Glasgow Math. J. **42** (2000) 247–253.

ON VECTOR-VALUED SPECTRA

ROBIN HARTE

Trinity College, Dublin, Ireland
e-mail:rharte@maths.tcd.ie

and CIARAN TAYLOR*

Institute of Technology, Tallaght, Ireland
e-mail:ciaran.taylor@it-tallaght.ie

(Received 27 July, 1998; revised 17 May, 1999)

Abstract. Elements $\alpha \in A \otimes E$ of the tensor product of a Banach algebra A and a Banach space E induce systems $\{\psi(\alpha) : \psi \in E^*\}$ of elements of A indexed by the dual space E^*, whose joint spectrum belongs to the second dual E^{**}. In this note we investigate when the spectrum actually lies in $E \subseteq E^{**}$, and extend the spectral mapping theorem $P\sigma_A(\alpha) = \sigma_A P(\alpha)$ to polynomial mappings $P : E \to F$ between Banach spaces. When the algebra A is commutative and the Banach space $E = B$ is another algebra we also reach a sort of vector-valued Gelfand theory.

1991 *Mathematics Subject Classification*. 47A13.

If A is a complex Banach algebra, with identity 1, and $(a_x)_{x\in X}$ a family of elements in A, then the left and the right spectrum of $a \in A^X$ are defined as subsets of the corresponding families of complex numbers ([**2**];[**3**, Definition 11.4.1]):

$$\sigma_A^{left}(a) = \{\lambda \in \mathbf{C}^X : 1 \notin \sum_{x\in X} A(a_x - \lambda_x)\}, \tag{0.1}$$

and

$$\sigma_A^{right}(a) = \{\lambda \in \mathbf{C}^X : 1 \notin \sum_{x\in X} (a_x - \lambda_x)A\}. \tag{0.2}$$

Thus if, for example, $\lambda \in \mathbf{C}^X$ is not in the left spectrum of $a \in A^X$ there is $b = (b_x)_{x\in X}$ in A, vanishing for all but finitely many $x \in X$, for which $\sum_{x\in X} b_x(a_x - \lambda_x) = 1$. The right and left spectra are in a sense the same, being interchanged by "reversal of products". We recall that they are compact subsets of $\mathbf{C}^X$, in the topology of pointwise convergence on X, possibly empty, and ([**3**, Theorem 11.4.2]) subject to the one-way spectral mapping theorem for 'polynomials':

$$p\sigma_A^{left}(a) \subseteq \sigma_A^{left} p(a), \;\; p\sigma_A^{right}(a) \subseteq \sigma_A^{right} p(a), \tag{0.3}$$

where a polynomial $p = (p_y)_{y\in Y} : A^X \to A^Y$ is a system of members of the free algebra generated by the coordinates z_x and the identity 1. If in particular the system $a = (a_x)_{x\in X}$ is commutative then by [**3**, Theorem 11.4.4] the spectra $\sigma_A^{left}(a)$, $\sigma_A^{right}(a)$ are nonempty, and there is equality in (0.3).

*The second author acknowledges partial support from an Enterprise Ireland basic research scholarship.

https://doi.org/10.1017/S0017089500020103 Published online by Cambridge University Press

If, for example, $X = \{1, 2, \ldots, n\}$ then these add up to the "so-called Harte spectrum" [**4**] of the n-tuple $a = (a_1, a_2, \ldots, a_n)$; more generally $\lambda \in \mathbf{C}^X$ is in $\sigma_A^{left}(a)$ if and only if every finite restriction of λ is in the left spectrum of the corresponding restriction of $a \in A^X$. Infinite indexing systems X have however the possibility of non-trivial structure: thus ([**2**, Lemma 1], [**3**, Theorem 11.4.3]) we have the following result.

