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NONLINEAR BOUNDARY VALUE PROBLEMS FOR ELLIPTIC SYSTEMS

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1. Introduction

The purpose of this paper is to discuss non-linear boundary value problems for elliptic systems of the type

$$\begin{aligned} A_k u_k - \lambda_1^k u_k &= g_k(x, u_1, \ldots, u_m), && x \in G, \quad 1 \leqq k \leqq m \\ B_k u_k &= 0, && x \in \partial G, \quad 1 \leqq k \leqq m \end{aligned} \tag{1.1}$$

where A_k is a second order uniformly elliptic operator and $\lambda_1^k \in R$ is such that the problem

$$\begin{aligned} A_k u_k &= \lambda_1^k u_k, && x \in G \\ B_k u_k &= 0, && x \in \partial G \end{aligned} \tag{1.2}$$

has a one-dimensional space of solutions that is generated by a non-negative function. The boundary ∂G is supposed to be smooth and the functions $g_k, 1 \leqq k \leqq m$, are defined on $\bar{G} \times R^m$ and are continuously differentiable (usually, B_k represents Dirichlet or Neumann conditions and λ_1^k is the first eigenvalue associated with A_k and such boundary conditions).

We shall consider classical solutions of (1.1) and thus no previous assumptions concerning the growth of non-linearities g_k are necessary. Moreover, this allows us to obtain some conditions of asymptotic nature which are expressed in terms of non-strict inequalities and extend to the vector case those given by Kazdan and Warner **[10]** and Landesman and Lazer **[11]**. At the same time, we generalize the main results in **[7, 12]**.

In Section 2 we obtain a general existence theorem for equations in some normed spaces which is related to a previous theorem of Ortega and the author **[5]**. In Section 3 we apply the results of Section 2 to the problem (1.1) and we give some examples which show the obtained generalizations.

The main tool we use is the Leray–Schauder degree.

2. Abstract results

Let X and Z be normed spaces and $L: \operatorname{dom} L \subset X \to Z$ a linear Fredholm mapping of index zero. We assume that the spaces X and Z are included into some space of functions $L^\infty(G, R^m)$, where G is a bounded domain in an euclidean space R^n.

Let us consider the equation

$$Lu = Nu \tag{2.1}$$

where $N: X \to Z$ is a given mapping.

It can be shown that u is a solution of this equation if and only if

$$u = Pu + K(I-Q)Nu + JQNu \tag{2.2}$$

where $P: X \to X$ and $Q: Z \to Z$ are projectors such that $\operatorname{Im} P = \ker L$, $\operatorname{Im} L = \ker Q$, K is the generalized inverse of L and J is an isomorphism from $\operatorname{Im} Q$ into $\ker L$. Suppose that the mappings $QN: X \to Z$ and $K(I-Q)N: X \to X$ are compact on bounded subsets of X (i.e., continuous and $QN(B)$, $K(I-Q)N(B)$ are relatively compact for each bounded $B \subset X$) and that K is continuous.

As a standing hypothesis on the $\ker L$ we have

(H) $$\ker L = \{(a_1\phi_1, \ldots, a_m\phi_m) : (a_1, \ldots, a_m) \in R^m\}$$

where the functions $\phi_k : \bar{G} \to R$ are continuous, positive on G and

$$|u_k(x)| \leqq c\phi_k(x)\|u\|_X \qquad \forall\, x \in \bar{G} \tag{2.3}$$

for all $u = (u_1, \ldots, u_m) \in X$ and each k, $1 \leqq k \leqq m$ (c is a positive constant independent of u and k). Also, we suppose that the functions ϕ_k are normalized in the sense that $\int_G \phi_k^2(x)\,dx = 1$.

The main result is the following theorem.

Theorem 2.1. *Suppose* (H) *holds and that:*

(i) *There exists a mapping $\xi: X \to Z^*$ and constants $\alpha \geqq 0$, $\beta \geqq 0$, such that*

$$\|Nu\|_Z \leqq \langle Nu, \xi u\rangle + \alpha\|u\|_X + \beta$$

where Z^* is the normed dual space of Z and $\langle Nu, \xi u\rangle = (\xi u)(Nu)$ for every $u \in X$.

(ii) *There exists a bounded (Φ takes bounded sets into bounded sets) and continuous mapping $\Phi: X \to \operatorname{Im} Q$ such that*

(ii.1) *Every possible solution $u \in X$ of the equation*

$$Lu = \lambda Nu + (1-\lambda)\Phi u, \qquad \lambda \in]0, 1[$$

satisfies the relation $\langle Nu, \xi u\rangle \leqq 0$.

(ii.2) *There exists $r > 0$ such that for all $u \in X$, $u = (u_1, \ldots, u_m)$ with $|u_k(x)| \geqq r\phi_k(x)$, $\forall\, x \in G$ for some k, $1 \leqq k \leqq m$, one has for some j, $1 \leqq j \leqq m$, that $(QN)_j(u)(\Phi)_j(u) \geqq 0$ and that the second factor is not zero.*

(ii.3) $d_B(J\Phi|_{\ker L}, B_X(s)\cap\ker L, 0)\neq 0$ *for every* $s\geqq r$, *where* d_B *means the Brouwer degree and* $B_X(s)$ *is the open ball of centre zero and radius* s *in* X.

