INDUCED AND PRODUCED MODULES

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Introduction. We shall consider here two generalizations to rings of the concept of induced representation as it occurs in the representation theory of finite groups **(6)**.

If A is a ring, S a subring of A, we shall associate with each S-module M an *induced pair* $(I(M), \kappa)$ consisting of an A-modulo $I(M)$ and an S-homomorphism κ: $M \to I(M)$. κ will be an isomorphism if and only if there exists an A-module having M as an S-submodule, and then $I(M)$ can be described as "the most general A-module generated by M"; so our definition is a generalization of the classical concept, consistent with that of other writers **(7; 11; 15)**.

Dually, we shall associate with M a *produced pair* $(P(M), \pi)$ consisting of an A-module $P(M)$ and an S-homomorphism $\pi: P(M) \to M$, the duality involved being an instance of the duality formalized by MacLane **(12)**.

We investigate certain dual properties of the induced and produced pairs. If M is given as an A-module, we can consider it as an S-module and form $(I(M), \kappa)$ and $(P(M), \pi)$. Then M is (A-isomorphic with) an A-quotient module of $I(M)$ and an A-submodule of $P(M)$. We determine when M is a direct summand of $I(M)$, or dually, of $P(M)$.

If A is a group ring, and S a subring by a subgroup of finite index, $I(M)$ and $P(M)$ are isomorphic A-modules for every unitary S-module M. The general question of when $I(M)$ and $P(M)$ are isomorphic leads to the consideration of certain generalized "Casimir operators" and to the notion of a ring *self-dual* with respect to a subring.

As applications of our general discussion we mention a characterization of separable algebras related to a result of Hochschild **(9)**, a characterization of the principal indecomposable representations of a Frobenius algebra, and certain theorems of Eckmann **(2)**, Gaschütz **(5)** and Higman **(7)**.

1. Induced and Produced Modules

1. The induced pair. Let S be a subring of a ring A. It is our intention to study some relations between representation modules for S and those for A. Our methods will not be complicated by extending the problem according to the following definitions, which also will be natural for the considerations of Part II:

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If S is a ring, a module M which is at the same time a left and a right S-module such that

$$s \cdot ut = su \cdot t \qquad (s, t \in S, u \in M)$$

is called an *S-bimodule*. We shall call a ring A an *S-ring* if it is an S-bimodule subject to the additional conditions

$$s \cdot ab = sa \cdot b, \quad a \cdot sb = as \cdot b, \quad a \cdot bs = ab \cdot s \qquad (a \in S; a, b \in A),$$

typical examples being the case in which S is a subring of A, or A a hypercomplex system over a commutative field S.

An S-bimodule M will be called a *right A,S-module* if it is a right A-module satisfying

$$s \cdot ua = su \cdot a, \quad u \cdot sa = us \cdot a, \quad u \cdot as = ua \cdot s \qquad (a \in S, u \in M, a \in A).$$

Interchanging left and right we obtain the definition of *left A,S-module*. We shall usually abbreviate "right A,S-module" to "A,S-module". Also, we shall usually state only one of a pair of left-right dual definitions or propositions.

The S-ring A itself can be considered in a natural way as an A,S-module and as a left A,S module.

If S is a subring of A, a right A-module M becomes an A,S-module if we let S operate trivially on the left of M; $su = 0$ for all $s \in S$, $u \in M$.

We write $M \simeq_A N$ to indicate the existence of an A,S-isomorphism of an A,S-module M onto an A,S-module N.

We shall now consider certain A,S-modules determined by S-bimodules. We claim that to each S-bimodule M there corresponds a pair $(I(M), \kappa)$, consisting of an A,S-module $I(M) = I_{A,S}(M)$ and an S-homomorphism $\kappa: M \rightarrow I(M)$, determined up to an A,S-isomorphism by the property:

(I) *If H is any A,S-module and $\delta: M \rightarrow H$ is an S-homomorphism then there exists one and only one A,S-homomorphism $\bar{\delta}: I(M) \rightarrow H$ such that $\delta = \kappa\bar{\delta}$, in other words, such that the following diagram is commutative:*

$$\begin{tikzcd} I(M) \arrow[dr, "\bar{\delta}"] & \\ M \arrow[u, "\kappa"] \arrow[r, "\delta"'] & H \end{tikzcd}$$

We shall say that the pair $(I(M), \kappa)$ is *induced* by M, writing $I(M) = I_{A,S}(M)$ when we wish to stress the role of A and S.

We postpone to §7 the exhibition of a pair satisfying (I). That there can exist essentially only one such pair is readily seen as follows: Suppose that the pair (H, δ) also satisfies (I). Then there exists by (I) an A,S-homomorphism $\bar{\delta}: I(M) \rightarrow H$ such that $\delta = \kappa\bar{\delta}$, and an A,S-homomorphism $\bar{\kappa}: H \rightarrow I(M)$ such that $\kappa = \delta\bar{\kappa}$. Then $\bar{\delta}\bar{\kappa}$ is an A,S-endomorphism of $I(M)$ and $\kappa\bar{\delta}\bar{\kappa} = \delta\bar{\kappa} = \kappa$. It follows from the uniqueness requirement of (I) that $\bar{\delta}\bar{\kappa}$ is the identity automorphism of $I(M)$. By symmetry $\bar{\kappa}\bar{\delta}$ is the identity automorphism of H, proving that $\bar{\delta} = \bar{\kappa}^{-1}$ is an A,S-isomorphism of $I(M)$ onto H, with $\delta = \kappa\bar{\delta}$.

Another characterization of the induced pair is as follows: We say that a pair (H, δ), consisting of an A,S-module H and an S-homomorphism $\delta: M \to H$, is *generated* by M if the smallest A,S-submodule of H containing $M\delta$ is H itself. If δ is an isomorphism, we say that H is *generated* by M. We have

THEOREM 1. *The induced pair $(I(M), \kappa)$ is determined up to an A,S-isomorphism by the properties*

(a) *$(I(M), \kappa)$ is generated by M.*

(b) *if (H, δ) is generated by M then there exists an A,S-homomorphism $\bar{\delta}: I(M) \to H$ such that $\delta = \kappa\bar{\delta}$.*

For the proof we use

LEMMA 1. *Let (H, δ) be generated by M and let κ be an A, S-module, $\epsilon: M \to K$ an S-homomorphism. Then there exists at most one A,S-homomorphism $\bar{\epsilon}: H \to K$ such that $\epsilon = \delta\bar{\epsilon}$.*

Proof. The equation $\epsilon = \delta\bar{\epsilon}$ means that the restriction of $\bar{\epsilon}$ to $M\delta$ is uniquely determined by ϵ. Since $\bar{\epsilon}$ is an A,S-homomorphism, we have for all a in A that $\delta a \cdot \bar{\epsilon} = \delta\bar{\epsilon} \cdot a = \epsilon a$, which means that the restriction of $\bar{\epsilon}$ to $M \cdot \delta a$ is uniquely determined by ϵ. Hence $\bar{\epsilon}$ is uniquely determined by ϵ since H is the l.u.b. of its submodules $M\delta$ and $M \cdot \delta a$ with a in A.