Lemma 1. *If $\lambda \in \mathbf{C}^X$ is in the left or the right spectrum of $a \in A^X$, and if $a : X \to A$ is either bounded, or continuous, or homomorphic, or linear, then so is $\lambda : X \to \mathbf{C}$.*

Proof. This is the one way spectral mapping theorem (0.3), together with the spectral theory of a single element: for arbitrary x, x', x'' in X we find that λ_x, $\lambda_{x'} - \lambda_x$ and $p(\lambda_x, \lambda_{x'}, \lambda_{x''})$ are in $\sigma(a_x)$, $\sigma(a_{x'} - a_x)$ and $\sigma\ p(a_x, a_{x'}, a_{x''})$, respectively, and hence we have

$$|\lambda_x| \leq \|a_x\| \ ; \ |\lambda_{x'} - \lambda_x| \leq \|a_{x'} - a_x\| \ ; \ |p(\lambda_x, \lambda_{x'}, \lambda_{x''})| \leq \|p(a_x, a_{x'}, a_{x''})\|. \tag{1.1}$$

The first of these inequalities transmits boundedness, the second continuity, and the third ensures that $(\lambda_x, \lambda_{x'}, \lambda_{x''})$ is subjected to any polynomial identity satisfied by $(a_x, a_{x'}, a_{x''})$. □

Thus, for example, if $X = \mathbf{N}$ and $a = (a_n) \in \ell_1(A)$ then

$$\lambda \in \sigma_A^{left}(a) \subseteq \mathbf{C}^{\mathbf{N}} \Longrightarrow \lambda \in \ell_1; \tag{1.2}$$

if instead $X = \Omega$ is a topological space and $a \in C(\Omega, A)$, then

$$\lambda \in \sigma_A^{left}(a) \subseteq \mathbf{C}^{\Omega} \Longrightarrow \lambda \in C(\Omega); \tag{1.3}$$

if, in particular, $X = F$ is a Banach space and $a \in BL(F, A)$ then

$$\lambda \in \sigma_A^{left}(a) \subseteq \mathbf{C}^F \Longrightarrow \lambda \in F^*. \tag{1.4}$$

In the ultimate special case ([**2**, Theorem 1]; [**3**, Theorem 11.4.3]) if $a_x = x$ for each $x \in X = A$, then the left and the right spectrum of a reduce to the Gelfand 'maximal ideal space' of multiplicative linear functionals:

$$\sigma_A^{left}(a) = \sigma_A^{right}(a) = \sigma(A). \tag{1.5}$$

In this note we would like to focus on the situation in which

$$\alpha = \sum_{n=1}^{\infty} a_n \otimes e_n \in A \otimes E \tag{1.6}$$

belongs to a uniform cross-normed tensor product [**3**, Definition 11.7.1] of A with a Banach space E; examples include operator matrices and continuous vector-valued functions. Waelbroeck [**5**],[**6**] has looked here for a functional calculus involving holomorphic functions in infinitely many variables. Evidently α induces a bounded linear operator $\alpha_\wedge : \psi \longmapsto \psi(\alpha)$ from the dual space E^* into A, where we write

https://doi.org/10.1017/S0017089500020103 Published online by Cambridge University Press

$$\alpha_\wedge(\psi) = \psi(\alpha) = \sum_{n=1}^{\infty} \psi(e_n)a_n \in A \tag{1.7}$$

if $\psi \in E^*$ and $\alpha \in A \otimes E$ is given by (1.6). The spectrum of $\alpha_\wedge \in BL(F, A)$, with $F = E^*$, lies by (1.4) in the second dual E^{**} of the space E, and would be a candidate for the spectrum of the system $\alpha \in A \otimes E$; we would however prefer a spectrum lying in the space E, writing more intuitively

$$\sigma_A^{left}(\alpha) = \{x \in E : 1 \notin \sum_{\psi \in E^*} A(\psi(\alpha) - \psi(x))\}, \tag{1.8}$$

$$\sigma_A^{right}(\alpha) = \{x \in E : 1 \notin \sum_{\psi \in E^*} (\psi(\alpha) - \psi(x))A\}. \tag{1.9}$$

Thus the spectrum of α is essentially the intersection of the space $E \cong E^\wedge \subseteq E^{**}$ with the spectrum of $\alpha_\wedge$. The good news is that, provided we stay in the 'projective' product, the spectra of α and $\alpha_\wedge$ coincide.

