

$$\mathbb{A}(\boldsymbol{\mu},\boldsymbol{\nu}) \ge \int_{\mathbb{T}^d} f(\widehat{j})\,\mathrm{d}x + \int_{\mathbb{T}^d} f^\infty(\widehat{j}^\perp)\,\mathrm{d}\pi_\#\boldsymbol{\sigma}(x). \tag{3.21}$$

We need to be careful here: the decomposition of $\widetilde{\boldsymbol{\nu}}$ in (3.20) may not be a Lebesgue decomposition, in the sense that $\mathrm{d}t\otimes\pi_\#\boldsymbol{\sigma}$ can have a nonzero absolutely continuous part. Let $\widetilde{\sigma}\in\mathcal{M}_+(\mathbb{T}^d)$ be singular

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w.r.t. $\mathscr{L}^d$ and such that $\mu_0, \mu_1, \pi_\# \boldsymbol{\sigma} \ll \mathscr{L}^d + \widetilde{\sigma}$. We can write the Lebesgue decompositions

$$\widetilde{\boldsymbol{\mu}} = \widetilde{\rho}\mathscr{L}^{d+1} + \widetilde{\rho}^\perp \,\mathrm{d}t \otimes \widetilde{\sigma}, \qquad \widetilde{\boldsymbol{\nu}} = \widetilde{j}\mathscr{L}^{d+1} + \widetilde{j}^\perp \,\mathrm{d}t \otimes \widetilde{\sigma}.$$

If we write

$$\pi_\# \boldsymbol{\sigma} = \alpha \mathscr{L}^d + \beta \widetilde{\sigma}$$

for some functions $\alpha, \beta : \mathbb{T}^d \to \mathbb{R}_+$, then

$$\widetilde{j} = \widehat{j} + \alpha \widehat{j}^\perp \quad \text{and} \quad \widetilde{j}^\perp = \beta \widehat{j}^\perp.$$

The inequality (3.21) becomes, recalling that f^∞ is 1-homogeneous,

$$\mathbb{A}(\boldsymbol{\mu}, \boldsymbol{\nu}) \geq \int_{\mathbb{T}^d} \big(f(\widehat{j}) + f^\infty(\alpha \widehat{j}^\perp)\big) \,\mathrm{d}x + \int_{\mathbb{T}^d} f^\infty(\beta \widehat{j}^\perp) \,\mathrm{d}\widetilde{\sigma}. \tag{3.22}$$

At this point, we need a lemma.

Lemma 3.13. *For every $j_1, j_2 \in \mathbb{R}^d$, we have that $f(j_1 + j_2) \leq f(j_1) + f^\infty(j_2)$.*

Proof. Let $g \leq f$ be a convex and Lipschitz continuous function. By convexity, for every $\epsilon \in (0, 1)$, we have

$$g(j_1 + j_2) = g\left((1-\epsilon)\frac{j_1}{1-\epsilon} + \epsilon \frac{j_2}{\epsilon}\right) \leq (1-\epsilon) g\left(\frac{j_1}{1-\epsilon}\right) + \epsilon g\left(\frac{j_2}{\epsilon}\right).$$

Let $j_0 \in \mathsf{D}(f)$. By the Lipschitz continuity of g,

$$g(j_1 + j_2) \leq (1-\epsilon)\left(g(j_1) + (\mathrm{Lip} g)\left(\frac{1}{1-\epsilon} - 1\right)|j_1|\right) + \epsilon g\left(\frac{j_2}{\epsilon} + j_0\right) + \epsilon(\mathrm{Lip} g)|j_0|$$

and, since $g \leq f$,

$$g(j_1 + j_2) \leq (1-\epsilon) f(j_1) + \epsilon f\left(\frac{j_2}{\epsilon} + j_0\right) + \epsilon(\mathrm{Lip} g)\big(|j_0| + |j_1|\big).$$

As we let $\epsilon \to 0$, we find

$$g(j_1 + j_2) \leq f(j_1) + f^\infty(j_2).$$

Since f is convex and lower semicontinuous, we conclude by an approximation argument. □

Applying this lemma with $j_1 = \widehat{j}(x)$ and $j_2 = \alpha \widehat{j}^\perp(x)$ for every $x \in \mathbb{T}^d$, (3.22) finally becomes

$$\mathbb{A}(\boldsymbol{\mu}, \boldsymbol{\nu}) \geq \int_{\mathbb{T}^d} f(\widetilde{j}) \,\mathrm{d}x + \int_{\mathbb{T}^d} f^\infty(\widetilde{j}^\perp) \,\mathrm{d}\widetilde{\sigma} = \mathbb{A}(\widetilde{\boldsymbol{\mu}}, \widetilde{\boldsymbol{\nu}}).$$

In other words, we have shown that an optimal curve $\boldsymbol{\mu}$ between two given boundary data is always given by the affine interpolation (and a constant-in-time flux). We conclude that

$$\begin{aligned} \mathbb{MA}(\mu_0, \mu_1) &= \mathbb{A}(\widetilde{\boldsymbol{\mu}}) \\ &= \inf_\nu \left\{ \int_{\mathbb{T}^d} f(\widetilde{j}) \,\mathrm{d}x + \int_{\mathbb{T}^d} f^\infty(\widetilde{j}^\perp) \,\mathrm{d}\widetilde{\sigma} : \nu = \widetilde{j}\mathscr{L}^d + \widetilde{j}^\perp \widetilde{\sigma},\ \mathscr{L}^d \perp \widetilde{\sigma} \ \text{ and } \ \nabla \cdot \nu = \mu_0 - \mu_1 \right\}. \end{aligned} \tag{3.23}$$

We refer to the latter expression as the *static formulation* of the boundary value problem described by $\mathbb{MA}(\mu_0, \mu_1)$ (in the case when f is of flow-based type).

Remark 3.14. Using this equivalence, the lower semicontinuity of $\mathbb{MA}$ directly follows from the semicontinuity of $\mathbb{A}$ given by [[23], Lemma 3.14].

Arguing in a similar way (in fact, via an even simpler argument, due to the lack of singularities), we obtain a static formulation of the discrete transport problem in terms of a discrete divergence equation, when $F(m, J) = F(J)$. Precisely, in this case we obtain

$$\mathcal{MA}_\varepsilon(m_0, m_1) = \inf\left\{\mathcal{F}_\varepsilon(J) : J \in \mathbb{R}_a^{\mathcal{E}_\varepsilon}, \quad \mathsf{div} J = m_0 - m_1\right\}.$$

The sought Γ-liminf inequality easily follows from such static formulations.