Then, if α *is sufficiently small, equation* (2.1) *has at least one solution.*

Proof. We know (from the Leray–Schauder degree theory) that equation (2.1) will have a solution if we prove that there exists an open bounded subset Ω of X of the form $\Omega=B_X(r_1)$, $r_1\geqq r$, such that

$$u-T(u,\lambda)\equiv u-Pu-\lambda K(I-Q)Nu-\lambda JQNu-(1-\lambda)J\Phi u\neq 0 \tag{2.4}$$

for each $\lambda\in[0,1[$ and each $u\in\partial\Omega$. In fact, if this is the case and $u-T(u,1)=0$ for some $u\in\partial\Omega$, we have, from the equivalence between (2.1) and (2.2) that (2.1) has a solution. If $u-T(u,1)\neq 0$ $\forall u\in\partial\Omega$, we deduce from the homotopy property of the Leray–Schauder degree that

$$d_{\text{L-S}}(I-T(\cdot,1),\Omega,0)=d_{\text{L-S}}(I-T(\cdot,0),\Omega,0)=d_{\text{L-S}}(I-P-J\Phi,\Omega,0).$$

But as $\operatorname{Im}(P+J\Phi)\subset\ker L$ is finite dimensional,

$$\begin{aligned}d_{\text{L-S}}(I-P-J\Phi,\Omega,0)&=d_B(I-P-J\Phi|_{\ker L},\Omega\cap\ker L,0)\\&=d_B(-J\Phi|_{\ker L},\Omega\cap\ker L,0)\end{aligned}$$

and this degree is different from zero because of (ii.3).

Also, we must remark that the previous assumptions do not depend upon the choice of P (see [**13**]) and, for convenience, we take $P:X\to X$ defined by $(Pu)_k(x)=\phi_k(x)\int_G u_k(x)\phi_k(x)\,dx$, $1\leqq k\leqq m$.

To prove (2.4), let $u\in X$ and $\lambda\in]0,1[$ such that

$$u-Pu=\lambda K(I-Q)Nu+\lambda JQNu+(1-\lambda)J\Phi u. \tag{2.5}$$

Then applying P to both parts of (2.5) we obtain

$$\begin{aligned}u-Pu&=\lambda K(I-Q)Nu\\ \lambda JQNu+(1-\lambda)J\Phi u&=0.\end{aligned} \tag{2.6}$$

By using assumption (i), we have

$$\|u-Pu\|_X\leqq k_1\|Nu\|_Z\leqq k_1[\langle Nu,\xi u\rangle+\alpha\|u\|_X+\beta] \tag{2.7}$$

where k_1 is the norm of the continuous linear operator $K(I-Q)$.

Now, if u satisfies (2.6), $Lu=\lambda Nu+(1-\lambda)\Phi u$ and therefore, from (2.7) and (ii.1),

$$\|u-Pu\|_X\leqq k_1\alpha\|u\|_X+k_1\beta. \tag{2.8}$$

Also, it follows from hypothesis (ii.2) that for all k, $1 \leqq k \leqq m$, there exists $x_k \in G$ such that $|u_k(x_k)| < r\phi_k(x_k)$. In fact, if it is not the case, then $|u_k(x)| \geqq r\phi_k(x)$, $\forall x \in G$, for some k, $1 \leqq k \leqq m$ and, by assumption (ii.2), $(QN)_j(u)(\Phi)_j(u) \geqq 0$ for some j, $1 \leqq j \leqq m$. As

$$(QN)_j(u)(\Phi)_j(u) = \frac{-\lambda}{1-\lambda}(QN)_j(u)(QN)_j(u) \leqq 0,$$

we have $(QN)_j(u) = (\Phi)_j(u) = 0$ and this contradicts (ii.2) ($(\Phi)_j(u)$ is not zero).

Therefore

$$\begin{aligned} |(Pu)_k(x)|\,\phi_k(x_k) &\leqq \left|\int_G u_k(x)\phi_k(x)\,dx\right| \phi_k(x)\phi_k(x_k) \\ &\leqq k_2|(Pu)_k(x_k)| \leqq k_2(|u_k(x_k)| + |(I-P)_k(u)(x_k)|) \qquad \text{(from (2.3))} \\ &\leqq k_2 r\phi_k(x_k) + c\|(I-P)u\|_X \phi_k(x_k), \qquad \forall x \in G. \end{aligned}$$

Thus

$$\sup_{x \in G} |(Pu)_k(x)| \leqq k_2 r + c\|(I-P)u\|_X,$$

and since all the norms in a finite dimensional space ($\operatorname{Im} P$) are equivalent, we deduce

$$\|Pu\|_X \leqq k_3 + k_3 c\|(I-P)u\|_X \tag{2.9}$$

for a certain $k_3 > 0$. Therefore

$$\begin{aligned} \|u\|_X &\leqq \|Pu\|_X + \|u - Pu\|_X \\ &\leqq k_1\alpha(1 + k_3 c)\|u\|_X + (k_1\beta + k_3 + k_1 k_3 c\beta). \end{aligned}$$

Then, if $k_1\alpha(1 + k_3 c) < 1$, we obtain

$$\|u\|_X \leqq \frac{k_1\beta + k_3 + k_1 k_3 c\beta}{1 - k_1\alpha(1 + k_3 c)} \equiv r_0.$$

Lastly, if $\lambda = 0$ in (2.5), then $u - Pu = J\Phi u$ and therefore, $u = Pu$, $\Phi u = 0$. As $u = Pu$, $u \in \ker L$, i.e., $u(x) = (a_1\phi_1(x), \ldots, a_m\phi_m(x))$ for some $(a_1, \ldots, a_m) \in R^m$. From (ii.2), $|a_k| < r$ for all $1 \leqq k \leqq m$.