THEOREM 2. *If* $\alpha = \sum_{n=1}^{\infty} a_n \otimes e_n \in A \otimes E$ *with* $\max_n \|a_n\| < \infty$ *and* $\sum_n \|e_n\| < \infty$, *then*

$$\sigma_A^{left}(\alpha_\wedge) = \sigma_A^{left}(\alpha)^\wedge \subseteq E^\wedge \subseteq E^{**},\ \ \sigma_A^{right}(\alpha_\wedge) = \sigma_A^{right}(\alpha)^\wedge \subseteq E^\wedge \subseteq E^{**}. \tag{2.1}$$

Proof. If we write $e^\wedge : E^* \to \ell_1$ for the operator which sends $\psi \in E^*$ to $(\psi(e_n)) \in \ell_1$, then each $\xi \in \sigma_A^{left}(\alpha)$ is 'majorized' [**3**, Definition 10.1.1] by $e^\wedge$:

$$|\xi(\psi)| \leq \|a\|_\infty |e^\wedge(\psi)|, \tag{2.2}$$

and hence by [**1**, Lemma 1] (cf. [**3**, Theorem 5.5.3]) factors through $e^\wedge$:

$$\xi = \theta \circ e^\wedge \text{ with } \theta \in \ell_1^* \tag{2.3}$$

obtained by Hahn-Banach extension from the functional $\theta_0 : e^\wedge(E^*) \to \mathbf{C}$, defined by setting $\theta_0(e^\wedge(\psi)) = \xi(\psi)$. Thus there is $\lambda \in \ell_\infty$ for which

$$\xi = \sum_{n=1}^{\infty} \lambda_n e_n^\wedge \in E^\wedge. \tag{2.4}$$

□

When the element $\alpha \in A \otimes E$ is commutative, in the sense that $\{\psi(\alpha) : \psi \in E^*\} \subseteq A$ is commutative, this extends to all uniform products.

THEOREM 3. *If* $\alpha \in A \otimes E$ *is commutative, then (2.1) holds. If, in particular, A is commutative then*

$$\sigma_A^{left}(\alpha_\wedge) = \sigma_A^{right}(\alpha_\wedge) = \{\varphi(\alpha) : \varphi \in \sigma(A)\}^\wedge, \tag{3.1}$$

where $\sigma(A) \subseteq A^*$ *is the 'maximal ideal space' of A and where if* $\alpha \in A \otimes E$ *and* $\varphi \in \sigma(A)$ *we have*

$$\varphi(\alpha) = \lim\{\sum_{j\in J}\varphi(a_j)e_j : \sum_{j\in J} a_j \otimes e_j \to \alpha\} \in E. \tag{3.2}$$

The Gelfand mapping

$$\alpha \mapsto \alpha^\wedge : A \otimes E \to C(\sigma(A), E), \tag{3.3}$$

is continuous, and the spectrum of $\alpha \in A \otimes E$ *is the range of the Gelfand transform* $\alpha^\wedge : \varphi \mapsto \varphi(\alpha)$.

Proof. Suppose first that the algebra A is commutative: if $\alpha \in A \otimes E$ and if $\xi \in E^{**}$ is in $\sigma_A^{left}(\alpha_\wedge)$ then

$$1 \notin \sum_{\psi \in E^*} A(\psi(\alpha) - \xi(\psi)). \tag{3.4}$$

By Gelfand theory [**3**, Theorem 9.6.3], there is $\varphi \in \sigma(A)$ for which

$$\{\psi(\alpha) - \xi(\psi) : \psi \in E^*\} \subseteq \varphi^{-1}(0), \tag{3.5}$$

which means, if $\alpha \in A \otimes E$, that

$$\lim\{\sum_{j\in J}\varphi(a_j)\psi(e_j) : \sum_{j\in J} a_j \otimes e_j \to \alpha\} = \xi(\psi) \text{ for each } \psi \in E^*. \tag{3.6}$$