Proof of Theorem 1. That $(I(M), \kappa)$ satisfies (b) follows at once from its defining property (I).

To prove (a) we show that the g.l.b. $\bar{M}$ of the A,S-submodules of $I(M)$ which contain $M\kappa$ coincides with $I(M)$. Let j be the injection of M into $I(M)$. Then $\kappa = \alpha j$ defines an S-homomorphism $\alpha: M \to \bar{M}$. Since (M, α) is generated by M, as is clear from its construction, Lemma 1 implies that j is the only A,S-homomorphism of M into $I(M)$ such that $\kappa = \alpha j$. By (I) there exists an A,S-homomorphism $\bar{\alpha}: I(M) \to \bar{M}$ such that $\alpha = \kappa\bar{\alpha}$. Then $\kappa = \alpha j = \kappa\bar{\alpha}j = \alpha j\bar{\alpha}j$. But $j\bar{\alpha}j: \bar{M} \to I(M)$ is an A,S-homomorphism. Hence $j\bar{\alpha}j = j$, so that $j\bar{\alpha}$ is the identity automorphism of $I(M)$. Similarly $\bar{\alpha}j$ is the identity automorphism of $\bar{M}$, proving that $\bar{\alpha} = j^{-1}$, and hence that $\bar{M} = I(M)$. Consequently (a) is satisfied by the induced pair.

That $(I(M), \kappa)$ is essentially uniquely determined by (a) and (b) is readily verified using Lemma 1, thus completing the proof of Theorem 1.

When κ is an isomorphism, so that M can be identified with the S-submodule $M\kappa$ of $I(M)$, we say the $I(M)$ *is induced by M*. By (I) we see that this is the case if and only if there exists an A,S-module having M as an S-submodule. By Theorem 1, when κ is an isomorphism we can describe $I(M)$ as "the most general A,S-module generated by M", and this is the only case in which there exists an A,S-module conforming to this description. Thus our definition of induced module is consistent with the classical one; for instance, if A is a ring with identity element e, S a subring of A containing e, and R a right ideal of S (turned into an S-bimodule by letting S operate trivially on the left) then $I(R) = RA$.

2. The duality and the produced pair. We want to subject our discussion to a dualization, the duality involved being an instance of the duality formalized by MacLane **(12)**.

If M and N are modules, we write

$$\alpha \colon M \to N$$

to mean that α is a homomorphism of M into N, and write $K(\alpha)$ for the kernel of α. If M is a submodule of N, $u \to u$ $(u \in M)$ defines the *injection* of M into N, while if M is a quotient module of N, $M = N/L$, $v \to L + v$ $(v \in N)$ defines the *projection* of N onto M.

To any statement $\mathfrak{S}$ about modules, referring to homomorphisms of one such into another, submodules, quotient modules, injections and projections, but not to elements of modules, we assign a statement $\bar{\mathfrak{S}}$ by carrying out in $\mathfrak{S}$ the interchanges indicated by the following table:

$\alpha \colon M \to N$	$\alpha \colon N \to M$
α an isomorphism into product $\alpha\beta$	α a homomorphism onto product $\beta\,\alpha$
submodule	quotient module
injection	projection

Note that "S-bimodule" remains unchanged, while "right A,S-module" is interchanged with "left A,S-module".

In the terminology of MacLane **(12)**, $\bar{\mathfrak{S}}$ is the dual of $\mathfrak{S}$. We shall call the left-right dual $\mathfrak{S}'$ of $\bar{\mathfrak{S}}$ the *dual* of $\mathfrak{S}$. Then "right A,S-module" and "left A,S-module" are self dual.

In particular, the dual of the (right) A,S-module $I(M)$ is an A,S-module. Precisely, the dual of our statement defining and asserting the existence and essential uniqueness of the induced pair is the following (as before, S is a ring, A an S-ring):

To each S-bimodule M there corresponds a pair $(P(M), \pi)$, consisting of an A,S-module $P(M)$ and an S-homomorphism $\pi \colon P(M) \to M$, determined up to an A,S-isomorphism by the property (*P*) *if H is any A,S-module, and $\delta \colon H \to M$ is an S-homomorphism, then there exists one and only one A,S-homomorphism $\bar{\delta} \colon H \to P(M)$ such that $\delta = \bar{\delta}\pi$*:

$$\begin{tikzcd} P(M) \arrow[d, "\pi"'] & \\ M & H \arrow[l, "\delta"'] \arrow[ul, "\bar{\delta}"'] \end{tikzcd}$$

We call $(P(M), \pi)$ the pair *produced* by M. Its existence will be established in §7 by the exhibition of a pair satisfying (P). The essential uniqueness of the produced pair can be shown by dualizing the demonstration of the corresponding fact for the induced pair, i.e., by replacing each statement of that demonstration by its dual in the sense defined above.

Actually, the uniqueness proof for $(I(M), \kappa)$ can easily be translated into a "categorical" one for the abelian bicategory of all modules and homomorphisms of modules into modules **(12)**. Hence we can infer at once the essential uniqueness of $(P(M), \pi)$ from that of $(I(M), \kappa)$ by invoking the *duality principle* of MacLane **(12**, p. 806**)**.

Except for the existence proofs for $(I(M), \kappa)$ and $(P(M), \pi)$, the same remark will apply to all the theorems of this Part I. These will occur in dual pairs $\mathfrak{S}$, $\mathfrak{S}'$, of which we shall prove only one, say $\mathfrak{S}$, phrasing the proof in such a way that it can easily be translated into a categorical one. Then we can conclude at once from MacLane's duality principle that $\overline{\overline{\mathfrak{S}}}$, and hence $\mathfrak{S}'$ is a theorem. If we do not wish to refer to the duality principle, we can obtain a proof of $\mathfrak{S}'$ by dualizing that of $\mathfrak{S}$.

Dualizing the definition of generated pair as given in §1, we say that a pair (H, δ), consisting of an A,S-module H and an S-homomorphism $\delta: H \rightarrow M$, is *permeated* by the S-bimodule M of the only A,S-quotient module of H containing $H/K(\delta)$ is H itself. The reader will readily obtain a characterization of the produced pair in terms of this notion by dualizing Theorem 1.

When π is onto, so that M can be identified with the S-quotient module $P(M)/K(\pi)$ of $P(M)$, we say that $P(M)$ is *produced* by M, thus dualizing the notion of induced module. According to (P), π is onto if and only if there exists an A,S-module of which M is an S-quotient module.