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Proof of the liminf inequality (flow-based type). Let $m_0^\varepsilon, m_1^\varepsilon \in \mathbb{R}_+^{\mathcal{X}_\varepsilon}$ be sequences of discrete nonnegative measures that converge weakly (via ι_ε in the usual sense) to μ_0, μ_1, and such that $m_0^\varepsilon(\mathcal{X}_\varepsilon) = m_1^\varepsilon(\mathcal{X}_\varepsilon)$ for every $\varepsilon > 0$. Let $(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) \in \mathcal{CE}_\varepsilon$ be (almost-)optimal solutions associated with $\mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon)$, namely

$$\liminf_{\varepsilon\to 0} \mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) = \liminf_{\varepsilon\to 0} \mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon) = \liminf_{\varepsilon\to 0} \mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon). \tag{3.24}$$

Consider the discrete equivalent of the measure constructed in (3.20), namely

$$\widetilde{m}_t^\varepsilon := m_0^\varepsilon + t(m_1^\varepsilon - m_0^\varepsilon) \quad \text{and} \quad \widetilde{J}_t^\varepsilon \equiv \widetilde{J}^\varepsilon := \int_0^1 J_s^\varepsilon \,\mathrm{d}s,$$

which still solves the continuity equation. By applying Jensen's inequality, the convexity of F ensures that

$$\mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) \ge \mathcal{A}_\varepsilon(\widetilde{\boldsymbol{m}}^\varepsilon, \widetilde{\boldsymbol{J}}^\varepsilon) = \mathcal{F}_\varepsilon(\widetilde{J}^\varepsilon) \quad \text{and} \quad (\widetilde{\boldsymbol{m}}^\varepsilon, \widetilde{\boldsymbol{J}}^\varepsilon) \in \mathcal{CE}_\varepsilon. \tag{3.25}$$

Thus $\mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) \ge \mathcal{A}_\varepsilon(\widetilde{\boldsymbol{m}}^\varepsilon)$. Note that, by construction, $\widetilde{\boldsymbol{m}}^\varepsilon \to \widetilde{\boldsymbol{\mu}}$ weakly, where

$$\widetilde{\boldsymbol{\mu}} := \widetilde{\mu}_t(\,\mathrm{d}x)\,\mathrm{d}t \quad \text{with} \quad \widetilde{\mu}_t := \mu_0 + t(\mu_1 - \mu_0).$$

Hence, from (3.24), (3.25), and the Γ-convergence of $\mathcal{A}_\varepsilon$ to $\mathbb{A}_{\mathrm{hom}}$ (cfr. [[23], Theorem 5.1]), we infer that

$$\liminf_{\varepsilon\to 0} \mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) \ge \mathbb{A}_{\mathrm{hom}}(\widetilde{\boldsymbol{\mu}}) \ge \mathbb{MA}_{\mathrm{hom}}(\mu_0, \mu_1),$$

which concludes the proof of the liminf inequality. □

3.3. About the lower semicontinuity of $\mathbb{MA}$

In view of our main result, whenever F satisfies either Assumption 3.1 or Assumption 3.2, the limit boundary-value problem $\mathbb{MA}_{\mathrm{hom}}(\cdot,\cdot)$ is necessarily jointly lower semicontinuous with respect to the weak topology on $\mathcal{M}_+(\mathbb{T}^d) \times \mathcal{M}_+(\mathbb{T}^d)$. This indeed follows from the general fact that any Γ-limit with respect to a given topology is always lower semicontinuous with respect to that same topology. Using a very similar proof to that of the Γ-liminf inequality, we can actually show that, if f is with linear growth or it is of flow-based type, then the associated $\mathbb{MA}$ is always lower semicontinuous (even if, a priori, f is not of the form $f = f_{\mathrm{hom}}$), thus providing a positive answer in this framework to the validity of (2.5). In the flow-based setting, this fact has been observed in Remark 3.14.

Proposition 3.15. *Assume that f is with linear growth, namely it satisfies one of the two equivalent conditions appearing in Lemma 3.4, and assume that $(\mu_0^n, \mu_1^n) \to (\mu_0, \mu_1) \in \mathcal{M}_+(\mathbb{T}^d) \times \mathcal{M}_+(\mathbb{T}^d)$ weakly. Then:*

$$\liminf_{n\to\infty} \mathbb{MA}(\mu_0^n, \mu_1^n) \ge \mathbb{MA}(\mu_0, \mu_1).$$

The proof goes along the same line of the proof of the Γ-liminf inequality for discrete energies F with linear growth. We sketch it here and add details whenever we encounter nontrivial differences between the two proofs.

Proof. Let $(\boldsymbol{\mu}^n, \boldsymbol{\nu}^n) \in \mathsf{CE}$ be (almost-)optimal solutions associated to $\mathbb{MA}(\mu_0^n, \mu_1^n)$, i.e.

$$\liminf_{n\to\infty} \mathbb{MA}(\mu_0^n, \mu_1^n) = \liminf_{n\to\infty} \mathbb{A}(\boldsymbol{\mu}^n, \boldsymbol{\nu}^n). \tag{3.26}$$

With no loss of generality, we can assume $\sup_n \mathbb{A}(\boldsymbol{\mu}^n, \boldsymbol{\nu}^n) < \infty$ and that the limits inferior are true limits. By the compactness of Remark 2.5, we know that, up to a non-relabelled subsequence, $\boldsymbol{\mu}^n \to \boldsymbol{\mu}$ weakly in $\mathcal{M}_+\big((0,1)\times\mathbb{T}^d\big)$. Moreover, we also have $\mathrm{d}\boldsymbol{\mu}(t,x) = \mu_t(\,\mathrm{d}x)\,\mathrm{d}t \in \mathrm{BV}_{\mathrm{KR}}\big((0,1);\mathcal{M}_+(\mathbb{T}^d)\big)$ for some measurable curve $t \mapsto \mu_t \in \mathcal{M}_+(\mathbb{T}^d)$ of constant, finite mass. Once again, due to the lack of continuity of the trace operators in BV, we cannot ensure that $\boldsymbol{\mu}_{t=0} = \mu_0$ and $\boldsymbol{\mu}_{t=1} = \mu_1$. To solve this issue, we rescale

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our measures $\boldsymbol{\mu}^n$ in time in the same spirit as in (3.12). For a given $\delta > 0$, we define $\mathcal{I}_\delta := (\delta, 1-\delta)$ and $\boldsymbol{\mu}^{n,\delta} \in \mathrm{BV}_{\mathrm{KR}}\big((0,1);\mathcal{M}_+(\mathbb{T}^d)\big)$ as

$$\mu_t^{n,\delta} := \begin{cases} \mu_0^n & \text{if } t \in (0,\delta] \\ \mu^n_{\frac{t-\delta}{1-2\delta}} & \text{if } t \in \mathcal{I}_\delta \\ \mu_1^n & \text{if } t \in [1-\delta, 1) \end{cases}, \quad \mathrm{d}\boldsymbol{\mu}^{n,\delta}(t,x) := \mu_t^{n,\delta}(\mathrm{d}x)\,\mathrm{d}t.$$

By construction, it is not hard to see that $\boldsymbol{\mu}^{n,\delta} \to \widehat{\boldsymbol{\mu}}^\delta$ weakly, where

$$\widehat{\mu}_t^\delta := \begin{cases} \mu_0 & \text{if } t \in (0,\delta] \\ \mu_{\frac{t-\delta}{1-2\delta}} & \text{if } t \in \mathcal{I}_\delta \\ \mu_1 & \text{if } t \in [1-\delta, 1) \end{cases}, \quad \mathrm{d}\widehat{\boldsymbol{\mu}}^{\delta}(t,x) := \widehat{\mu}_t^{\delta}(\mathrm{d}x)\,\mathrm{d}t.$$