But this means that

$$\xi = \lim\{(\sum_{j\in J}\varphi(a_j)e_j)^\wedge : \sum_{j\in J} a_j \otimes e_j \to \alpha\} \in E^\wedge \subseteq E^{**}. \tag{3.7}$$

For the continuity observe that, with $\alpha \in A \otimes E$ and arbitrary $\varphi \in \sigma(A)$, we have

$$|\varphi(\alpha)| \le \sup_{\|\psi\|\le 1}|\psi(\alpha)|. \tag{3.8}$$

If more generally $\alpha \in A \otimes E$ is commutative, then this argument applies to the closed (unital) subalgebra $B \subseteq A$ generated by the elements $\{\psi(\alpha) : \psi \in E^*\}$, so that

$$\sigma_A^{left}(\alpha) \subseteq \sigma_B^{left}(\alpha) \subseteq E^\wedge.$$

□

Theorem 3 survives if the algebra A is commutative modulo its radical, or if $\alpha \in A \otimes E$ is for example ‘quasi-commutative’ [**3**, Definition 11.8.3], in the sense that all the commutators $\psi(\alpha)\theta(\alpha) - \theta(\alpha)\psi(\alpha)$ commute with each $\phi(\alpha)$. Theorem 2 sometimes holds for non commutative A, nonreflexive E and arbitrary products.

Example 4. *If* $E = c_0$, *so* $E^{**} \cong \ell_\infty$, *and if*

$$\alpha \in A \otimes c_0 \subseteq A \odot c_0 \cong c_0(A) \subseteq \ell_\infty(A), \tag{4.1}$$

https://doi.org/10.1017/S0017089500020103 Published online by Cambridge University Press

then there is implication

$$y \in \sigma_A^{left}(\alpha) \cup \sigma_A^{right}(\alpha) \subseteq \ell_\infty \Longrightarrow y \in c_0. \tag{4.2}$$

Proof. This result follows from Lemma 1.

The ‘polynomials’ of (0.3), specialised to the systems $\alpha_\wedge = (\psi(\alpha))_{\psi \in E^*}$, are generated by “co-ordinates” $(z_\psi)_{\psi \in E^*}$, and continue to act on the “E-valued” spectra (1.8) and (1.9). In this context however there are more serious polynomials $P : E \to F$, induced by symmetric bounded multilinear operators:

$$P = \sum_{n=0}^{N} P_n \in Poly(E, F) \text{ with } P_n(x) = P_n^\vee(x, x, \ldots, x) \in Poly_n(E, F)(x), \tag{4.3}$$

where $P_n^\vee : E^n \to F$ is bounded symmetric n-linear. Thus $Poly_1(E, F) = BL(E, F)$ is just the space of bounded linear operators and $Poly_0(E, F) = F$ is the constants; a product $\psi \cdot T : x \mapsto \psi(x)T(x)$ with $\psi \in E^*$ and $T \in BL(E, F)$ is a rather special kind of element of $Poly_2(E, F)$. These ‘polynomials’ also act on the (projective) product $A \otimes E$ if we define (cf [**5**, Chapter VIII p. 127], [**6**, p. 106])

$$P(\alpha) = \sum_{n=0}^{N} \sum_{|k|=n} a_k \otimes P_n^\vee(e_k) \text{ if } \alpha = \sum_{m=0}^{\infty} a_m \otimes e_m, \tag{4.4}$$

where we write

$$e_k = (e_{k_1}, e_{k_2}, \ldots, e_{k_n}) \text{ and } a_k = a_{k_1} a_{k_2} \ldots a_{k_n} \text{ if } k = k_1 k_2 \ldots k_n. \tag{4.5}$$

It has to be checked that $P(\alpha)$ is well-defined; then the spectral mapping theorem holds.

THEOREM 5. *If $\alpha \in A \otimes E$ is arbitrary and if $P : E \to F$ is a polynomial, there is inclusion*

$$P\sigma_A^{left}(\alpha) \subseteq \sigma_A^{left} P(\alpha) \text{ and } P\sigma_A^{right}(\alpha) \subseteq \sigma_A^{right} P(\alpha)\ , \tag{5.1}$$

with equality if α is commutative.