3. Iteration and direct sums. When M is given as a right S-module, it can be turned into an S-bimodule by letting S operate trivially on the left; $su = 0$ for all s in S, u in M. Then it is easily seen that S operates trivially on the left of the A,S-modules $I(M)$ and $P(M)$.

Suppose that T is a subring of S, then the S-ring A is also a T-ring. Let M be a right T-module and consider the induced pairs $(I_{S,T}(M), \kappa_T)$ and $(I_{A,S}[I_{S,T}(M)], \kappa_s)$. Then it is seen by verifying (I) that $(I_{A,S}[I_{S,T}(M)], \kappa_T\kappa_S)$ is induced by M. We have the formula

$$I_{A,S} I_{S,T} = I_{A,T} \tag{3}$$

and dually

$$P_{A,S} P_{S,T} = P_{A,T}. \tag{3'}$$

Similarly it is proved that if an S-bimodule M is a direct sum $M = M_1 \oplus M_2$ of S-submodules, then we may choose

$$I(M_1 \oplus M_2) = I(M_1) \oplus I(M_2) \tag{4}$$

$$P(M_1 \oplus M_2) = P(M_1) \oplus P(M_2). \tag{4'}$$

4. A,S-injective and A,S-projective modules. Let H be an A,S-module. We shall call H an *A,S-protract* of an A,S-module H' if there exists commutative diagram

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$$\begin{tikzcd} H' \arrow[dr, "\sigma"] & \\ H \arrow[u, "\epsilon"] \arrow[r, "i"'] & H \end{tikzcd}$$

where $\sigma: H' \to H$ is an A,S-homomorphism, $\epsilon: H \to H'$ is an S-homomorphism, and i is the identity automorphism of H. This implies that σ is onto, $H \simeq_A H'/K(\sigma)$, ϵ is an isomorphism, and, as an S-bimodule, $H' = K(\sigma) \oplus H\epsilon$. We shall also refer to H as an A,S-protract of (H', σ) or of (H', σ, ϵ).

An A,S-module K will be called *A,S-projective* if, whenever α is an A,S-homomorphism of K into an A,S-module H, and H is an A,S-protract of (H', σ), then there exists an A,S-homomorphism $\bar{\alpha}: K \to H'$ such that $\alpha = \bar{\alpha}\sigma$:

$$\begin{tikzcd} & H' \arrow[d, dash, "\sigma"] \\ K \arrow[ur, "\bar{\alpha}"] \arrow[r, dash, "\alpha"'] & H \end{tikzcd}$$

The interest of this definition for our present purposes arises from the following fact:

THEOREM 2. *For any S-bimodule M, the A,S-module $I(M)$ is A,S-projective.*

Proof. Let H be an A,S-protract of (H', σ, ϵ) and let $\alpha: I(M) \to H$ be an A,S-homomorphism. Now $\phi = \kappa\alpha\epsilon: M \to H'$ is an S-homomorphism, hence by (I) there exists an A,S-homomorphism $\bar{\alpha}: I(M) \to H'$ such that $\phi = \kappa\bar{\alpha}$. Then $\kappa\bar{\alpha}\sigma = \phi\sigma = \kappa\alpha\epsilon\sigma = \kappa\alpha$. But $\bar{\alpha}\sigma: I(M) \to H'$ is an A,S-homomorphism, hence by the uniqueness part of (I), $\alpha = \bar{\alpha}\sigma$, proving that $I(M)$ is A,S-projective.

From the notion of A,S-protract we obtain by dualization that of *A,S-retract*; that an A,S-module H is an A,S-retract of an A,S-module H' means the existence of a commutative diagram

$$\begin{tikzcd} H' \arrow[d, "\epsilon"'] & \\ H & H \arrow[ul, "\sigma"'] \arrow[l, "i"] \end{tikzcd}$$

where $\sigma: H \to H'$ is an A,S-homomorphism, $\epsilon: H' \to H$ is an S-homomorphism, and as before, i is the identity automorphism of H. Then σ is an isomorphism, ϵ is onto, and, as an S-bimodule, $H' = H\sigma \oplus K(\epsilon)$. We shall also refer to H as an A,S-retract of (H', σ), or of (H', σ, ϵ).

Dually to A,S-projective modules we have *A,S-injective* modules, an A,S-module K being called A,S-injective if, whenever H is an A,S-retract of

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(H', σ), and $\alpha: H \rightarrow K$ is an A,S-homomorphism, there exists an A,S-homomorphism $\bar{\alpha}: H' \rightarrow K$ such that $\alpha = \sigma\bar{\alpha}$. Dual to the fact that $I(M)$ is A,S-projective we have

THEOREM 2$'$. *For any S-bimodule M, $P(M)$ is A,S-injective.*

5. The case M an A,S-module. Now we consider the case in which M is given as an A,S-module to begin with. Then we can consider M as an S-bimodule in order to form the induced pair $(I(M), \kappa)$ and the produced pair $(P(M), \pi)$.

Let i be the identity automorphism of M. By (I) there exists one and only one A,S-homomorphism $t: I(M) \rightarrow M$ such that $i = \kappa t$. It follows that κt must be an isomorphism and t must be onto. We have

THEOREM 3. *If M is an A,S-module, then $I_{A,S}(M)$ is induced by M (considered as an S-bimodule), and there exists one and only one A,S-homomorphism of $I(M)$ onto M such that M is an A,S-protract of $(I(M), t, \kappa)$.*

Dually

THEOREM 3$'$. *$P_{A,S}(M)$ is produced by M, and there exists one and only one A,S-isomorphism j: $M \rightarrow P(M)$ such that M is an A,S-retract of $(P(M), j, \pi)$.*

$$\begin{tikzcd} I(M) \arrow[dr, dash, "t"] & \\ M \arrow[u, "\kappa"] \arrow[r, dash, "i"'] & M \end{tikzcd} \qquad \begin{tikzcd} P(M) \arrow[d, "\pi"'] & \\ M & M \arrow[l, "i"] \arrow[ul, "j"'] \end{tikzcd}$$

Thus an A,S-module M is always an A,S-protract of $I(M)$ and an A,S-retract of $P(M)$. When is it isomorphic with a direct summand of one of these A,S-modules? We shall prove[1]

THEOREM 4. *For an A,S-module M, the following conditions imply each other:*

(a) *$K(t)$ is a direct summand of $I_{A,S}(M)$, so that $I_{A,S}(M) \simeq {}_AK(t) \oplus M$.*

(b) *M is (A,S-isomorphic with) a direct summand of $I_{A,S}(M)$.*

(c) *M is A,S-projective.*

(d) *if M is an A,S-protract of (H, σ), then $K(\sigma)$ is a direct summand of H, so that $H \simeq K(\sigma) \oplus M$.*

(e) *M is A,S-isomorphic with a direct summand of every A,S-protract of M.*

Proof. (a) implies (b) at once.