We stress that, as in (3.13), the rescaled curve $t \mapsto \widehat{\mu}_t^\delta$ could have discontinuities at $t=\delta$ and $t = 1-\delta$, corresponding to the possible jumps in the limit as $n \to \infty$ for $\boldsymbol{\mu}^n$ at $\{0,1\}$. Nevertheless, $\widehat{\boldsymbol{\mu}}^\delta$ is a competitor for $\mathbb{MA}(\mu_0,\mu_1)$, which, by lower semicontinuity of $\mathbb{A}$, ensures that

$$\liminf_{n\to\infty} \mathbb{A}(\boldsymbol{\mu}^{n,\delta}) \ge \mathbb{A}(\widehat{\boldsymbol{\mu}}^\delta) \ge \mathbb{MA}(\mu_0,\mu_1). \tag{3.27}$$

In order to estimate the left-hand side of the latter displayed equation, we seek a suitable vector meausure $\boldsymbol{\nu}^{n,\delta}$ so that $(\boldsymbol{\mu}^{n,\delta}, \boldsymbol{\nu}^{n,\delta}) \in \mathsf{CE}$ and whose action $\mathbb{A}(\boldsymbol{\mu}^{n,\delta}, \boldsymbol{\nu}^{n,\delta})$ is comparable with $\mathbb{A}(\boldsymbol{\mu}^n, \boldsymbol{\nu}^n)$ for small $\delta > 0$.

It is useful to introduce the following notation: for $\delta \in (0, 1/2)$,

$$r_\delta\colon(0,1) \to \mathcal{I}_\delta, \quad r_\delta(s) := (1-2\delta)s + \delta,$$

$$R_\delta\colon(0,1)\times\mathbb{T}^d \to \mathcal{I}_\delta \times \mathbb{T}^d, \quad R_\delta(s,x) := (r_\delta(s), x).$$

Define $\widehat{\iota}_\delta\colon\mathcal{M}^d(\mathcal{I}_\delta \times \mathbb{T}^d) \to \mathcal{M}^d\big((0,1)\times\mathbb{T}^d\big)$ as the natural embedding obtained by extending to 0 any measure outside $\mathcal{I}_\delta$, and set

$$\boldsymbol{\nu}^{n,\delta} := \widehat{\iota}_\delta\big[(R_\delta)_\# \boldsymbol{\nu}^n\big] \in \mathcal{M}^d((0,1)\times\mathbb{T}^d).$$

The proof that $(\boldsymbol{\mu}^{n,\delta}, \boldsymbol{\nu}^{n,\delta}) \in \mathsf{CE}$ works as in (3.16). In the same spirit as in (3.15), we claim that

$$\mathbb{A}(\boldsymbol{\mu}^{n,\delta}, \boldsymbol{\nu}^{n,\delta}) \le \big(1 + C(f)\delta\big)\mathbb{A}(\boldsymbol{\mu}^n, \boldsymbol{\nu}^n) + C(f)\delta\Big(1 + \boldsymbol{\mu}^n((0,1)\times\mathbb{T}^d)\Big), \tag{3.28}$$

where $C(f) \in \mathbb{R}_+$ only depends on f. The combination of (3.26), (3.27) and (3.28), and the arbitrariness of δ would then suffice to conclude the proof.

We are left with the proof of the claim (3.28), which is a bit more involved, compared to that of (3.15), due to the presence of the singular part at the continuous level. We need the following.

Lemma 3.16. *Let $\boldsymbol{\sigma} \in \mathcal{M}_+\big((0,1)\times\mathbb{T}^d\big)$ be a singular measure with respect to $\mathscr{L}^{d+1}$. Then, the measure $(R_\delta)_\#\boldsymbol{\sigma} \in \mathcal{M}_+(\mathcal{I}_\delta \times \mathbb{T}^d)$ is also singular with respect to $\mathscr{L}^{d+1}$. Moreover, for every measure $\boldsymbol{\xi} = f\mathscr{L}^{d+1} + f^\perp\boldsymbol{\sigma} \in \mathcal{M}^n((0,1)\times\mathbb{T}^d)$, we have the decomposition*

$$(R_\delta)_\#\boldsymbol{\xi} = f^\delta \mathscr{L}^{d+1} + f^{\delta,\perp}(R_\delta)_\#\boldsymbol{\sigma},$$

where the respective densities are given by the formulas

$$f^\delta(t,x) = \frac{1}{1-2\delta} f\big(r_\delta^{-1}(t), x\big) \quad \text{and} \quad f^{\delta,\perp}(t,x) = f^\perp\big(r_\delta^{-1}(t), x\big).$$

Proof. By assumption, $\boldsymbol{\sigma}$ is singular with respect to $\mathscr{L}^{d+1}$, which means there exists a set $A \subset (0,1)\times\mathbb{T}^d$ such that $\mathscr{L}^{d+1}(A) = 0 = \boldsymbol{\sigma}(A^c)$. By the very definition of push-forward and the bijectivity of R_δ, we then have that

$$(R_\delta)_\#\boldsymbol{\sigma}\big((R_\delta(A))^c\big) = \boldsymbol{\sigma}\Big(R_\delta^{-1}\big(R_\delta(A^c)\big)\Big) = \boldsymbol{\sigma}(A^c) = 0,$$

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whereas, by the scaling properties of the Lebesgue measure, we have that $\mathscr{L}^{d+1}(R_\delta(A)) = (1-2\delta)\mathscr{L}^{d+1}(A) = 0$, which shows the claimed singularity. The second part of the lemma follows from the fact that $(R_\delta)_\#\mathscr{L}^{d+1} = (1-2\delta)^{-1}\mathscr{L}^{d+1}$ and the following statement: for every $\boldsymbol{\xi}' = f'\boldsymbol{\sigma}'$ with $\boldsymbol{\sigma}' \in \mathcal{M}_+((0,1)\times\mathbb{T}^d)$, we claim that

$$\frac{\mathrm{d}(R_\delta)_\#\boldsymbol{\xi}'}{\mathrm{d}(R_\delta)_\#\boldsymbol{\sigma}'}(t,x) = f'(R_\delta^{-1}(t,x)), \quad \forall (t,x) \in \mathcal{I}_\delta \times \mathbb{T}^d. \tag{3.29}$$

Indeed, by definition of push-forward, we have for every test function $\varphi \in C_b$

$$\int \varphi \,\mathrm{d}(R_\delta)_\#\boldsymbol{\xi}' = \int (\varphi\circ R_\delta)\,\mathrm{d}\boldsymbol{\xi}' = \int (\varphi\circ R_\delta) f'\,\mathrm{d}\boldsymbol{\sigma}' = \int \varphi\cdot(f'\circ R_\delta^{-1})\,\mathrm{d}(R_\delta)_\#\boldsymbol{\sigma}',$$

which indeed shows (3.29). □

Let

$$\boldsymbol{\mu}^n = \rho^n\,\mathrm{d}\mathscr{L}^{d+1} + \rho^{n,\perp}\,\mathrm{d}\boldsymbol{\sigma} \quad \text{and} \quad \boldsymbol{\nu}^n = j^n\,\mathrm{d}\mathscr{L}^{d+1} + j^{n,\perp}\,\mathrm{d}\boldsymbol{\sigma}$$

be Lebesgue decompositions. We apply Lemma 3.16 to both $\boldsymbol{\mu}^n$ and $\boldsymbol{\nu}^n$ and find that, on $\mathcal{I}_\delta\times\mathbb{T}^d$, we have