Proof. We claim that, acting on $A \otimes E$, a weak remainder theorem (cf [**3**, Theorem 11.2.1]) is valid for polynomials $P \in Poly(E, F)$: for arbitary $\theta \in F^*$ we have

$$\theta(P(\alpha) - P(x)) \in \text{cl} \sum_{\psi \in E^*} A(\psi(\alpha) - \psi(x)). \tag{5.2}$$

It is clear that (5.2) holds if P is either a constant or a linear operator, and holds if $P = Q + R$ where Q and R are polynomials for which (5.2) holds. It is therefore sufficient to establish (5.2) for $P \in Poly_n(E, F)$ for each $n \in \mathbf{N}$. Generally if $P \in Poly_{n+1}(E, F)$, if $\alpha = \sum_{m=1}^{\infty} a_m \otimes e_m \in A \otimes E$ and $x \in E$, we have

https://doi.org/10.1017/S0017089500020103 Published online by Cambridge University Press

$$P(\alpha) - P(x) = \sum_{|k|+|j|=n} (a_k \otimes 1)\Big(\sum_{m=1}^{\infty} a_m \otimes P^{\vee}(e_k, x^j, e_m) - 1 \otimes P^{\vee}(e_k, x^j, x)\Big). \tag{5.3}$$

This remainder theorem gives the one way inclusion (5.1); if, in particular, α is commutative then by [**3**, Theorem 11.4.4] together with (2.1)

$$y \in \sigma_A^{left} P(\alpha) \subseteq F \Longrightarrow \exists x \in E \text{ with } (x, y) \in \sigma_A^{left}(\alpha, P(\alpha)) \tag{5.4}$$

and by [**3**, Theorem 11.2.6]

$$(x, y) \in \sigma_A^{left}(\alpha, P(\alpha)) \Longrightarrow y = P(x). \tag{5.5}$$

To prove (5.5) we write $Q(z, w) = w - P(z)$ and notice that

$$Q(x, y) \in Q\sigma_A^{left}(\alpha, P(\alpha)) \subseteq \sigma_A^{left} Q(\alpha, P(\alpha)) = \sigma_A^{left}(0) = \{0\}.$$

□

When A is commutative and we stay in the projective product, Waelbroeck [**6**] establishes a functional calculus $f \mapsto f(\alpha)$ from functions 'holomorphic' near $\sigma_A(\alpha) \subseteq E$ to A. If, in particular, $E = B$ is another Banach algebra and the cross-norm on $A \otimes E$ is such that $A \otimes E = A \otimes B$ is again a Banach algebra, then there are three kinds of 'spectrum' induced on $\alpha \in A \otimes B$:

$$\sigma_A(\alpha) \subseteq B\ ;\ \ \sigma_B(\alpha) \subseteq A\ ;\ \ \sigma_{A\otimes B}(\alpha) \subseteq \mathbf{C}\ . \tag{5.6}$$

These are sometimes related. See [**2**, (2.4)].

THEOREM 6. *If A and B are Banach algebras with A commutative, then for arbitrary $\alpha \in A \otimes B$*

$$\sigma_{A\otimes B}^{left}(\alpha) = \bigcup\{\sigma_B^{left}(b) : b \in \sigma_A^{left}(\alpha)\},\ \ \sigma_{A\otimes B}^{right}(\alpha) = \bigcup\{\sigma_B^{right}(b) : b \in \sigma_A^{right}(\alpha)\}. \tag{6.1}$$