(b) implies (c): By Theorem 2, $I(M)$, and hence every direct summand thereof, is A,S-projective. Hence (b) implies (c).

[1]The reader will readily see the relation of our result to results of Baer **(1)** and Eckmann and Schopf **(4)**.

(c) implies (d): Suppose M is an A,S-retract of (H, σ). Let i be the identity automorphism of M. Then (c) implies the existence of an A,S-homomorphism $\bar{\imath}: M \to H$ such that $\bar{\imath}\sigma = i$. Hence $\bar{\imath}$ is an isomorphism and $H = K(\sigma) \oplus M\bar{\imath}$, proving that (c) implies (d).

(d) implies (e) at once.

(e) implies (a): By Theorem 3, M is an A,S-protract of $I(M)$. Hence (e) implies that M is a direct summand of $I(M)$, so that M is A,S-projective by Theorem 2. By Theorem 3, M is an A,S-protract of $(I(M), t)$, hence $K(t)$ is a direct summand of $I(M)$ by the implication of (c) by (d) already proved. Thus (e) implies (a), completing the proof of Theorem 4.

The dual of Theorem 4 is

THEOREM 4′. *For an A,S-module M the following are mutually equivalent conditions:*

(a′) *Mj is a direct summand of $P_{A,S}(M)$, so that $P_{A,S}(M) \simeq {}_AP(M)/Mj \simeq M$.*

(b′) *M is (A,S-isomorphic with) a direct summand of $P_{A,S}(M)$.*

(c′) *M is A,S-injective.*

(d′) *if M is an A,S-retract of (H, σ), then $M\sigma$ is a direct summand of H, so that $H \simeq {}_AH/M\sigma \oplus M$.*

(e′) *M is (A,S-isomorphic with) a direct summand of every A,S-retract of M.*

COROLLARY. *Let H be an A,S-projective (injective), indecomposable A,S-module such that the S-submodules of H and the A,S-submodules of $I_{A,S}(H)$ ($P_{A,S}(H)$) satisfy the descending chain condition. Then there exists an indecomposable S-bimodule M such that $I_{A,S}(M)$ is induced by $M(P_{A,S}(M)$ is produced by $M)$ and contains a direct summand A,S-isomorphic with H.*

Proof. As an Bimodule, H has a Remak decomposition $H = M_1 \oplus \ldots \oplus M_k$, M_i an indecomposable S-bimodule. By (4), $I(H) \simeq {}_AI(M_1) \oplus \ldots \oplus I(M_k)$, and this decomposition can be refined to a Remak decomposition. If H is A,S-projective, we infer from Theorem 4 that H is A,S-isomorphic with a direct summand of $I(H)$. Hence, since H is indecomposable, it is A,S-isomorphic with a direct summand of $I(M_i)$ for some i by the Remak theorem. $I(M_i)$ is induced by M since $I(H)$ is induced by H. Thus the corollary is proved with $M = M_i$.

6. The Casimir operators. Now assume the existence of an A,S-homomorphism

$$\lambda: P_{A,S}(M) \to I_{A,S}(M)$$

for some fixed A,S-module M. Note that this assumption is self dual.

If N is a second A,S-module, and $\alpha: M \to N$ is an S-homomorphism, there exists by (I) one and only one A,S-homomorphism $\bar{\alpha}: I(M) \to N$ such that $\alpha = \kappa\bar{\alpha}$. If $j: M \to P(M)$ is the A,S-isomorphism of Theorem 3′, then

(5) $\hat{\alpha} = j\lambda\bar{\alpha}: M \to N$ is an A,S-homomorphism.

$$\begin{tikzcd} P(M) \arrow[r, "\lambda"] & (I)M \arrow[d, "\bar{\alpha}"] \\ M \arrow[u, "j"] \arrow[ur, "\kappa"] \arrow[r, "\alpha"'] & N \end{tikzcd}$$

Dually, if $\beta: N \to M$ is an A,S-homomorphism,

(5′) $\tilde{\beta} = \bar{\beta}\lambda t: N \to M$

is an A,S-homomorphism, where $\bar{\beta}: N \to P(M)$ is the one and only A,S-homomorphism such that $\beta = \bar{\beta}\pi$, whose existence is guaranteed by (P), and $t: I(M) \to P(M)$ is the homomorphism onto of Theorem 3.

The mappings $\hat{\alpha}$ and $\tilde{\beta}$ are generalizations of the Casimir operators which occur in the theory of separable algebras as will be seen in §8. If α and β are A,S-endomorphisms of an A,S-module M, then $\hat{\alpha}$ and $\tilde{\beta}$ are A,S-endomorphisms of M. We have the following self-dual theorem:

THEOREM 5. *Let i be the identity automorphism of M. Then each of the conditions*

(1) *There exists an S-endomorphism α of M such that $\hat{\alpha} = i$.*

(2) *There exists an S-endomorphism β of M such that $\tilde{\beta} = i$.*

Implies all the conditions (a) *through* (e) *and* (a′) *through* (e′) *of Theorems* 4, 4′.

If λ is an isomorphism onto, the conditions (a) *through* (e) *and* (a′) *through* (e′) *are mutually equivalent and imply* (1) *and* (2).

Proof. If $\hat{\alpha} = j\lambda\bar{\alpha} = i$ then $\lambda\bar{\alpha}: P(M) \to M$ is an A,S-homomorphism onto and $P(M) = Mj \oplus K(\lambda\bar{\alpha})$, proving (a′) of Theorem 4′. Furthermore, $j\lambda: M \to I(M)$ is an A,S-isomorphism, $\bar{\alpha}: I(M) \to M$ is an A,S-homomorphism onto, and $I(M) = Mj\lambda \oplus K(\bar{\alpha})$, proving (b) of Theorem 4. The remaining conditions of Theorems 4 and 4′ follow from those theorems.

Next assume that λ is an isomorphism onto. That the conditions (a) through (e′) are mutually equivalent follows at once from this assumption and Theorems 4, 4′. If we now assume (a′) of Theorem 4′, i.e., that Mj is a direct summand of $P(M)$, then there exists an A,S-homomorphism $\bar{\imath}: P(M) \to M$ such that $j\bar{\imath} = i$. Then $\alpha = \kappa\lambda^{-1}\bar{\imath}$ is an S-endomorphism of M, and $\bar{\alpha} = \lambda^{-1}i$. Hence $\hat{\alpha} = j\lambda\bar{\alpha} = j\bar{\imath} = i$, proving (1).

The rest of Theorem 5 is dual to the part already proved.

7. Construction of induced and produced pairs. For the construction of the induced and produced pairs we need the following definitions, also used in later sections.