Taking $\Omega = B_X(r_1)$, $r_1 > \max\{r_0, r\}$, the proof is finished.

Remarks. (1) As we shall see, the mappings L that satisfy (H) include a great class of linear elliptic operators together with suitable boundary conditions.

(2) Theorem 2.1 is very general and we may take different functions Φ to obtain some

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existence theorems for equation (2.1). In view of the applications we consider only two cases. We begin with a corollary which is related to a result obtained by the author and Ortega [5] and the author and Martinez–Amores [4].

Corollary 2.1. *Assume* (H) *holds and that*:

(a) *Conditions* (i) *and* (ii.3) *of Theorem* 2.1 *hold.*

(b) $\langle Lu, \xi u\rangle \leqq 0$ *for all* $u \in \operatorname{dom} L$.

(c) *There exists* $r>0$ *such that for all* $u\in X$, $u=(u_1,\dots,u_m)$ *with* $|u_k(x)|\geqq r\phi_k(x)$, $\forall x\in G$ *for some* k, $1\leqq k\leqq m$, *one has* $QNu\neq 0$.

Then, if α *is sufficiently small, equation* (2.1) *has at least one solution.*

Proof. Take $\Phi = QN$ in Theorem 2.1.

Corollary 2.2. *Assume* (H) *holds and that* N *is uniformly bounded, i.e.,* $\|Nu\|_Z\leqq\beta$ *for every* $u\in X$. *Also, we suppose that*

$$(\mathrm{H}_1)\qquad \operatorname{Im} L=\left\{z=(z_1,\dots,z_m)\in Z:\int_G z_k(x)\psi_k(x)\,dx=0,\ 1\leqq k\leqq m\right\}$$

where the functions $\psi_k:\bar G\to R$, $1\leqq k\leqq m$, *are continuous, positive on* G *and normalized* $(\int_G \psi_k^2(x)\,dx=1)$.

Then, if hypothesis

(a) *There is* $r>0$ *such that for all* $u\in X$, $u=(u_1,\dots,u_m)$ *with* $|u_k(x)|\geqq r\phi_k(x)$ $\forall x\in G$, *for some* k, $1\leqq k\leqq m$, *one has* $\operatorname{sign} u_k(QNu)_{\sigma(k)}\leqq 0$ (*where* σ *is a permutation of the indices* $1,\dots,m$), *is satisfied, equation* (2.1) *has at least one solution.*

Proof. (H_1) allows us to take $Q:Z\to Z$ defined by

$$(Qz)_k(x)=\psi_k(x)\int_G z_k(x)\psi_k(x)\,dx,\qquad 1\leqq k\leqq m.$$

Now, take in Theorem 2.1 $\Phi:X\to\operatorname{Im}Q$ defined by $(\Phi u)_k=-(Qu)_{\sigma-1(k)}$, $1\leqq k\leqq m$. Trivially hypotheses (i), (ii.1) are verified with $\xi\equiv 0$.

On the other hand, if $|u_k(x)|\geqq r\phi_k(x)$ $\forall x\in G$, then either $u_k(x)\geqq r\phi_k(x)$ $\forall x\in G$ or $u_k(x)\leqq -r\phi_k(x)$ $\forall x\in G$. In the first case by taking $j=\sigma(k)$ in (ii.2), we have that (ii.2) becomes

$$\begin{aligned}(QNu)_{\sigma(k)}(\Phi u)_{\sigma(k)}&=(QNu)_{\sigma(k)}(-Qu)_k\\ &=-(QNu)_{\sigma(k)}\int_G u_k(x)\psi_k(x)\,dx\,\psi_k(x)\geqq 0.\end{aligned}$$

This implies that $(QNu)_{\sigma(k)}\leqq 0$.

Analogously if $u_k(x)\leqq -r\phi_k(x)$ $\forall x\in G$, one obtains $(QNu)_{\sigma(k)}\geqq 0$. Therefore (a) of

Corollary 2.2 becomes the same as (ii.2) of Theorem 2.1. Also

$$d_B(J\Phi|_{\ker L}, B_X(s)\cap \ker L, 0)\neq 0 \Leftrightarrow d_B(F, B_{R^m}(s), 0)\neq 0$$

where $F: R^m \to R^m$ is defined by

$$(F(a))_k = a_{\sigma^{-1}(k)} \int_G \phi_k(x)\psi_k(x)\,dx, \quad 1\leqq k\leqq m.$$

As $(F(a))_k = a_{\sigma^{-1}(k)} c_k$, $c_k>0$, $1\leqq k\leqq m$, the last degree is different from zero and the corollary is proved.

Let us think now in the non-resonance case, i.e., $\ker L$ is trivial. Then $P=0$, $Q=0$ and $K=L^{-1}$. Assuming that L^{-1} is continuous and that $L^{-1}N$ is compact on bounded subsets of X, we have the following result. (The proof is very similar to the proof of Theorem 2.1.)

Theorem 2.2 ([5]). *Let us suppose that there exists a mapping $\xi: X\to Z^*$ verifying $\langle Lu, \xi u\rangle \leqq 0$ for all $u\in \operatorname{dom} L$ and that $\|Nu\|_Z \leqq \langle Nu, \xi u\rangle + \alpha\|u\|_X + \beta$ for all $u\in X$ and constants $\alpha\geqq 0$, $\beta\geqq 0$. Then, if α is sufficiently small, equation* (2.1) *has at least one solution.*

3. Applications

In this section we use the results of the previous one to study some non-linear boundary value problems for elliptic systems.