$$\boldsymbol{\mu}^{n,\delta} = \rho^{n,\delta}\,\mathrm{d}\mathscr{L}^{d+1} + \rho^{n,\delta,\perp}\,\mathrm{d}(R_\delta)_\#\boldsymbol{\sigma} \quad \text{and} \quad \boldsymbol{\nu}^{n,\delta} = j^{n,\delta}\,\mathrm{d}\mathscr{L}^{d+1} + j^{n,\delta,\perp}\,\mathrm{d}(R_\delta)_\#\boldsymbol{\sigma},$$

with $(R_\delta)_\#\boldsymbol{\sigma}$ singular with respect to $\mathscr{L}^{d+1}$ and

$$\begin{aligned} \rho^{n,\delta}(t,x) &= \big(\rho^n\circ R_\delta^{-1}\big)(t,x), & \rho^{n,\delta,\perp}(t,x) &= (1-2\delta)\big(\rho^{n,\perp}\circ R_\delta^{-1}\big)(t,x), \\ j^{n,\delta}(t,x) &= \frac{1}{1-2\delta}\big(j^n\circ R_\delta^{-1}\big)(t,x), & j^{n,\delta,\perp}(t,x) &= \big(j^{n,\perp}\circ R_\delta^{-1}\big). \end{aligned} \tag{3.30}$$

Further consider the Lebesgue decompositions

$$\mu_i^n = \rho_i^n\,\mathrm{d}\mathscr{L}^d + \rho_i^{n,\perp}\,\mathrm{d}\sigma_i, \qquad i\in\{0,1\}$$

for some $\sigma_1,\sigma_2\in\mathcal{M}_+(\mathbb{T}^d)$ singular w.r.t. $\mathscr{L}^d$. The action of $(\boldsymbol{\mu}^{n,\delta},\boldsymbol{\nu}^{n,\delta})$ is given by[2]

$$\mathbb{A}(\boldsymbol{\mu}^{n,\delta},\boldsymbol{\nu}^{n,\delta}) = \mathbb{A}^{\mathcal{I}_\delta}(\boldsymbol{\mu}^{n,\delta},\boldsymbol{\nu}^{n,\delta}) + \sum_{i=0,1}\delta\Big(\int_{\mathbb{T}^d} f(\rho_i^n,0)\,\mathrm{d}\mathscr{L}^d + \int_{\mathbb{T}^d} f^\infty(\rho_i^{n,\perp},0)\,\mathrm{d}\sigma_i\Big),$$

where we used the notation

$$\mathbb{A}^{\mathcal{I}_\delta}(\boldsymbol{\mu}^{n,\delta},\boldsymbol{\nu}^{n,\delta}) := \int_{\mathcal{I}_\delta\times\mathbb{T}^d} f(\rho^{n,\delta},j^{n,\delta})\,\mathrm{d}\mathscr{L}^{d+1} + \int_{\mathcal{I}_\delta\times\mathbb{T}^d} f^\infty(\rho^{n,\delta,\perp},j^{n,\delta,\perp})\,\mathrm{d}(R_\delta)_\#\boldsymbol{\sigma}.$$

Making use of the formulas (3.30) and the homogeneity of f^∞, we find

$$\begin{aligned} \mathbb{A}^{\mathcal{I}_\delta}(\boldsymbol{\mu}^{n,\delta},\boldsymbol{\nu}^{n,\delta}) &= (1-2\delta)\int_{(0,1)\times\mathbb{T}^d} f\Big(\rho^n,\frac{j^n}{1-2\delta}\Big)\,\mathrm{d}\mathscr{L}^{d+1} + \int_{(0,1)\times\mathbb{T}^d} f^\infty\big((1-2\delta)\rho^{n,\perp},j^{n,\perp}\big)\,\mathrm{d}\boldsymbol{\sigma} \\ &= (1-2\delta)\Big(\int_{(0,1)\times\mathbb{T}^d} f\Big(\rho^n,\frac{j^n}{1-2\delta}\Big)\,\mathrm{d}\mathscr{L}^{d+1} + \int_{(0,1)\times\mathbb{T}^d} f^\infty\Big(\rho^{n,\perp},\frac{j^{n,\perp}}{1-2\delta}\Big)\,\mathrm{d}\boldsymbol{\sigma}\Big). \end{aligned} \tag{3.31}$$

Furthermore, it follows from the linear growth assumption that, for $i=0,1$,

$$\int_{\mathbb{T}^d} f(\rho_i^n,0)\,\mathrm{d}\mathscr{L}^d + \int_{\mathbb{T}^d} f^\infty(\rho_i^{n,\perp},0)\,\mathrm{d}\sigma_i \le C(\mu_i^n(\mathbb{T}^d)+1) = C\big(\boldsymbol{\mu}^n((0,1)\times\mathbb{T}^d)+1\big)$$

as well as, by (3.31), the nonnegativity of f, and Assumption 2.2,

$$\begin{aligned} \mathbb{A}^{\mathcal{I}_\delta}(\boldsymbol{\mu}^{n,\delta},\boldsymbol{\nu}^{n,\delta}) &\le \mathbb{A}(\boldsymbol{\mu}^n,\boldsymbol{\nu}^n) + 2\delta(\mathrm{Lip}f)\Big(\int_{(0,1)\times\mathbb{T}^d}|j^n|\,\mathrm{d}\mathscr{L}^{d+1} + \int_{(0,1)\times\mathbb{T}^d}|j^{n,\perp}|\,\mathrm{d}\boldsymbol{\sigma}\Big) \\ &\le \mathbb{A}(\boldsymbol{\mu}^n,\boldsymbol{\nu}^n) + \frac{2\delta(\mathrm{Lip}f)}{c}\Big(\mathbb{A}(\boldsymbol{\mu}^n,\boldsymbol{\nu}^n) + C(1+\|\boldsymbol{\mu}^n\|_{\mathrm{TV}})\Big). \end{aligned}$$

We thus conclude (3.28). □

[2]Note that the definition of the action does not depend on the choice of the measure which is singular with respect to $\mathscr{L}^{d+1}$, therefore we can use $(R_\delta)_\#\boldsymbol{\sigma}$ instead of $\boldsymbol{\sigma}$.

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4. Analysis of the cell problem with examples

This section is devoted to the characterisation and illustration of f_{hom} in the case where the function F is of the form

$$F(m, J) = F(J) = \sum_{(x,y)\in\mathcal{E}^Q} \alpha_{xy} |J(x, y)| \tag{4.1}$$

for some strictly positive function $\alpha : \mathcal{E}^Q \ni (x, y) \mapsto \alpha_{xy} > 0$. A natural problem of interest is to determine whether/when the Γ-limit $\mathbb{MA}_{\mathrm{hom}}$ can be the W_1-distance. The analogous problem for the W_2-distance has been extensively studied in [24] and [23] in the case where the graph stucture is associated with finite-volume partitions.