Let M be an S-bimodule. By $M \otimes_S A$ we denote the tensor product of M and A over S, that is, the module generated by all pairs $u \otimes a$, with u in M, a in A, with the relations

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$$us \otimes a = u \otimes sa \qquad (s \in S)$$

$$(u_1 + u_2) \otimes a = u_1 \otimes a + u_2 \otimes a, \quad u \otimes (a_1 + a_2) = u \otimes a_1 + u \otimes a_2$$

into an A,S-module by

$$s(u \otimes a) = su \otimes a, \quad (u \otimes a)s = u \otimes as,$$
$$(u \otimes a)b = u \otimes ab \qquad (s \in S, b \in A).$$

By $S\text{-Hom}_r(A, M)$ we shall denote the module of all module homomorphisms $f: A \to M$ such that

$$as \cdot f = af \cdot s \qquad (s \in S)$$

turned into an A,S-module by

$$a(s^*f) = s(af), \quad a(f^*s) = (sa)f,$$
$$a(f^*b) = (ba)f \qquad (s \in S, b \in A).$$

For the construction of the induced pair let us assume first that A is a ring with identity element e, S a subring of A containing e, and let the S-bimodule M be *right unitary*, i.e., such that $ue = u$ for all u in M. In this case we can choose

$$I_{A,S}(M) = M \otimes {}_S A, \quad \kappa: u \to u \otimes e \qquad (u \in M)$$

for the induced pair. For we can check that if H is any A,S-module and $\delta: M \to H$ is an S-homomorphism, then

$$\bar{\delta}: u \otimes a \to u\delta \cdot a \tag{6}$$

determines the one and only one A,S-homomorphism $\bar{\delta}: I(M) \to H$ such that $\delta = \kappa\bar{\delta}$.

Furthermore, in this case we can choose

$$P_{A,S}(M) = S\text{-Hom}_r(A, M), \quad \pi: f \to ef \quad (f \in S\text{-Hom}_r(A, M)$$

as a realization of the produced pair. If $\delta: H \to M$ is an A,S-homomorphism,

$$a \cdot h\bar{\delta} = ha \cdot \delta \qquad (h \in H) \tag{6'}$$

defines the one and only A,S-homomorphism $\bar{\delta}: H \to S\text{-Hom}_r(A, M)$ such that $\delta = \bar{\delta}\kappa$.

In particular, if M is given as an A,S-module, then the A,S-homomorphism onto $t: I(M) \to M$ of Theorem 3 is given by

$$t: u \otimes a \to ua \tag{7}$$

while the A,S-isomorphism into $j: M \to P(M)$ of Theorem 3′ is given by

$$a \cdot uj = ua. \tag{7'}$$

The general case of an S-ring A and an S-bimodule M is easily reduced to the one just considered.

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II. Self-Dual S-Rings

8. S-bilinear forms. For an S-ring A, $A' = S\text{-Hom}_r(A, S)$ as defined in §7 is an A,S-module. Let $\phi: A \to A'$ be an A,S-homomorphism, (considering A as an A,S-module). Then

$$(a, b) = b \cdot a\phi \qquad (a, b \in A) \tag{1}$$

defines a mapping (a, b) of the set of pairs of elements of A into S with the properties

$$\begin{aligned}&(a_1 + a_2, b) = (a_1, b) + (a_2, b), (a, b_1 + b_2) = (a, b_1) + (a, b_2)\\ &(as, b) = (a, sb), s(a, b) = (sa, b), (a, b)s = (a, bs) \qquad (s \in S)\end{aligned} \tag{2}$$

and

$$(ab, c) = (a, bc) \qquad (b \in A). \tag{3}$$

Such a mapping satisfying conditions (2) is called an *S-bilinear form* on A. If in addition it satisfies (3) we call it *associative.* A 1.1 correspondence between the S-bilinear forms (a, b) on A and the A,S-homomorphisms $\phi: A \to A'$ is established by (1).

Interchanging left and right in the definition of A', we define $A'' = S\text{-Hom}_l(A, S)$ as the module of all module homomorphisms $f: M \to S$ such that $sa \cdot f = s \cdot af$ $(s \in S, a \in A)$, turned into a left A,S-module by

$$a(f^*s) = (af)s, a(s^*f) = (sa)f, a(b^*f) = (ba)f \qquad (s \in S, a, b \in A).$$

By the symmetry of the conditions (2) and (3) we have that

$$(a, b) = a \cdot b\phi' \tag{4}$$

defines an A,S-homomorphism $\phi': A \to A''$.

Note that ϕ is an isomorphism if and only if

$$(a, A) = 0 \text{ implies } a = 0 \tag{5}$$

while ϕ' is an isomorphism if and only if

$$(A, a) = 0 \text{ implies } a = 0. \tag{6}$$

An S-bilinear form satisfying (5) and (6) is called *non-singular.*

We shall call an S-bilinear form (a, b) on A *invariant* if the A,S-homomorphisms $\phi: A \to A'$ and $\phi': A \to A''$ defined by (1) and (4) respectively are both isomorphisms onto.

An S-ring A will be called *self-dual* (with respect to S) if it admits an invariant S-bilinear form. Note that if A is an algebra over a field S, this condition implies that the dimension of A over S is finite and hence is much stronger than the assumption of the existence of a non-singular form.

Let $[a, b]$ be a second invariant S-bilinear form on A determining A,S-isomorphisms $\psi: A \to A'$, $\psi': A \to A''$ according to (1) and (4). Then

$$[a, b] = (a\sigma, b) = (a, b\sigma') \tag{7}$$

where $\sigma = \psi\phi^{-1}$ and $\sigma' = \psi'\phi'^{-1}$ are A,S-automorphisms of A considered as a right and a left A,S-module respectively.

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9. Casimir operators. By a *left S-basis* of an S-ring A we shall mean a finite set $a_1, \ldots, a_n$ of elements of A such that each a in A can be written uniquely in the form

$$a = \sum_{j=1}^{n} s_j a_j \qquad (s \in S)$$

i.e., $a_1, \ldots, a_n$ is a finite free basis of A as a left S-module. A *right basis* is defined similarly.

Throughout §9 *we shall assume that A is a ring with identity element e, S a subring of A containing e.* If then M is an S-bimodule such that $ue = u$ for all u in M (called right unitary), we can, according to §7, take $I_{A,S}(M) = M \otimes_S A$ and $P_{A,S}(M) = S\text{-Hom}_r(A, M)$. In particular, $A = I(S)$ and $A' = P(S)$.