Let $G\subset R^n$ be a bounded domain, p a natural number, $p>n$, and $\alpha\in R$ such that $0<\alpha<1-n/p$. We shall deal with the classical solvability of non-linear elliptic boundary value problems of the type

$$\begin{aligned} A_k u_k - \lambda_1^k u_k &= g_k(x, u_1, \dots, u_m), \qquad x\in G, \quad 1\leqq k\leqq m \\ B_k u_k &= 0, \qquad x\in \partial G, \quad 1\leqq k\leqq m \end{aligned} \tag{3.1}$$

where A_k is a second-order uniformly elliptic operator

$$A_k = -\sum_{i,j=1}^n a_{ij}^k(x)\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_{i=1}^n a_i^k(x)\frac{\partial}{\partial x_i}, \qquad 1\leqq k\leqq m$$

with $a_{ij}^k = a_{ji}^k$. The boundary ∂G and the coefficients of A_k, $1\leqq k\leqq m$, are supposed to be smooth, let us say a_{ij}^k, $a_i^k\in C^{2,\alpha}(\bar G)$, $\partial G\in C^{2,\alpha}$, with

$$\mu_k\sum_{i=1}^n \xi_i^2 \leqq \sum_{i,j=1}^n a_{ij}^k(x)\xi_i\xi_j, \qquad \forall x\in G, \quad \forall \xi_i\in R, \quad 1\leqq i\leqq n$$

where $\mu_k>0$ is a constant and $1\leqq k\leqq m$.

We restrict ourselves to linear homogeneous boundary conditions $B_k u_k = 0$, which will be either Dirichlet type ($u_k = 0$ on ∂G) or Neumann type $\partial u_k/\partial \nu_{A_k} = 0$ on ∂G, where ν_{A_k} is the unit outer conormal vector field on ∂G and $\partial/\partial \nu_{A_k}$ is the outer conormal derivative on ∂G).

For the sake of simplicity we shall assume that the functions $g_k: \bar{G} \times R^m \to R$, $1 \leqq k \leqq m$, are continuously differentiable.

From the above hypotheses it follows that the problem

$$\begin{aligned} A_k u_k &= \lambda u_k, \qquad x \in G \\ B_k u_k &= 0, \qquad x \in \partial G \end{aligned} \tag{3.2}$$

has a unique normalized positive solution ϕ_k on G which corresponds to a simple real eigenvalue λ_1^k ($\lambda_1^k = 0$ and $\phi_k = \text{constant}$ under the Neumann boundary condition).

Taking into account the properties of ϕ_k it is possible to prove the following statement ([**10**]): Given any bounded set X_0 of functions in $C^1(\bar{G}, R)$, if $B_k u_k = 0$, for all $u_k \in X_0$, then

$$\alpha \phi_k(x) < u_k(x) < \beta \phi_k(x) \text{ on } G \tag{3.3}$$

for some constants α, β independent of u_k.

Now, the following important result is easily established:

Lemma 3.1. *There exists $c_k > 0$ such that*

$$|u_k(x)| \leqq c_k \phi_k(x) \|u_k\|_1 \quad \text{on } \bar{G} \tag{3.4}$$

where

$$\|u_k\|_1 = \sum_{|\beta| \leqq 1} \sup_{x \in G} |D^\beta u_k(x)|$$

for all $u_k \in C^1(\bar{G}, R)$ satisfying $B_k u_k = 0$.

Proof. Take

$$X_0 = \left\{ \frac{u_k}{\|u_k\|_1} : u_k \in C^1(\bar{G}, R), u_k \neq 0, B_k u_k = 0 \right\}$$

in the inequality (3.3).

The adjoint problem to $A_k u_k - \lambda_1^k u_k = 0$, $x \in G$; $B_k u_k = 0$, $x \in \partial G$, has also a one-dimensional space of solutions generated by a normalized positive function ψ_k on G and such that if $f \in C^{0,\alpha}(\bar{G}, R)$, the problem

$$\begin{aligned} A_k u_k - \lambda_1^k u_k &= f, \qquad x \in G \\ B_k u_k &= 0, \qquad x \in \partial G \end{aligned}$$

has a solution $u_k \in C^{2,\alpha}(\bar{G}, R)$ if and only if $\int_G f(x)\psi_k(x)\,dx = 0$ (also, ψ_k = constant under the Neumann boundary condition).

We put

$$X = \{u \in C^{1,\alpha}(\bar{G}, R) \times \overset{(m)}{\cdots} \times C^{1,\alpha}(\bar{G}, R), u = (u_1, \dots, u_m) : B_k u_k = 0, 1 \leqq k \leqq m\}$$

with the norm

$$\|u\|_{1,\alpha} = \left(\sum_{k=1}^{m} \|u_k\|_{1,\alpha}^p\right)^{1/p}$$

where for each k, $1 \leqq k \leqq m$,

$$\|u_k\|_{1,\alpha} = \max_{|\beta| \leqq 1} \sup_{x \in G} |D^\beta u_k(x)| + \max_{|\beta| \leqq 1} \sup_{\substack{x,y \in G \\ x \neq y}} \frac{|D^\beta u_k(x) - D^\beta u_k(y)|}{|x-y|^\alpha}$$

and

$$Z = \{z \in C^{0,\alpha}(\bar{G}, R) \times \overset{(m)}{\cdots} \times C^{0,\alpha}(\bar{G}, R)\}$$

with the norm

$$\|z\|_p = \left(\sum_{k=1}^{m} \|z_k\|_p^p\right)^{1/p}, \quad z = (z_1, \dots, z_m),$$

where for each k, $1 \leqq k \leqq m$,

$$\|z_k\|_p = \left(\int_G |z_k(x)|^p\,dx\right)^{1/p}.$$

If we define $L : \operatorname{dom} L \subset X \to Z$ by

$$\operatorname{dom} L = \{u \in X : u \in C^{2,\alpha}(\bar{G}, R) \times \overset{(m)}{\cdots} \times C^{2,\alpha}(\bar{G}, R)\},$$