4.1. Discrete 1*-Wasserstein distance*

We start the analysis of this special setting by observing that, in this case, the discrete functional $\mathcal{MA}_\varepsilon$ actually coincides with the $\mathbb{W}_1$ distance associated to a natural induced metric structure. In order to prove this fact, we first define $\widetilde{\alpha}^\varepsilon : \mathcal{E}_\varepsilon \to \mathbb{R}_+$ as the unique function such that

$$\frac{\tau_\varepsilon^z \widetilde{\alpha}^\varepsilon}{\varepsilon}\bigg|_{\mathcal{E}^Q} := \alpha \qquad z \in \mathbb{Z}_\varepsilon^d.$$

It is easy to check that this definition is well-posed and determines the value of $\widetilde{\alpha}_{xy}$ for every $(x, y) \in \mathcal{E}_\varepsilon$. Further let

$$\alpha_{xy}^\varepsilon = \frac{\widetilde{\alpha}_{xy}^\varepsilon + \widetilde{\alpha}_{yx}^\varepsilon}{2}, \qquad (x, y) \in \mathcal{E}_\varepsilon$$

be the symmetrisation of $\widetilde{\alpha}^\varepsilon$. Given $J \in \mathbb{R}_a^{\mathcal{E}_\varepsilon}$, we can write $\mathcal{F}_\varepsilon(J)$ in terms of α^ε. Precisely,

$$\begin{aligned}
\mathcal{F}_\varepsilon(J) &= \sum_{z\in\mathbb{Z}_\varepsilon^d} \varepsilon^d F\left(\frac{\tau_\varepsilon^z J}{\varepsilon^{d-1}}\right) = \sum_{z\in\mathbb{Z}_\varepsilon^d} \varepsilon^d \sum_{(\widehat{x},\widehat{y})\in\mathcal{E}^Q} \alpha_{\widehat{x}\widehat{y}} \frac{|\tau_\varepsilon^z J(\widehat{x}, \widehat{y})|}{\varepsilon^{d-1}} \\
&= \sum_{z\in\mathbb{Z}_\varepsilon^d} \sum_{(\widehat{x},\widehat{y})\in\mathcal{E}^Q} \tau_\varepsilon^z \widetilde{\alpha}^\varepsilon(\widehat{x}, \widehat{y}) |\tau_\varepsilon^z J(\widehat{x}, \widehat{y})| = \sum_{(x,y)\in\mathcal{E}_\varepsilon} \widetilde{\alpha}_{xy}^\varepsilon |J(x, y)| \\
&= \sum_{(x,y)\in\mathcal{E}_\varepsilon} \alpha_{xy}^\varepsilon |J(x, y)|,
\end{aligned}$$

where in the last passage we used that $|J|$ is symmetric. We define a distance on $\mathcal{X}_\varepsilon$ given by

$$d_\varepsilon(x, y) := \mathcal{MA}_\varepsilon(\delta_x, \delta_y), \qquad \forall x, y \in \mathcal{X}_\varepsilon.$$

One can easily show that d_ε indeed defines a metric on $\mathcal{X}_\varepsilon$. In fact, d_ε can be seen as a weighted graph distance.

Proposition 4.1. *For every $x, y \in \mathcal{X}_\varepsilon$, we have*

$$d_\varepsilon(x, y) = \inf \left\{ \sum_{i=0}^{k-1} 2\alpha_{x_i x_{i+1}}^\varepsilon : x_0 = x,\ x_k = y,\ (x_i, x_{i+1}) \in \mathcal{E}_\varepsilon \quad \forall i,\ k \in \mathbb{N} \right\}.$$

Proof. The inequality $\leq$ directly follows by choosing unit fluxes along admissible paths: let $x_0 = x, x_1, \ldots, x_{k-1}, x_k = y$ be a path, i.e. $(x_i, x_{i+1}) \in \mathcal{E}_\varepsilon$ for every $i = 0, 1, \ldots, k$, and consider

$$J^P := \sum_{i=0}^{k-1} \big(\delta_{(x_i,x_{i+1})} - \delta_{(x_{i+1},x_i)}\big), \tag{4.2}$$

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which has divergence equal to $\delta_x - \delta_y$. Then,

$$\begin{aligned}
d_\varepsilon(x,y) = \mathcal{MA}_\varepsilon(\delta_x,\delta_y) &= \inf\left\{\mathcal{F}_\varepsilon(J) : \mathsf{div}J = \delta_x - \delta_y\right\} \\
&= \inf\left\{\sum_{(x,y)\in\mathcal{E}_\varepsilon} \alpha^\varepsilon_{xy}|J(x,y)| : \mathsf{div}J = \delta_x - \delta_y\right\} \\
&\le \sum_{(x,y)\in\mathcal{E}_\varepsilon} \alpha^\varepsilon_{xy}|J^P(x,y)| \le \sum_{i=0}^{k-1} 2\alpha^\varepsilon_{x_i x_{i+1}},
\end{aligned}$$

where in the last inequality we used that α^ε is symmetric.

To prove the converse, let $\bar{J} \in \mathbb{R}^{\mathcal{E}_\varepsilon}_a$ be an optimal flux for $\mathcal{MA}_\varepsilon(\delta_x,\delta_y)$, that is,

$$\mathsf{div}\bar{J} = \delta_x - \delta_y \quad \text{and} \quad \mathcal{MA}_\varepsilon(\delta_x,\delta_y) = \sum_{(x,y)\in\mathcal{E}_\varepsilon} \alpha^\varepsilon_{xy}|\bar{J}(x,y)|.$$

Since the graph $\mathcal{E}_\varepsilon$ is finite, in order for $\bar{J}$ to satisfy the divergence condition, there must exist a simple path $x_0 = x, x_1, \ldots, x_k = y$ such that $(x_i, x_{i+1}) \in \mathcal{E}_\varepsilon$ and $\bar{J}(x_i, x_{i+1}) > 0$ for every i. Let J^P be the associated vector field as in (4.2). Note that, for every $\lambda \in \mathbb{R}$, we have $\mathsf{div}\big((1-\lambda)\bar{J} + \lambda J^P\big) = \delta_x - \delta_y$. Furthermore, the function

$$\lambda \mapsto \mathcal{F}_\varepsilon((1-\lambda)\bar{J} + \lambda J^P\big) = \sum_{(x,y)\in\mathcal{E}_\varepsilon} \alpha^\varepsilon_{xy}|(1-\lambda)\bar{J}(x,y) + \lambda J^P(x,y)|$$

is differentiable at $\lambda = 0$, since $(\bar{J}(x,y) = 0) \Rightarrow (J^P(x,y) = 0)$. By optimality, the derivative at $\lambda = 0$ must equal 0, i.e.

$$0 = \sum_{(x,y)\in\mathcal{E}_\varepsilon} \alpha^\varepsilon_{xy}\big(J^P(x,y) - \bar{J}(x,y)\big)\mathrm{sgn}\bar{J}(x,y) = \sum_{(x,y)\in\mathcal{E}_\varepsilon} \alpha^\varepsilon_{xy} J^P(x,y)\mathrm{sgn}\bar{J}(x,y) - d_\varepsilon(x,y),$$

and, since $(J^P(x,y) \neq 0) \Rightarrow (\mathrm{sgn}\, J^P(x,y) = \mathrm{sgn}\, \bar{J}(x,y))$, we have

$$d_\varepsilon(x,y) = \sum_{(x,y)\in\mathcal{E}_\varepsilon} \alpha^\varepsilon_{xy}|J^P(x,y)| = 2\sum_{i=1}^{k} \alpha^\varepsilon_{x_i x_{i+1}},$$

where, in the last equality, we used that the path is simple (and the symmetry of α^ε). This shows the inequality $\ge$ and concludes the proof. □