If A has both a left and a right S-basis, and if there exists an associative S-bilinear form (a, b) on A such that the A,S-homomorphism $\phi: A \to A'$ defined by (1) is an A,S-isomorphism onto, we see that to each left S-basis $a_1, \ldots, a_n$ of A there corresponds a right S-basis $\bar{a}_1, \ldots, \bar{a}_n$ of A determined by

$$(a_i, \bar{a}_j) = \delta_{ij} \tag{8}$$

where $\delta_{ij} = 0$ if $i \neq j$, $\delta_{ii} = e$. Consequently (a, b) is invariant and A is self-dual with respect to S.

Given an invariant S-bilinear form (a, b) on A, a left S-basis $a_1, \ldots, a_n$ and a right S-basis $\bar{a}_1, \ldots, a_n$ satisfying (8) will be called *dual.* By (8) and the associativity of (a, b) we have

$$a_i a = \sum_{j=1}^{n} a_j a_{ji}, \ (a \in A, a_{ji} \in S) \text{ implies } a\bar{a}_i = \sum_{j=1}^{n} a_{ij}\bar{a}_j. \tag{9}$$

One proves easily

LEMMA 2. *If $a_1, \ldots, a_n$ is a left S-basis, $\bar{a}_1, \ldots, \bar{a}_n$ a set of elements of A satisfying* (9), *then*

$$\lambda: f \to \sum_{i=1}^{n} \bar{a}_i f \otimes a_i \tag{10}$$

defines an A,S-homomorphism λ: $P(M) \to I(M)$ for any right unitary S-bimodule M. If $\bar{a}_1, \ldots, \bar{a}_n$ is a right S-basis of A, then λ is an isomorphism onto, and A is self-dual.

Combining this with the above remarks we obtain

PROPOSITION 1. *Let A be a ring with identity element e, S a subring of A containing e. If A has both a left and a right S-basis, each of the following conditions is necessary and sufficient for A to be self-dual:*

(a) $A \simeq {}_A A' = S\text{-Hom}_r(A, S)$.

(b) $I_{A,S}(M) \simeq {}_A P_{A,S}(M)$ *for every right unitary S-bimodule M.*

(c) *to each left S-basis $a_1, \ldots, a_n$ of A there corresponds a right S-basis $\bar{a}_1, \ldots, \bar{a}_n$ satisfying* (9).

Now we compute the Casimir operators of §6 which are determined by the mapping λ defined by (10). Accordingly, we assume that A is an S-ring having a left S-basis $a_1, \ldots, a_n$ and a set of elements $\bar{a}_1, \ldots, \bar{a}_n$ satisfying (9). Let M and N be right unitary A,S-modules, and let $\alpha: M \to N$ be an S-homomorphism, $\hat{\alpha}: M \to N$ the corresponding A,S-homomorphism determined by $\lambda: P(M) \to I(M)$ according to (5) of Part I.

By (5), (6′), and (7) of Part I, and (10) we have for u in M

$$u\hat{\alpha} = uj\lambda\tilde{\alpha} = (\sum \bar{a}_i \cdot uj \otimes a_i)\bar{\alpha} = (\sum u\bar{a}_i \otimes a_i)\bar{\alpha} = \sum u\bar{a}_i\,\alpha\, a_i$$

i.e.,

$$\tilde{\alpha} = \sum_{i=1}^{n} \hat{\alpha}_i\,\alpha\, a_i. \tag{11}$$

Similarly, if $\beta: N \to M$ is an S-homomorphism and $\tilde{\beta}: N \to M$ is the corresponding A,S-homomorphism given by (5′) of Part I, we have

$$\tilde{\beta} = \sum \bar{a}_i\,\beta\, a_i = \hat{\beta}. \tag{12}$$

These formulas (11) and (12) justify our reference to $\hat{\alpha}$ and $\tilde{\beta}$ as generalized Casimir operators. Let A be a separable algebra over a field S, and let $a_1, \ldots, a_n$ be a basis of A, $\bar{a}_1, \ldots, \bar{a}_n$ a dual basis of A with respect to a discriminant matrix. If α is a linear transformation of a representation module for A over S, $\hat{\alpha}$ is the Casimir operator of classical theory **(8)**.

By (11) and Theorem 5 we have

THEOREM 6. *Let A be a ring with identity element e, S a subring of A containing e. Suppose that there exists a left S-basis $a_1, \ldots, a_n$ of A and a set of elements $\bar{a}_1, \ldots, \bar{a}_n$ of A satisfying* (9). *Let M be a right unitary A,S-module. Then the condition*

(*) *There exists an S-homomorphism α of M such that*

$$\sum_{i=1}^{n} \bar{a}_i\,\alpha\, a_i = i$$

(*the identity automorphism of M*)*is sufficient for all the conditions* (a) *through* (e′) *of Theorems* 4, 4′. *If $\bar{a}_1, \ldots, \bar{a}_n$ is a right S-basis of A, each of these conditions is equivalent to* (*).

This theorem contains results of Eckmann, Gaschütz and the author (cf. §13).

10. The case of commutative S. *Throughout* §10 *we shall assume that S is a commutative ring, and that A is a self-dual S-ring such that $sa = as$ for all $a \in A$, $s \in S$.* In this case, considered as S-bimodules, $\mathrm{Hom}_r(A,S) = \mathrm{Hom}_l(A,S)$. The assumption that A is self-dual with respect to S means that there exists an invarient S-bilinear form (a, b) on A, determining A,S-isomorphisms ϕ of A onto $\mathrm{Hom}_r(A,S)$ and ϕ' of A onto $\mathrm{Hom}_l(A,S)$. Since ϕ and ϕ' both have the same range, there exists for each a in A elements $a\dagger$ and a^* in A such that $a^*\phi = a\phi'$ and $a\phi = a\dagger\phi'$, i.e., such that

$$(a^*, b) = (b, a) \text{ and } (a, b) = (b, a\dagger). \tag{13}$$

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Since (a, b) is invariant,

$$\phi^*: a \to a^* \text{ and } \phi\dagger: a \to a\dagger \tag{14}$$

are reciprocal S-automorphisms of the S-bimodule A. Furthermore,

$$((ab)^*, c) = (c, ab) = (ca, b) = (b^*c, a) = (a^*, b^*c) = (a^*b^*, c),$$

which implies that $(ab)^* = a^* b^*$. This proves that ϕ^* is an automorphism off the S-ring A, the inverse of ϕ^* being $\phi\dagger$.

