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Small G-varieties

Hanspeter Kraft, Andriy Regeta, and Susanna Zimmermann

Abstract. An affine variety with an action of a semisimple group G is called "small" if every nontrivial G-orbit in X is isomorphic to the orbit of a highest weight vector. Such a variety X carries a canonical action of the multiplicative group $\mathbb{K}^*$ commuting with the G-action. We show that X is determined by the $\mathbb{K}^*$-variety X^U of fixed points under a maximal unipotent subgroup $U \subset G$. Moreover, if X is smooth, then X is a G-vector bundle over the algebraic quotient $X/\!\!/G$.

If G is of type A_n ($n \geq 2$), C_n, E_6, E_7, or E_8, we show that all affine G-varieties up to a certain dimension are small. As a consequence, we have the following result. If $n \geq 5$, every smooth affine SL_n-variety of dimension $< 2n-2$ is an SL_n-vector bundle over the smooth quotient $X/\!\!/\mathrm{SL}_n$, with fiber isomorphic to the natural representation or its dual.

1 Introduction

Our base field $\mathbb{K}$ is algebraically closed of characteristic zero. If G is an algebraic group, then a *G-variety* is an affine variety X with an action of G such that the corresponding map $G \times X \to X$ is a morphism. If G is semisimple, then the closure of an orbit Gx is a union of G-orbits and contains a unique closed orbit. A very interesting special case is when the closure is the union of the orbit Gx and a fixed point $x_0 \in X$: $\overline{Gx} = Gx \cup \{x_0\}$. Such an orbit is called a *minimal orbit*. It turns out that this condition does not depend on the embedding of the orbit Gx into an affine G-variety. In fact, the minimal orbits are isomorphic to highest weight orbits O_λ in irreducible representations V_λ of G. If all orbits in X are either minimal or fixed points, then the variety X is called *small*.

The following result shows that smooth small G-varieties have a very special structure. The proof is given at the end of Section 5.4. Recall that the *algebraic quotient* $\pi\colon X \to X/\!\!/G$ is the morphism corresponding to the inclusion $\mathcal{O}(X)^G \subseteq \mathcal{O}(X)$. If G is reductive, then $\mathcal{O}(X)^G$ is finitely generated and so $X/\!\!/G$ is an affine variety. In general, $X/\!\!/G$ is just an affine scheme.

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Theorem 1.1 *Let G be a simple group, and let X be a smooth irreducible small G-variety. Then $G \simeq \mathrm{SL}_n$ or $G \simeq \mathrm{Sp}_{2n}$, and the algebraic quotient $X \to X /\!\!/ G$ is a G-vector bundle with fiber:*

- *the standard representations $\mathbb{K}^n$ or its dual $(\mathbb{K}^n)^\vee$ if $G = \mathrm{SL}_n$,*
- *the standard representation $\mathbb{K}^{2n}$ if $G = \mathrm{Sp}_{2n}$.*

In particular, every fiber is the closure of a minimal orbit.

For $G = \mathrm{SL}_n$ or $G = \mathrm{Sp}_{2n}$, it turns out that an affine G-variety is small if its dimension is small enough. More precisely, we have the following result.

Theorem 1.2

(1) *For $n \geq 5$, an irreducible affine SL_n-variety X of dimension $< 2n - 2$ is small. In particular, if X is also smooth, then X is an SL_n-vector bundle over $X /\!\!/ \mathrm{SL}_n$ with fiber $\mathbb{K}^n$ or $(\mathbb{K}^n)^\vee$.*

(2) *For $n \geq 3$, an irreducible affine Sp_{2n}-variety X of dimension $< 4n - 4$ is small. In particular, if X is also smooth, then it is an Sp_{2n}-vector bundle over $X /\!\!/ \mathrm{Sp}_{2n}$ with fiber $\mathbb{K}^{2n}$.*

In general, we have the following theorem about the structure of a small G-variety where G is a semisimple algebraic group. As usual, we fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$, and denote by $U \subset B$ the maximal unipotent subgroup and by $U^- \subset G$ the opposite one. For a simple G-module V_λ of highest weight λ, we denote by $O_\lambda \subset V_\lambda$ the orbit of highest weight vectors, and by P_λ the corresponding parabolic subgroup, i.e., the normalizer of V_λ^U.

For any minimal orbit O, there is a well-defined cyclic covering $O_\lambda \to O$ where λ is an *indivisible* dominant weight, i.e., λ is not an integral multiple of another dominant weight. This λ is called the *type* of the minimal orbit O.