Consider the 1-Wasserstein distance associted to d_ε, that is,

$$\mathbb{W}_{1,\varepsilon}(m_0,m_1) = \inf\left\{\int_{\mathcal{X}_\varepsilon\times\mathcal{X}_\varepsilon} d_\varepsilon(x,y)\,\mathrm{d}\pi(x,y) : (e_0)_\#\pi = m_0, \quad (e_1)_\#\pi = m_1\right\},$$

as well as, by Kantorovich duality,

$$\mathbb{W}_{1,\varepsilon}(m_0,m_1) = \sup\left\{\int_{\mathcal{X}_\varepsilon} \varphi\,\mathrm{d}(m_0 - m_1) : \mathrm{Lip}_{d_\varepsilon}(\varphi) \le 1\right\},$$

for every $m_0, m_1 \in \mathscr{P}(\mathcal{X}_\varepsilon)$.

Proposition 4.2. *For every $m_0, m_1 \in \mathscr{P}(\mathcal{X}_\varepsilon)$, we have*

$$\mathcal{MA}_\varepsilon(m_0,m_1) = \mathbb{W}_{1,\varepsilon}(m_0,m_1). \tag{4.3}$$

Proof of $\ge$. Fix $m_0, m_1 \in \mathscr{P}(\mathcal{X}_\varepsilon)$ and set $m := m_0 - m_1$. Let $\bar{J} \in \mathbb{R}^{\mathcal{E}_\varepsilon}_a$ be an optimal flux for $\mathcal{MA}_\varepsilon(m_0,m_1)$, that is,

$$\mathsf{div}\bar{J} = m \quad \text{and} \quad \mathcal{MA}_\varepsilon(m_0,m_1) = \sum_{(x,y)\in\mathcal{E}_\varepsilon} \alpha^\varepsilon_{xy}|\bar{J}(x,y)|.$$

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Let $\varphi : \mathcal{X}_\varepsilon \to \mathbb{R}$ be such that $\mathrm{Lip}_{d_\varepsilon} \varphi \le 1$, i.e. $|\varphi(y) - \varphi(x)| \le d_\varepsilon(x, y)$ for $x, y \in \mathcal{X}_\varepsilon$. Then,

$$\begin{aligned}\int_{\mathcal{X}_\varepsilon} \varphi \, \mathrm{d}m &= \int_{\mathcal{X}_\varepsilon} \varphi \, \mathrm{d}\mathsf{div}\bar{J} = \sum_{x\in\mathcal{X}_\varepsilon} \varphi(x) \sum_{y\sim x} \bar{J}(x, y) = \frac{1}{2} \sum_{(x,y)\in\mathcal{E}_\varepsilon} \varphi(x)\big(\bar{J}(x, y) - \bar{J}(y, x)\big) \\ &= \frac{1}{2} \sum_{(x,y)\in\mathcal{E}_\varepsilon} \big(\varphi(y) - \varphi(x)\big)\bar{J}(x, y) \le \frac{1}{2} \sum_{(x,y)\in\mathcal{E}_\varepsilon} d_\varepsilon(x, y)|\bar{J}(x, y)|.\end{aligned}$$

In order to conclude, we make the following crucial observation: as a consequence of the optimality of $\bar{J}$, we claim that

$$\bar{J}(x, y) \ne 0 \quad \Longrightarrow \quad d_\varepsilon(x, y) = 2\alpha^\varepsilon_{xy}. \tag{4.4}$$

To this end, assume that $\bar{J}(x, y) \ne 0$ and consider an optimal $J^{(x,y)}$ for $d_\varepsilon(x, y) = \mathcal{MA}_\varepsilon(\delta_x, \delta_y)$. Note that, by construction,

$$\mathsf{div}\big(J^{(x,y)}\big) = \delta_x - \delta_y = \mathsf{div}\widetilde{J}, \qquad \text{where } \widetilde{J} := \delta_{(x,y)} - \delta_{(y,x)},$$

which in turns also implies that

$$\mathsf{div}\big(\bar{J} + \bar{J}(x, y)\big(J^{(x,y)} - \widetilde{J}\big)\big) = \mathsf{div}\bar{J}.$$

By optimality of $J^{(x,y)}$, we have

$$\mathcal{F}_\varepsilon(\widetilde{J}) = 2\alpha^\varepsilon_{xy} \ge \mathcal{F}_\varepsilon\big(J^{(x,y)}\big) = \sum_{(\tilde{x},\tilde{y})\in\mathcal{E}_\varepsilon} \alpha^\varepsilon_{\tilde{x}\tilde{y}} |J^{(x,y)}(\tilde{x}, \tilde{y})|, \tag{4.5}$$

whereas the optimality of $\bar{J}$ yields

$$\begin{aligned}\mathcal{F}_\varepsilon\big(\bar{J} + \bar{J}(x, y)\big(J^{(x,y)} - \widetilde{J}\big)\big) = &\sum_{(\tilde{x},\tilde{y})\in\mathcal{E}_\varepsilon\setminus\{(x,y),(y,x)\}} \alpha^\varepsilon_{\tilde{x}\tilde{y}} |\bar{J}(\tilde{x}, \tilde{y}) + \bar{J}(x, y)J^{(x,y)}(\tilde{x}, \tilde{y})| \\ &+ \alpha^\varepsilon_{xy}|\bar{J}(x, y)J^{(x,y)}(x, y)| + \alpha^\varepsilon_{yx}|\bar{J}(x, y)J^{(x,y)}(y, x)| \\ &\ge \mathcal{F}_\varepsilon(\bar{J}) = \sum_{(\tilde{x},\tilde{y})\in\mathcal{E}_\varepsilon} \alpha^\varepsilon_{\tilde{x}\tilde{y}} |\bar{J}(\tilde{x}, \tilde{y})|.\end{aligned}$$

By applying the triangle inequality and simplifying the latter formula, we find

$$\sum_{(\tilde{x},\tilde{y})\in\mathcal{E}_\varepsilon} \alpha^\varepsilon_{\tilde{x}\tilde{y}} |\bar{J}(x, y)J^{(x,y)}(\tilde{x}, \tilde{y})| \ge 2\alpha^\varepsilon_{xy}|\bar{J}(x, y)|. \tag{4.6}$$