Let $[a, b]$ be a second invariant S-bilinear form on A, and ψ the corresponding A,S-isomorphism of A onto $\mathrm{Hom}_r(A,S)$. The relation between ϕ^* and ψ^* is readily seen from (7). Let us consider in particular the case in which A has an identity element, then the A,S-automorphisms of A considered as an A,S-module are the left multiplications by units of A. In particular $\sigma = \psi\phi^{-1}$ is effected by left multiplication by a unit, say c_0, of A, and hence by (7)

$$[a, b] = (c_0 a, b). \tag{15}$$

Hence $[a\phi\dagger\psi^*, b] = [b, a\phi\dagger] = (c_0 b, a\phi\dagger) = (a, c_0 b) = [c_0^{-1}a, c_0 b] = [c_0^{-1}ac_0, b]$, so that $\phi\dagger\psi^*$ is the inner automorphism of A induced by c_0. It is easy to see that all the inner automorphisms of A are obtained in this way from invariant S-bilinear forms, hence

PROPOSITION 2. *If A has an identity element, the automorphisms ϕ^* of A determined by invariant S-bilinear forms on A according to* (13) *and* (14) *constitute a full residue class modulo the inner automorphisms of A.*

COROLLARY. *There exists a symmetric invariant S-bilinear form on A if and only if there exists an invariant S-bilinear form on A such that the corresponding automorphism ϕ^* is inner, and then all the automorphisms of A determined by invariant S-bilinear forms are inner.*

Proposition 3 and its Corollary were obtained by Nakayama for Frobenius algebras **(14)**.

Suppose that A possesses a right identity element e. Then $(a, b) = (a, be) = (e^*, ab) = (e^*a, b)$, so that $a = e^*a$ and e^* is a left identity. Hence $e = e^*$ is an identity element for A.

LEMMA 2. *If A possesses a one sided identity element then it possesses an identity element.*

Now we shall record some further criteria for the existence of an identity element in A, assuming the descending chain condition for left S-ideals. We shall use the following notation: For a subset X of A,

$$p(X) = \{a \in A | (X, a) = 0\}$$
$$q(X) = \{a \in A | (a, X) = 0\}$$

Then p and q induce lattice endomorphisms of the lattice of submodules of A. If L is a left ideal, the associativity of (a, b) implies that $p(L)$ is a right ideal. Also by (13), $q(L) = p(L\dagger)$ so that $q(L)$ is also a right ideal. Hence p and q induce homomorphisms of the lattice of left S-ideals into the lattice of right S-ideals. By symmetry, they also induce lattice homomorphisms in the opposite direction.

From the definitions of p and q we have $p(X) \subseteq$ the right annihilator $I_r(X)$ of X, and $q(X) \subseteq$ the left annihilator $I_l(X)$. If A has an identity we have equality.

PROPOSITION 3. *If the left S-ideals of A satisfy the descending chain condition, each of the following conditions is necessary and sufficient for the existence of an identity element in A. N denotes the radical of the S-ring A.*

(a) $I_r(L) = p(L)$ *for every nilpotent minimal left S-ideal L, and* $I_r(N) = p(N)$.

(b) $p(L) = p(AL)$ *for every nilpotent minimal left S-ideal L, and* $p(N) = p(AN)$.

(c) $L \subseteq LA$ *for every nilpotent minimal left S-ideal L.*

Proof. Suppose that A possesses an identity element. That the condition (a) is satisfied has been observed already, while (b) and (c) follow at once.

We shall complete the proof by showing that (b) implies (a), and that each of the conditions (a) and (c) implies the existence of a right identity; the existence of an identity follows from Lemma 3, since we are continuing under the assumptions made concerning A and S at the beginning of §10.

Since we have assumed the descending chain condition for left S-ideals, A is a semi-primary S-ring unless it is nilpotent. In any case there exists an element $e = e^2$ in A such that

$$A = eA \oplus R_0 = Ae \oplus L_0$$

where R_0 and L_0 are respectively a left and a right S-ideal of A contained in N such that $eR_0 = L_0e = 0$. If $L_0 \neq 0$, we infer from the descending chain condition that there exists a minimal left S-ideal L of A such that $L \subseteq L_0$. Since L is minimal, $NL = 0$, i.e., $L \subseteq I_r(N)$. Moreover, $LA = LeA + LR_0 = LR_0 \subseteq LN$, so that $LA = LN$.

Assuming (a) we have $L \subseteq I_r(N) = p(N) = q(N^*) = q(N) \subseteq I_l(N)$. Hence $LN = 0$ which implies that $LA = 0$. Then $A \subseteq I_r(L) = p(L)$, hence $L = 0$. But this contradicts the assumption that $L_0 \neq 0$, proving that e is a right identity of A.

Next assume (b). For any subset X we have the implications $XI_r(X) = 0$, $(A, XI_r(X)) = 0$, $(AX, I_r(X)) = 0$, $I_r(X) \subseteq p(AX)$. Hence if $p(X) = p(AX)$, $I_r(X) = p(X)$. Thus (*b*) implies (a).

Finally assume (c). Then we have the following implications: $NL = 0$, $(NL, A) = 0$, $(N, LA) = 0$, $L \subseteq LA \subseteq p(N) = q(N^*) = q(N) \subseteq I_l(N)$, $LN = 0$, $LA = 0$, $L = 0$. This again contradicts the assumption that $L_0 \neq 0$, proving that e is a right identity, and completing the proof of the proposition.

III. Applications

In this last part we mention some applications of the general discussion of the preceding parts, in particular relating our results to certain known results.

11. Algebras. For a first application of our general theory we consider a finite dimensional algebra A over a field F, assuming that A has an identity element e, so that F can be identified with a subring of the center of A.

A module M is called a *unitary representation module* for A if M is a (right) A-module such that $ue = u$ for all u in M, and has finite dimension over F. Then M corresponds to a representation Γ of A by matrices in F such that $\Gamma(e)$ is the identity matrix, i.e., to a *unitary representation* Γ *of* A.

We can apply the discussion of Parts I and II, reading "unitary representation module" for "A,F-module", and "finite vector module" for "F-bimodule". Note that a simplification takes place because of the fact that, if M is a subrepresentation module of a unitary representation module H of A (in which case Γ is called a top constituent of the representation Δ corresponding to H) then M is automatically an A,F-retract of H since there exists a vector submodule of H complementary to M. If M is a quotient representation module of H (in which case Γ is called a bottom constituent of Δ) then M is an A,F-protract of H.

If a unitary representation module M has dimension m over F then it decomposes as a vector module into a direct sum of m copies of F

$$M \simeq m \times F.$$

Hence, by §7 and formulas (4) and (4′) of Part I we have

$$I_{A,S}(M) \simeq {}_A m \times I_{A,S}(F) \simeq {}_A m \times (F \otimes_S A) \simeq {}_A m \times A \tag{1}$$

$$P_{A,S}(M) \simeq {}_A m \times S\text{-Hom}_r(A,S) = m \times A'. \tag{1′}$$

In (1), A is to be considered as the A-module corresponding to the first regular representation of A. In (1′), A is to be considered as the left A-module corresponding to the regular anti-representation of A, so that $A' = S\text{-Hom}_r(A,S)$ corresponds to the second regular representation of A.