$(Lu)_k = A_k u_k - \lambda_1^k u_k$, and $N : X \to Z$ by $(Nu)_k(x) = g_k(x, u_1(x), \dots, u_m(x))$, for all $u \in X$, and $x \in G$, then our problem (3.1) is equivalent to solving the operator equation

$$Lu = Nu. \tag{3.5}$$

Now $\ker L=\{u\in X:u=(a_1\phi_1,\dots,a_m\phi_m),(a_1,\dots,a_m)\in R^m\}$,

$$\operatorname{Im} L=\left\{z\in Z, z=(z_1,\dots,z_m):\int_G z_k(x)\psi_k(x)\,dx=0,\ 1\leqq k\leqq m\right\}$$

and L is a linear Fredholm mapping of index zero.

It is trivial that $P:X\to X$, $(Pu)_k(x)=\phi_k(x)\int_G u_k(x)\phi_k(x)\,dx$, $1\leqq k\leqq m$ and $Q:Z\to Z$, $(Qz)_k(x)=\psi_k(x)\int_G z_k(x)\psi_k(x)\,dx$, $1\leqq k\leqq m$, are projectors such that $\operatorname{Im} P=\ker L$, $\operatorname{Im} L=\ker Q$ and if K is the generalized inverse of $L(K:\operatorname{Im} L\to\operatorname{dom} L\cap\ker P)$, there exists $c(p)>0$ such that

$$\|(Kf)_k\|_{W^{2,p}(\bar{G},R)}\leqq c(p)\|f_k\|_p,\quad 1\leqq k\leqq m$$

(see [2]). As the inclusion $W^{2,p}(\bar{G},R)\hookrightarrow C^{1,\alpha}(\bar{G},R)$ is compact ([1]), $K:(\operatorname{Im} L,\|\cdot\|_p)\to(X,\|\cdot\|_{1,\alpha})$ is compact.

Now, we are in position to apply the results of the previous section to system (3.1). Firstly we assume that g_k, $1\leqq k\leqq m$, are bounded.

Theorem 3.1. *Suppose that:*

(i) *There is $r>0$ such that for all $u\in X$, $u=(u_1,\dots,u_m)$ with $|u_k(x)|\geqq r\phi_k(x)$, $\forall\, x\in G$, for some k, $1\leqq k\leqq m$, one has*

$$\operatorname{sign} u_k\int_G(g_{\sigma(k)}(x,u_1(x),\dots,u_m(x)))\psi_k(x)\,dx\leqq 0$$

where σ is a permutation of the indices $1,\dots,m$.

Then, (3.1) *has at least one solution.*

The proof is trivial from Corollary 2.2.

Remarks. (1) Theorem (3.1) is still true if the inequality in hypothesis (i) is reversed.

(2) Let $m=1$ and $g=g(x,u)$ satisfying:

(i′) There is $r>0$ such that $\int_G g(x,u(x))\psi_1(x)\,dx\cdot\int_G g(x,v(x))\psi_1(x)\,dx<0$ for all $u,v\in X$ with $u(x)\geqq r\phi_1(x)$, $v(x)\leqq -r\phi_1(x)$, $\forall\, x\in G$.

Then, assumption (i) of Theorem 3.1 is satisfied. In fact, if $u_0\in X$ verifies $u_0(x)\geqq r\phi_1(x)$, $\forall\, x\in G$ and $\int_G g(x,u_0(x))\psi_1(x)\,dx<0$, one has $\int_G g(x,u(x))\psi_1(x)\,dx<0$ for all $u\in X$ verifying $u(x)\geqq r\phi_1(x)$, $\forall\, x\in G$. (The set $A=\{u\in X:u(x)\geqq r\phi_1(x),\forall\, x\in G\}$ is connected and the mapping from X into R defined by $u\to\int_G g(x,u(x))\psi_1(x)\,dx$ is continuous.) Hence, $\int_G g(x,v(x))\psi_1(x)\,dx>0$ for all $v\in X$ verifying $v(x)\leqq -r\phi_1(x)$, $\forall\, x\in G$, and therefore (i) of Theorem 3.1 is verified with a strict inequality.

If $\int_G g(x,u_0(x))\psi_1(x)\,dx>0$, we should consider the previous remark.

Hypothesis (i′) has been considered by de Figueiredo and Ni [7]. Thus, our Theorem 3.1 generalizes, yet in the scalar case, the main result of these authors.

(3) Let g_k, $1 \leqq k \leqq m$, be such that $g_{\sigma(k)}(x, u_1, \ldots, u_m) \cdot u_k \leqq 0$ for all $x \in G$, $(u_1, \ldots, u_m) \in R^m$, $1 \leqq k \leqq m$. Then, assumption (i) of Theorem 3.1 is satisfied. (The inequality can be reversed.)