The combination of (4.5) and (4.6) implies $d_\varepsilon(x, y) = 2\alpha^\varepsilon_{xy}$. With (4.4) at hand, we can write

$$\int_{\mathcal{X}_\varepsilon} \varphi \, \mathrm{d}m \le \sum_{(x,y)\in\mathcal{E}_\varepsilon} \alpha^\varepsilon_{xy}|\bar{J}(x, y)| = \mathcal{MA}_\varepsilon(m_0, m_1),$$

and we conclude by arbitrariness of φ. □

Proof of $\le$. Let π be such that $(e_i)_\#\pi = m_i$ for $i = 0, 1$. Further, for every $x, y \in \mathcal{X}_\varepsilon$, let $J^{(x,y)} \in \mathbb{R}^{\mathcal{E}_\varepsilon}_a$ be optimal for $\mathcal{MA}_\varepsilon(\delta_x, \delta_y)$. It follows from a direct computation that the divergence of the asymmetric flux

$$J := \sum_{x,y\in\mathcal{X}_\varepsilon} \pi(x, y)J^{(x,y)}$$

is equal to $m_0 - m_1$. Thus,

$$\mathcal{MA}_\varepsilon(m_0, m_1) \le \sum_{(\tilde{x},\tilde{y})\in\mathcal{E}_\varepsilon} \alpha^\varepsilon_{\tilde{x}\tilde{y}} |J(\tilde{x}, \tilde{y})| \le \sum_{x,y\in\mathcal{X}_\varepsilon} \pi(x, y) \sum_{(\tilde{x},\tilde{y})\in\mathcal{E}_\varepsilon} \alpha^\varepsilon_{\tilde{x}\tilde{y}} |J^{(x,y)}(\tilde{x}, \tilde{y})| = \int_{\mathcal{X}_\varepsilon\times\mathcal{X}_\varepsilon} d_\varepsilon \, \mathrm{d}\pi,$$

and we conclude by arbitrariness of π. □

In view of the equality $\mathcal{MA}_\varepsilon = \mathbb{W}_{1,\varepsilon}$, it is worth noting that for cost functions of the form (4.1) there are (at least) two different possible methods to show discrete-to-continuum limits for $\mathcal{MA}_\varepsilon$. One such

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method is provided by the current work and makes use of the Γ-convergence of $\mathcal{A}_\varepsilon$ to $\mathbb{A}_{\text{hom}}$ proved in [[23], Theorem 5.4]. The convergence of the "weighted graph distance" d_ε follows *a posteriori*. Another approach is to study directly the scaling limits of the distance d_ε as $\varepsilon \to 0$ and, from that, infer the convergence of the associated 1-Wasserstein distances, in a similar spirit as in [4].

4.2. General properties of f_{hom}

For $j \in \mathbb{R}^d$, recall that

$$f_{\text{hom}}(j) := \inf \{F(J) : J \in \mathsf{Rep}(j)\}, \tag{4.7}$$

where $\mathsf{Rep}(j)$ is the set of all $\mathbb{Z}^d$-periodic functions $J \in \mathbb{R}^{\mathcal{E}}_a$ such that

$$\mathsf{Eff}(J) := \frac{1}{2} \sum_{(x,y)\in\mathcal{E}^Q} J(x,y)(y_{\mathsf{z}} - x_{\mathsf{z}}) = j \quad \text{and} \quad \mathsf{div} J \equiv 0.$$

As noted in [[23], Lemma 4.7], we may as well write min in place of inf in (4.7).

Our first observation is that, indeed, the homogenised density is a norm. This has already been proved in [[23], Corollary 5.3]; for the sake of completeness, we provide here a simple proof in our setting.

Proposition 4.3. *The function f_{hom} is a norm.*

Proof. Finiteness follows from the nonemptiness of the set of representatives proved in [[23], Lemma 4.5]. To prove positiveness, take any $j \in \mathbb{R}^d$ and $J \in \mathsf{Rep}(j)$. For every norm $\|\cdot\|$, we have

$$\|j\| = \|\mathsf{Eff}(J)\| \le \frac{1}{2} \sum_{(x,y)\in\mathcal{E}^Q} |J(x,y)| \|y_{\mathsf{z}} - x_{\mathsf{z}}\| \le \frac{F(J)}{2} \max_{(x,y)\in\mathcal{E}^Q} \frac{\|y_{\mathsf{z}} - x_{\mathsf{z}}\|}{\alpha_{xy}}. \tag{4.8}$$

The constant that multiplies $F(J)$ at the right-hand side is finite because every α_{xy} is strictly positive and the graph $(\mathcal{X}, \mathcal{E})$ is locally finite. Absolute homogeneity and the triangle inequality follow from the absolute homogeneity and subadditivity of F and the affinity of the constraints. □

Hence, $\mathbb{MA}_{\text{hom}}$ is *always* (i.e. for any choice of $(\alpha_{xy})_{x,y}$ and of the graph $(\mathcal{X}, \mathcal{E})$) the W_1-distance w.r.t. some norm. However, the norm f_{hom} can equal the 2-norm $|\cdot|_2$ only in dimension $d = 1$. In fact, the unit ball for f_{hom} has to be a polytope, namely the associated sphere is contained in the union of finitely many hyperplanes. These types of norms are also known as *crystalline norms*.

Proposition 4.4. *The unit ball associated to the norm f_{hom}, namely*

$$B := \left\{ j \in \mathbb{R}^d : f_{\text{hom}}(j) \le 1 \right\},$$

is the convex hull of finitely many points. In particular, the associated unit sphere is contained in the union of finitely many hyperplanes, i.e. f_{hom} is a crystalline norm.

Proof. Let X be the vector space of all $\mathbb{Z}^d$-periodic functions $J \in \mathbb{R}^{\mathcal{E}}_a$ such that $\mathsf{div} J \equiv 0$. The sublevel set

$$X_1 := \{J \in X : F(J) \le 1\}$$

is clearly compact (due to the strict positivity of $(\alpha_{xy})_{x,y}$) and can be written as finite intersection of half-spaces, namely

$$X_1 = \bigcap_{r\in\{-1,1\}^{\mathcal{E}^Q}} \left\{ J \in X : \sum_{(x,y)\in\mathcal{E}^Q} \alpha_{xy} r_{xy} J(x,y) \le 1 \right\}.$$

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Thus, X_1 is the convex hull of some finite set of points A, that is, $X_1 = \mathrm{conv}(A)$. Since f_{hom} is defined as a minimum, we have

$$B = \left\{ j \in \mathbb{R}^d : \exists J \in \mathsf{Rep}(j),\ F(J) \le 1 \right\} = \mathsf{Eff}(X_1) = \mathsf{Eff}(\mathrm{conv}(A)) = \mathrm{conv}(\mathsf{Eff}(A)),$$

where the last equality is due to the linearity of Eff. □

4.3. Embedded graphs

To visualise some examples, we shall now focus on the case where $(\mathcal{X}, \mathcal{E})$ is embedded, in the sense that V is a subset of $[0,1)^d$, and we use the identification $(z, v) \equiv z + v$ (see also [[23], Remark 2.2]). It has been proved in [[23], Proposition 9.1] that, for embedded graphs, the identity

$$\mathsf{Eff}(J) = \frac{1}{2} \sum_{(x,y)\in\mathcal{E}^Q} J(x,y)(y-x) \tag{4.9}$$

holds for every $\mathbb{Z}^d$-periodic and divergence-free vector field $J \in \mathbb{R}^{\mathcal{E}}_a$. In what follows, we also make the choice

$$\alpha_{xy} := \frac{1}{2}|x-y|_2, \qquad (x,y) \in \mathcal{E}^Q.$$

4.3.1. One-dimensional case with nearest-neighbour interaction

Assume $d = 1$, let $x_1 < x_2 < \cdots < x_k$ be an enumeration of V, and set

$$\mathcal{E} := \{(x,y) \in \mathcal{X} \times \mathcal{X} \text{ s.t. there is no } z \in \mathcal{X} \text{ strictly between } x \text{ and } y\}.$$