M corresponds to a unitary representation Γ of degree m.

By Theorems 3, 3′ we have

Theorem 7. *A unitary representation Γ of A of degree m is a bottom constituent of $m \times$ the first regular representation of A and a top constituent of $m \times$ the second regular representation of A.*

Let us call Γ *projective* or *injective* according as M is A,F-projective or A,S-injective, in the sense of 4.

If M happens to be indecomposable, then by the Remak theorem and (1), M is isomorphic with a direct summand of $I(M)$ if and only if it is isomorphic with a direct summand of A. Hence by Theorem 4 we have

THEOREM 8. *If Γ is an indecomposable unitary representation of A in F, the following conditions imply each other:*

(a) *Γ is a principal indecomposable representation of A, i.e., Γ is equivalent to an indecomposable component of the first regular representation of A.*

(b) *Γ is projective.*

(c) *whenever Γ is a bottom constituent of a unitary representation Δ of A, Γ is equivalent to a component of Δ.*

The dual of Theorem 8 corresponding to Theorem 4′ is obtained by interchanging "first" with "second" regular representation, "projective" with "injective", and "bottom" with "top" constituent.

COROLLARY. *A unitary representation Γ is projective if and only if it is a direct sum of principal indecomposable representations, while Γ is injective if and only if it is a direct sum of indecomposable components of the second regular representation.*

An algebra A over a field F is a self-dual F-ring in the sense of Part II if and only if its two regular representations are equivalent (Proposition 1). Thus the algebras with identity elements which are self-dual with respect to the ground field are precisely the Frobenius algebras **(13)**. Lemma 3 and Proposition 3 provide necessary and sufficient conditions for an algebra A having its two regular representations equivalent to possess an identity element.

By Theorem 6 and Theorem 8 and its dual we have

THEOREM 9. *Let A be an algebra over a field F, and assume that A possesses an identity element. Let Γ be an indecomposable unitary representation of A in F of degree m. Let $a_1, \ldots, a_n$ be a basis of A, and assume that there exists a set $\bar{a}_1, \ldots, \bar{a}_n$ of elements of A satisfying* (9) *of Part II. Then the condition*

(**) *there exists a matrix X in F of degree m such that*

$$\sum_{i=1}^{n} \Gamma(\bar{a}_i)X\,\Gamma(a_i) = I_m$$

(*where I_m is the identity matrix of degree m*)

implies that Γ is a component of both the first and second regular representations of A.

If A is a Frobenius algebra, and $a_1, \ldots, a_n$ and $\bar{a}_1, \ldots, \bar{a}_n$ are dual bases of A, then (**) *characterizes the direct sums of principal indecomposable representations of A.*

12. Separable algebras. We shall also record the following characterization of separable algebras. The equivalence of (b) of our theorem with separability was proved by Hochschild **(10**, Theorem 5**)**; that condition (b) below is equivalent with Hochschild's condition (ii) is fairly obvious.

THEOREM 10. *Each of the following conditions is necessary and sufficient for an algebra A with identity element e to be separable:*

(a) *A is a Frobenius algebra, and there exists an element a in A such that $\sum \bar{a}_i a a_i = e$, where $a_1, \ldots, a_n$ and $\bar{a}_1, \ldots, \bar{a}_n$ are dual bases of A.*

(b) *to each basis $a_1, \ldots, a_n$ of A there corresponds a set $\tilde{a}_1, \ldots, \tilde{a}_n$ of elements of A satisfying condition* (9) *of Part II such that $\sum \tilde{a}_i a_i = e$.*

Proof. As is well known, a separable algebra is Frobenius, and even symmetric, for it has an identity element, and its reduced trace induces an invariant bilinear form. Hence to prove that (a) is satisfied by separable algebras A we have to show the existence of an element a in A such that $\sum \bar{a}_i a a_i = e$. By a standard device, this question is reduced to the case where A is a direct sum of full matrix rings in F,

$$A = \sum_{\alpha=1}^{k} {}_o F_{n_\alpha}.$$

Let $E^\alpha{}_{ij}$ $(\alpha = 1, \ldots, k;\ i, j = 1, \ldots, n_\alpha)$ be the matrix units. Then $(E^\alpha{}_{ij}, E^\beta{}_{kl}) = \delta_{\alpha\beta}\delta_{il}\delta_{jk}$ determines an invariant F-bilinear form on A such that the basis of A dual to the basis consisting of the $E^\alpha{}_{ij}$ is defined by

$$\overline{E^\alpha{}_{ij}} = E^\alpha{}_{ji}.$$

Let $T = \sum E^\alpha{}_{ii}$ so that $TE^\alpha{}_{ij} = 0$ if $i \neq 1$, $TE^\alpha{}_{ij} = E^\alpha{}_{ij}$. Then

$$\sum_{\alpha, i, j} \overline{E^\alpha{}_{ij}} T E^\alpha{}_{ij} = \sum_{i,\alpha} E^\alpha{}_{ii} = I$$

proving (a) in this case.

Assume (a) and write $\tilde{a}_i = \bar{a}_i a$. Then $a_1, \ldots, a_n$ and $\tilde{a}_1, \ldots, \tilde{a}_n$ satisfy condition (9) of Part II since this is the case for $a_1, \ldots, a_n$ and $\bar{a}_1, \ldots, \bar{a}_n$ according to Proposition 3. Hence (a) implies (b).

By Theorem 6, condition (b) implies that every unitary representation module of A is a direct summand of every unitary representation module of which it is a representation submodule. Hence A is semi-simple. Since condition (b) survives any extension of the ground field, this implies that A is separable. Thus (a) and (b) are each equivalent to separability.

13. Groups. Let G be a group, and denote by A the ring of all linear combinations

$$\sum_{g \in G} a_g g,$$

where a_g is a rational integer, and almost all a_g are zero. Let S be a subring of A spanned by a subgroup H of finite index. A set L of representatives of the cosets xH is a left S-basis of A and the set L^{-1} of inverses of elements of L is a dual right S-basis. Therefore A is a self-dual S-ring in the sense of Part II. The generalized Casimir operators have in this case the form

$$\sum_{x \in L} x^{-1} \alpha x$$

(cf. §9), and application of Theorem 6 yields at once Theorems 1, 1′ of a previous note of Higman **(7)**, which includes a theorem of Eckmann **(2)** and coincides with Gaschütz's generalization **(5)** of the Maschke theorem in case $S = 1$.

Assume that G is finite, and let M be a representation module for G in a field of prime characteristic p. If the subgroup H contains a p-Sylow subgroup of G, multiplication by the index $G{:}\,H$ induces an automorphism of M, and taking α to be the inverse of this automorphism we have $\sum x^{-1}\alpha x = 1$. We thus obtain Theorem 2 of **(7)**. See **(8)** for another application.

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