(4) Let g_k, $1 \leqq k \leqq m$, satisfy the following conditions:

(A+) There exist $S_1 \in R$ and a bounded continuous mapping $g_+^k : \bar{G} \times R^m \to R$, $1 \leqq k \leqq m$, such that for all $u \in X$, $u = (u_1, \ldots, u_m)$ with $u_k(x) > S_1 \phi_k(x)$, $\forall x \in G$, one has $g_k(x, u_1(x), \ldots, u_m(x)) \leqq g_+^k(x, u_1(x), \ldots, u_m(x))$, $\forall x \in G$ and

$$\int_G g_+^k(x, u_1(x), \ldots, u_m(x)) \psi_k(x)\, dx \leqq 0.$$

(A−) There exist $S_2 \in R$ and a bounded continuous mapping $g_-^k : \bar{G} \times R^m \to R$, $1 \leqq k \leqq m$, such that for all $u \in X$, $u = (u_1, \ldots, u_m)$ with $u_k(x) < S_2 \phi_k(x)$, $\forall x \in G$, one has $g_k(x, u_1(x), \ldots, u_m(x)) \geqq g_-^k(x, u_1(x), \ldots, u_m(x))$, $\forall x \in G$ and

$$\int_G g_-^k(x, u_1(x), \ldots, u_m(x)) \psi_k(x)\, dx \geqq 0.$$

Then, condition (i) of Theorem 3.1 is again satisfied (see [6], where the details are given for the ordinary case).

Therefore, Theorem 3.1 extends and generalizes, for systems with bounded nonlinearities, the results given by Kazdan and Warner [10] in the scalar case.

Example 1. The B.V.P.

$$-\Delta u_1 - \lambda_1 u_1 = g_1(u_2) + f_1(x), \qquad x \in G$$

$$-\Delta u_2 - \lambda_1 u_2 = g_2(u_1) + f_2(x), \qquad x \in G$$

$$u_1(x) = u_2(x) = 0, \qquad x \in \partial G$$

has at least one solution assuming that $\int_G f_i(x) \psi(x)\, dx = 0$, $1 \leqq i \leqq 2$, and $g_1(u_2) u_2 \geqq 0$, $g_2(u_1) u_1 \geqq 0$.

This example cannot be studied from the results mentioned in the previous remarks, because it is a vector problem. Yet in the scalar case, we may consider the B.V.P.

$$\begin{aligned} -\Delta u - \lambda_1 u &= h(u) + f(x), \qquad x \in G \\ u &= 0, \qquad x \in \partial G \end{aligned} \tag{3.6}$$

where $h: R \to R$ is of class C^1 and bounded, $f: \bar{G} \to R$ is continuous and $\int_G f(x) \psi(x)\, dx = 0$.

If $h(u) \cdot u > 0$ for all $u \in R$, all conditions of Theorem 3.1 are satisfied (with the reversed inequality) and (3.6) has a solution. However, (3.6) does not satisfy (A+) and (A−). In fact, if $g(x, u) = h(u) + f(x)$ and $u \in X$ is such that $u(x) > S_1 \phi(x)$, $\forall x \in G$ with $u(x) > 0$

$\forall x \in G$, one obtains

$$0 \geqq \int_G g_+(x, u(x))\psi(x)\,dx \geqq \int_G g(x, u(x))\psi(x)\,dx = \int_G h(u(x))\psi(x)\,dx$$

which is a contradiction. Also (A −) is not satisfied.

Let us consider now the case where g_k, $1 \leqq k \leqq m$, can have a growth of some superlinear type. First we need a lemma.

Lemma 3.2. *Let A be a second-order uniform operator of the form*

$$A = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(x)\frac{\partial}{\partial x_j}\right)$$

where the coefficients of A satisfy the same regularity assumptions considered at the beginning of this section for A_k and let $u \in C^{2,\alpha}(\bar{G}, R)$ be such that $\partial u/\partial v_A = 0$ on ∂G. Then, if $w \in C^1(R, R)$ verifies $w'(t) \leqq 0$ for all $t \in R$, we have that $w \circ u \in C^{0,\alpha}(\bar{G}, R)$ and

$$\int_G Au(x)w(u(x))\,dx \leqq 0 \tag{3.7}$$

Proof. It is trivial that $w \circ u \in C^1(\bar{G}, R)$. But as $\partial G \in C^{2,\alpha}$, $C^1(\bar{G}) \hookrightarrow C^{0,\alpha}(\bar{G})$ (see [8]) and then $w \circ u \in C^{0,\alpha}(\bar{G})$.

On the other hand, by using the Green's formula ([9, p. 69]), we deduce (v_i denotes the ith component of the unit vector of the outward normal v to ∂G),

$$\begin{aligned}
\int_G Au(x)w(u(x))\,dx &= -\sum_{i=1}^{n}\int_G \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\right) w(u(x))\,dx \\
&= -\sum_{i=1}^{n}\left[-\int_G w'(u(x))\frac{\partial u(x)}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\right)dx\right. \\
&\quad \left. + \int_{\partial G} w(u(x))\left(\sum_{j=1}^{n} a_{ij}(x)\frac{\partial u(x)}{\partial x_j} v_i(x)\,dS\right)\right] \\
&= \int_G w'(u(x))\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial u(x)}{\partial x_j}\,dx \\
&\quad - \int_{\partial G} w(u(x))\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u(x)}{\partial x_j} v_i(x)\,ds \\
&\leqq \int_G w'(u(x))\mu\sum_{i=1}^{n}\left(\frac{\partial u(x)}{\partial x_i}\right)^2 dx - \int_{\partial G} w(u(x))\frac{\partial u(x)}{\partial v_A}\,ds \leqq 0.
\end{aligned}$$

When the boundary conditions are of Dirichlet type, an analogous result may be proved by the same way.