In other words, denoting $x_0 = x_k - 1$ and $x_{k+1} = x_1 + 1$,

$$\mathcal{E} = \bigcup_{z\in\mathbb{Z}} \bigcup_{i=1}^{k} \{(x_i, x_{i+1})\} \cup \{(x_i, x_{i-1})\}.$$

By rewriting (4.8) using (4.9), and by the definition of f_{hom}, we find

$$|j| \le f_{\mathrm{hom}}(j), \qquad j \in \mathbb{R}^d.$$

On the other hand, given $j \in \mathbb{R}^d$, choose

$$J(x,y) := j\,\mathrm{sgn}(y-x), \qquad (x,y) \in \mathcal{E}.$$

This vector field is in $\mathsf{Rep}(j)$ because

$$\mathsf{div}J(x_i) = J(x_i, x_{i+1}) + J(x_i, x_{i-1}) = j - j = 0$$

for every i, and

$$\begin{aligned}
\mathsf{Eff}(J) &= \frac{1}{2} \sum_{i=1}^{k} \big(J(x_i, x_{i+1})(x_{i+1} - x_i) + J(x_i, x_{i-1})(x_{i-1} - x_i) \big) \\
&= \frac{j}{2} \sum_{i=1}^{k} \big(|x_{i+1} - x_i| + |x_i - x_{i-1}| \big) \\
&= \frac{j}{2} \big(x_{k+1} - x_1 + x_k - x_0 \big) \\
&= j.
\end{aligned}$$

A similar computation shows that $F(J) = |j|$, from which $f_{\mathrm{hom}}(j) = |j|$.

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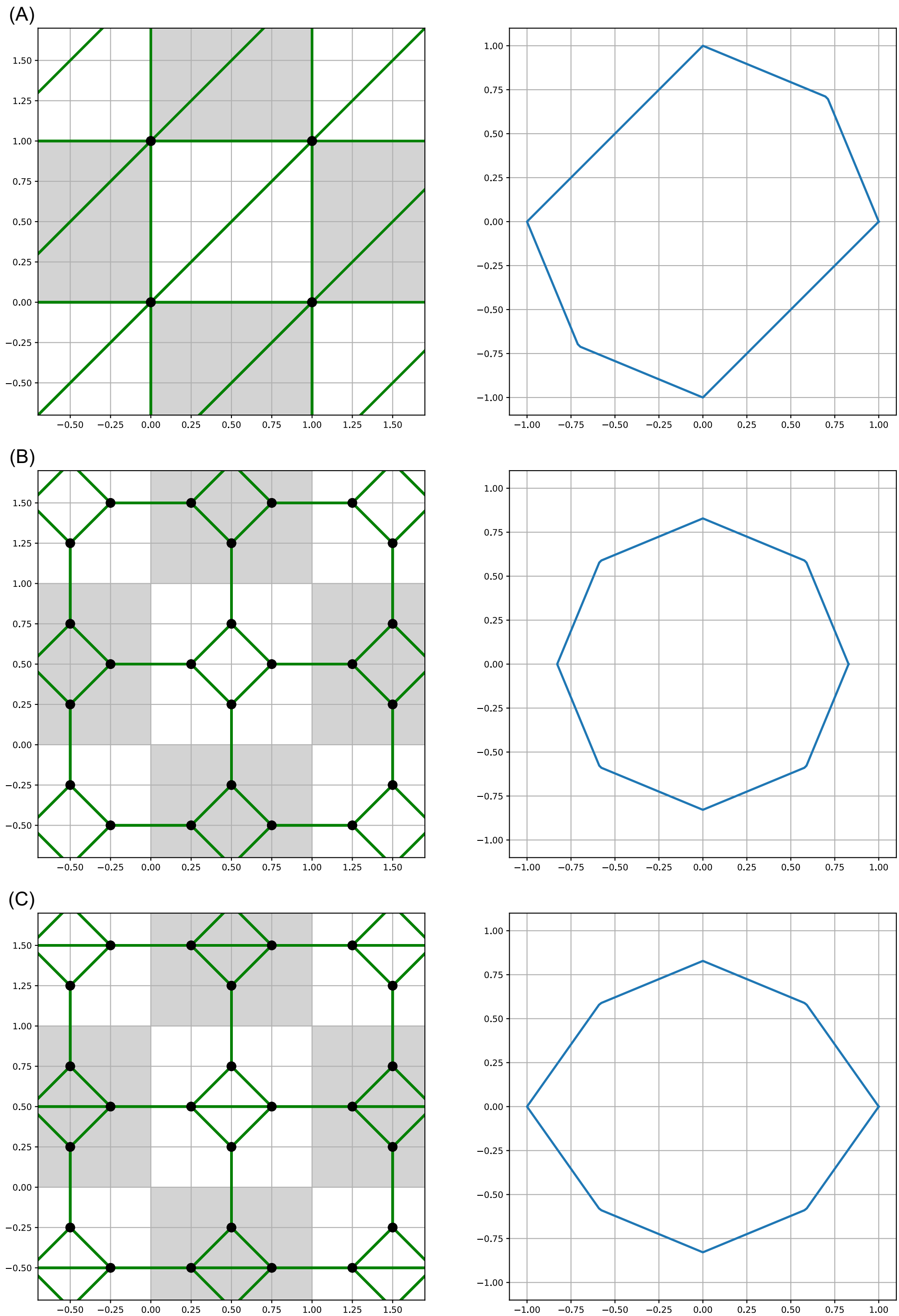

Figure 2. *Examples of graphs in* $\mathbb{R}^2$ *and corresponding unit balls for* f_{hom}.

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4.3.2. Cubic partition

Consider the case where $\mathcal{X} = \mathbb{Z}^d$ and

$$\mathcal{E} := \left\{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d : |x - y|_\infty = 1\right\}.$$

It is a result of [[23], Section 9.2] that

$$f_{\text{hom}}(j) = |j|_1, \qquad j \in \mathbb{R}^d.$$

Notice that, in this case, the 2-norm is evaluated only at vectors on the coordinate axes. Therefore, the same result holds when $\alpha_{xy} = \frac{1}{2}|x - y|_p$, for any p.

4.3.3. Graphs in $\mathbb{R}^2$

A few other examples in dimension $d = 2$ are shown in Figure 2: for each one, we display the graph and the unit ball in the corresponding norm f_{hom}. To algorithmically construct the unit balls, we solve the variational problem (4.7) for every j on a discretisation of the circle $\mathbb{S}^1$. In turn, this is achieved with the help of the Python library CVXPY [7], [3]. For visualisation, we make use of the library matplotlib [28].

Acknowledgements. The authors are very grateful to Jan Maas for his useful feedback on a preliminary version of this work. The authors also thank the anonymous reviewers for the careful reading of the manuscript and for useful suggestions. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper

Financial support. L.P. gratefully acknowledges fundings from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – GZ 2047/1, Projekt-ID 390685813. F.Q. gratefully acknowledges support from the Austrian Science Fund (FWF) project 10.55776/F65.

Competing interests. None.

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Cite this article: Portinale L. and Quattrocchi F. Discrete-to-continuum limits of optimal transport with linear growth on periodic graphs. *European Journal of Applied Mathematics*, https://doi.org/10.1017/S0956792524000810