Proof. Recall [**3**, Theorem 11.7.5] that if $a \in A^X$ and $b \in B^Y$ are arbitrary there is equality

$$\sigma_{A\otimes B}^{left}(a \otimes 1, 1 \otimes b) = \sigma_A^{left}(a) \times \sigma_B^{left}(b) : \tag{6.2}$$

inclusion one way is obvious, since if for example $\sum_{x\in X} a'_x(a_x - \lambda_x) = 1$ then

$$\sum_{x\in X}(a'_x \otimes 1)((a_x - \lambda_x) \otimes 1) + \sum_{y\in Y}(1 \otimes b'_y)(1 \otimes (b_y - \mu_y)) = 1 \otimes 1$$

with $b'_y = 0$. Conversely if (λ, μ) is in the right hand side, so that $M = \mathrm{cl} \sum_{x\in X} A(a_x - \lambda_x)$ and $N = \mathrm{cl} \sum_{y\in Y} B(b_y - \mu_y)$ are proper closed left ideals of A and B, then by the Hahn-Banach theorem there are $\varphi \in A^*$ and $\theta \in B^*$ for which $\varphi(1) = 1 = \theta(1)$ with $\varphi(M) = \{0\} = \theta(N)$. But now since the product $A \otimes B$ is uniform, the linear functional $\varphi \otimes \theta$ is well-defined and bounded on $A \otimes B$, and satisfies

$$(\varphi \otimes \theta)(1 \otimes 1) = 1 \text{ with } (\varphi \otimes \theta)(M \otimes B + A \otimes N) = \{0\}.$$

Towards (6.1) suppose $b \in \sigma_A^{left}(\alpha)$, so that by Theorem 3 there is $\varphi \in \sigma(A)$ for which $b = \varphi(\alpha)$. By (6.2) it follows from $\mu \in \sigma_A^{left}\varphi(\alpha)$ that

$$(\varphi, \mu) \in \sigma_{A\otimes B}^{left}(A \otimes 1, 1 \otimes \varphi(\alpha)) \subseteq A^* \times \mathbf{C}, \tag{6.3}$$

or equivalently

$$(\varphi, \mu) \in \sigma_{A\otimes B}^{left}(A \otimes 1, \alpha), \tag{6.4}$$

since by [**3**, Theorem 11.3.5] $\alpha - 1 \otimes \varphi(\alpha)$ is in the closed left ideal of $A \otimes B$ generated by

$$A \otimes 1 - \varphi \otimes 1 = \{(c - \varphi(c)) \otimes 1 : c \in A\}.$$

By the one way spectral mapping theorem (0.3) this gives $\mu \in \sigma_{A\otimes B}^{left}(\alpha)$. Conversely this implies by the two way spectral mapping theorem that there is $\varphi \in \sigma(A)$ for which (6.4) holds; hence we have established (6.3), which by (6.2) gives $\mu \in \sigma_B^{left}\varphi(\alpha)$, with of course $b = \varphi(\alpha) \in \sigma_A^{left}(\alpha)$. □

From Theorem 6 we can deduce that the spectrum of a commutative operator matrix, an upper triangular operator matrix, or a continuous vector-valued function, is what it ought to be [**3**, Theorem 11.7.7; (11.7.7.13), (11.7.7.16)], and also give an alternative proof of Allen's theorem [**3**, Theorem 11.7.9] about holomorphic one sided inverses. Specifically Theorem 6 offers a sort of vector-valued Gelfand theorem for $A \otimes B$: if A is commutative then α is left, or right, invertible in $A \otimes B$ if and only if $\varphi(\alpha)$ is left, or right, invertible in B for every $\varphi \in \sigma(A)$.

REFERENCES

1. M. R. Embry, Factorization of operators on Banach spaces, *Proc. Amer. Math. Soc.* **38** (1973), 587–590.

2. R. E. Harte, The spectral mapping theorem in many variables in *Proc. Seminar 'Uniform algebras', University of Aberdeen* 1973, 59–63; **MR** 50:#14226.

3. R. E. Harte, *Invertibility and singularity* (Dekker, New York, 1988); **MR** 89d:47001.

4. F.-H. Vasilescu, *Math. Reviews*. **MR** 95d:47003.

5. L. Waelbroeck, *Topological vector spaces and algebras*, Lecture Notes in Mathematics #230 (Springer-Verlag, 1971).

6. L. Waelbroeck, The holomorphic functional calculus and infinite dimensional holomorphy in *Proc. on infinite dimensional holomorphy 1973,* 101–108 Lecture Notes in Mathematics #364 (Springer-Verlag, 1974).