Lemma 3.3. *Let $u \in C^{2,\alpha}(\bar{G}, R)$ be such that $u=0$ on ∂G and $w \in C^1(R, R)$ satisfying $w'(t) \leqq 0$ for all $t \in R$ and $w(0)=0$. Then $w \circ u \in C^{0,\alpha}(\bar{G}, R)$ and*

$$\int_G Au(x)w(u(x))\,dx \leqq 0. \tag{3.8}$$

Remarks. (1) A similar inequality to (3.7), (3.8) has been proved by Brezis and Strauss [3] for elliptic B.V.P. in L^1.

(2) It is not possible to obtain an analogous result to the previous lemmas for problems whose linear part is of the form $Au(x)-\lambda_1 u(x)$, $x \in G$, $u(x)=0$, $x \in \partial G$. (Take the Dirichlet problem for $n=1$, $G=(0,\pi)$. Then, $Au(x)-\lambda_1 u(x) = -u''(x)-u(x)$, $u(0)=u(\pi)=0$, and $w: R \to R$ defined by

$$w(t) = \begin{cases} 0 \text{ if } t<0 \\ -t^2 \text{ if } t \geqq 0. \end{cases}$$

Then, if $u(x) = -(x^2-\pi x)$, we have

$$\int_G Au(x)w(u(x))\,dx = \int_0^\pi -(x^2-\pi x)[-2(2x-\pi)^2+(x^2-\pi x)^2]\,dx = \pi^5\left(\frac{\pi^2}{140}-\frac{1}{15}\right)>0).$$

Theorem 3.2. *Let us suppose that:*

(i) *There exists a mapping $w: R^m \to R^m$ of class C^1, $w(u_1, \dots, u_m) = (w_1(u_1), \dots, w_m(u_m))$ such that $w_k: R \to R$, $1 \leqq k \leqq m$, verifies $w_k'(t) \leqq 0$ for all $t \in R$, and constants $\alpha_1' \geqq 0$, $\beta_1' \geqq 0$, satisfying:*

$$\sum_{k=1}^m |g_k(x, u_1, \dots, u_m)|^p \leqq (w(u_1, \dots, u_m), g(x, u_1, \dots, u_m)) + \alpha_1'|u| + \beta_1'$$

for all $(x, u) = (x, u_1, \dots, u_m) \in \bar{G} \times R^m$ (where $(\cdot,\cdot)$ denotes the usual inner product in R^m), $g=(g_1, \dots, g_m)$.

(ii) *There exists $r>0$ such that for all $u \in X$, $u=(u_1, \dots, u_m)$ with $|u_k(x)| \geqq r$ for each $x \in G$ and some k, $1 \leqq k \leqq m$, one has $\int_G g(x, u_1(x), \dots, u_m(x))\,dx \neq 0$.*

(iii) *$d_B(F, B_{R^m}(s), 0) \neq 0$ for every $s \geqq r$, where $F: R^m \to R^m$ is defined by $F(c) = \int_G g(x, c)\,dx$.*

Then, the B.V.P. (3.1), *where each A_k has the form*

$$A_k = -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sum_{j=1}^n a_{ij}^k(x)\frac{\partial}{\partial x_j}\right), \quad 1 \leqq k \leqq m$$

and $B_k u_k = \partial u_k/\partial v_{A_k}$, $1 \leqq k \leqq m$, has at least one solution provided α_1' is sufficiently small.

Proof. Take in Corollary 2.1 $\xi: X \to Z^*$ defined by

$$(\xi u)(z) = \sum_{k=1}^{m} \int_G w_k(u_k(x)) z_k(x)\, dx.$$

It is obvious that (H) is satisfied with $\phi_k = 1/\mathrm{vol}(G)$, $1 \leqq k \leqq m$. On the other hand, using (i) we have

$$\|Nu\|_p^p = \sum_{k=1}^{m} \|(Nu)_k\|_p^p = \sum_{k=1}^{m} \int_G |g_k(x, u_1, \ldots, u_m)|^p\, dx \leqq \langle Nu, \xi u\rangle + \alpha_1'' \|u\|_{1,\alpha} + \beta_1'',$$

for all $u \in X$. Since there is $\gamma > 0$ such that $t \leqq t^p + \gamma$ for all $t \geqq 0$ we have condition (i) of Theorem 2.1.

Also, taking into account Lemma 3.2 we have hypothesis (b) of Corollary 2.1.

Lastly, condition (ii.3) of Theorem 2.1 and (c) of Corollary 2.1 are respectively the same as (iii) and (ii) in Theorem 3.2.

Then all conditions of Corollary 2.1 are satisfied and the theorem is proved.

Example 2. Let us consider the B.V.P.

$$\begin{aligned} -\Delta u_1 &= -e^{u_1} + h_1(x, u_1, u_2) + f_1(x), && x \in G \\ -\Delta u_2 &= -e^{u_2} + h_2(x, u_1, u_2) + f_2(x), && x \in G \\ \frac{\partial u_1}{\partial v} &= \frac{\partial u_2}{\partial v} = 0, && x \in \partial G, \end{aligned} \tag{3.9}$$

where $f_i: \bar{G} \to R$, $h_i: \bar{G} \times R^2 \to R$ are of class C^1, $h_i(x, u) \leqq 0$ for all $x \in \bar{G}$, $u \in R^2$ and h_i are bounded and satisfy $\lim_{|u_1| + |u_2| \to \infty} h_i(x, u_1, u_2) = 0$ uniformly in $x \in \bar{G}$, $1 \leqq i \leqq 2$.

Then, (3.9) has one solution if, and only if, $\int_G f_i(x)\, dx > 0$, $i = 1, 2$.

The above condition is clearly necessary and is also sufficient. In fact, let $p > n$ be an odd number and $w_i: R \to R$, $1 \leqq i \leqq 2$, defined by

$$w_i(t) = \begin{cases} 0 & \text{if} \quad t \leqq 0 \\ (-e^t + t + 1)^p & \text{if} \quad t > 0. \end{cases}$$

Then

$$|g_i(x, u_1, u_2)|^p = |-e^{u_i} + h_i(x, u_1, u_2) + f_i(x)|^p \leqq \beta \qquad \text{if } u_i \leqq 0$$

$$\begin{aligned} |g_i(x, u_1, u_2)|^p &= |-e^{u_i} + h_i(x, u_1, u_2) + f_i(x)|^p \\ &\leqq (-e^{u_i} + u_i + 1)^p (-e^{u_i} + h_i(x, u_1, u_2) + f_i(x)) + \beta' \qquad \text{if} \quad u_i > 0 \end{aligned}$$

and therefore, (i) of Theorem 3.2 is satisfied with $\alpha_1' = 0$.

Hypothesis (ii) of this theorem is trivially verified. Also $F: R^2 \to R^2$ is now defined by $F(c_1, c_2) = (F_1(c_1, c_2), F_2(c_1, c_2))$, where

$$F_1(c_1, c_2) = -e^{c_1} + (1/\mathrm{vol}(G))\left[\int_G h_1(x, c_1, c_2)\,dx + \int_G f_1(x)\,dx\right]$$

$$F_2(c_1, c_2) = -e^{c_2} + (1/\mathrm{vol}(G))\left[\int_G h_2(x, c_1, c_2)\,dx + \int_G f_2(x)\,dx\right].$$

It is easily proved that there exists $r_1 > 0$ such that $F_i(c_1, c_2)c_i \leqq 0$, with $|c_i| \geqq r_1$, $i = 1, 2$. Then, taking into account the Poincaré–Bohl theorem [**14**], (iii) of Theorem 3.2 is also verified.

Remarks, (1) If $w_k = 0$, $1 \leqq k \leqq m$, condition (i) may be substituted by

(i′) $$|g_k(x, u_1, \ldots, u_m)| \leqq \alpha'_k |u_k| + \beta'_k, \qquad 1 \leqq k \leqq m.$$

Therefore, Theorem 3.2 generalizes the main result obtained by Mawhin in [**12**], not only with respect to the growth of the non-linear term but also because we consider systems of equations.

(2) In the scalar case, B.V.P. of type (3.9) may be studied using the method of upper and lower solutions [**10**], but the use of this method in the case of vector problems is very restrictive. (We know that either certain monotone properties on the components g_k of the non-linearity g are required or a stronger definition of upper and lower solutions is needed.)

Finally and as an application of Theorem 2.2 we can study the existence of solutions of the Dirichlet problem

$$\begin{aligned} A_k u_k &= g_k(x, u_1, \ldots, u_m), && x \in G, \quad 1 \leqq k \leqq m \\ u_k &= 0, && x \in \partial G, \quad 1 \leqq k \leqq m. \end{aligned} \tag{3.10}$$

Theorem 3.3. *Let us assume that there exists a mapping $w: R^m \to R^m$ verifying the hypotheses of the previous theorem and, moreover, $w_k(0) = 0$, $1 \leqq k \leqq m$. Then, if* (i) *of Theorem* 3.2 *is satisfied, equation* (3.10) *has at least one solution provided α'_1 is sufficiently small.*

Example 3. The B.V.P.

$$\begin{aligned} -\Delta u_1 + \lambda u_1 &= -u_1^{p_1} + h_1(x, u_1, u_2), && x \in G \\ -\Delta u_2 + \mu u_2 &= -u_2^{p_2} + h_2(x, u_1, u_2), && x \in G \\ u_1 = u_2 &= 0, && x \in \partial G \end{aligned}$$

where p_i, $1 \leqq i \leqq 2$, are odd numbers greater than one, h_i, $1 \leqq i \leqq 2$, are bounded and $\lambda, \mu \in R$ has at least one solution. (We must take $w_1(t) = (-t^{p_1})^p$, $w_2(t) = (-t^{p_2})^p$ and p an odd number, $p > n$.)

Final remark. Our main result (Theorem 2.1) can always be applied to problems where condition (H) is satisfied and not only to the problems considered in this section. For instance, if we have the B.V.P. (ordinary case)

$$u_1'' - 2u_1 - 3u_2 = g_1(x, u_1, u_2), \qquad x \in (0, \pi)$$

$$u_2'' + 2u_2 + u_1 = g_2(x, u_1, u_2), \qquad x \in (0, \pi)$$

$$u_1(0) = u_1(\pi) = u_1'(0) + u_1'(\pi) = 0; \quad u_2(0) = u_2(\pi) = u_2'(0) + u_2'(\pi) = 0$$

then, it is not difficult to prove that $\phi_1(x) = \phi_2(x) = (\sin x)\sqrt{(2/\pi)}$ and consequently (2.3) is satisfied.

Also we may consider systems of the form (3.1) with some boundary conditions of Dirichlet type and the others of Neumann type.

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