In Section 2.4, we define the *canonical $\mathbb{K}^*$-action* on a minimal orbit O. For $O = O_\lambda \subset V_\lambda$ with an indivisible λ, it is the scalar multiplication.

For a reductive group H and H-varieties X and Y, we denote by $X \star^H Y$ the algebraic quotient $(X \times Y) /\!\!/ H$. There are two projections: $X \star^H Y \to X /\!\!/ H$ and $X \star^H Y \to Y /\!\!/ H$.

A similar construction is the following, called *associated bundle*. Let $H \subset G$ be a closed subgroup of an algebraic group G, and let Y be an H-variety. Consider the free action of H on $G \times Y$ defined by $h(g, y) := (gh^{-1}, hv)$. Then the orbit space $G \times^H Y := (G \times Y)/H$ has a canonical structure of an algebraic variety and the projection $G \times^H Y \to G/H$ is a bundle with fiber Y, locally trivial in the étale topology. If H is reductive and Y affine, then $G \times^H Y = G \star^H Y$.

An action of a reductive group G on an affine variety X is called *fix-pointed* if the closed orbits are fixed points.

Theorem 1.3 *Let X be an irreducible small G-variety. Then the following holds.*

(1) *The G-action is fix-pointed and in particular $X^G \xrightarrow{\sim} X /\!\!/ G$.*
(2) *All minimal orbits in X have the same type λ, called the type of X.*
(3) *The quotient $X \to X /\!\!/ U^-$ restricts to an isomorphism $X^U \xrightarrow{\sim} X /\!\!/ U^-$. In particular, X is normal if and only if X^U is normal.*

(4) *There is a unique $\mathbb{K}^*$-action on X which induces the canonical $\mathbb{K}^*$-action on each minimal orbit of X and commutes with the G-action. Its action on X^U is fix-pointed, and $X^U /\!\!/ \mathbb{K}^* \xrightarrow{\sim} X /\!\!/ G \xleftarrow{\sim} X^G$.*

(5) *The morphism $G \times X^U \to X$, $(g, x) \mapsto gx$, induces a G-equivariant isomorphism*

$$\Phi: \overline{O_\lambda} \star^{\mathbb{K}^*} X^U := (\overline{O_\lambda} \times X^U) /\!\!/ \mathbb{K}^* \xrightarrow{\simeq} X,$$

where $\mathbb{K}^*$ acts on $\overline{O_\lambda}$ by $(t, v) \mapsto t^{-1} \cdot v$ and on X^U by the action from (4).

(6) *We have* $\operatorname{Norm}_G(X^U) = P_\lambda$*, and the G-equivariant morphism*

$$\Psi: G \times^{P_\lambda} X^U \to X, \quad [g, x] \mapsto gx,$$

is proper, surjective, and birational, and induces an isomorphism between the algebras of regular functions.

The proofs are given in Proposition 4.3 for the statements (1)–(3) and in Proposition 4.4 for the statements (4)–(6).

As a consequence, we obtain the following one-to-one correspondence between irreducible small G-varieties of a given type and certain irreducible fix-pointed affine $\mathbb{K}^*$-varieties. The proof is given at the end of Section 4.2. A $\mathbb{K}^*$-action on a variety Y is called *positively fix-pointed* if for every $y \in Y$ the limit $\lim_{t \to 0} ty$ exists and is therefore a fixed point.

Corollary 1.4 *For any indivisible highest weight $\lambda \in \Lambda_G$, the functor $F: X \mapsto X^U$ defines an equivalence of categories*

$$\left\{ \begin{array}{l} \text{irreducible small } G\text{-varieties} X \\ \text{of type } \lambda \end{array} \right\} \xrightarrow{F} \left\{ \begin{array}{l} \text{irreducible positively fix-pointed} \\ \text{affine } \mathbb{K}^*\text{-varieties } Y \end{array} \right\}.$$

The inverse of F is given by $Y \mapsto \overline{O_\lambda} \star^{\mathbb{K}^*} Y$, where the $\mathbb{K}^*$-action on $\overline{O_\lambda} \times Y$ is defined as $t(v, y) \mapsto (t^{-1} \cdot v, ty)$.

Our Theorem 1.1 is a consequence of the following description of smooth small G-varieties.

Theorem 1.5 (See Theorem 4.11) *Let X be an irreducible small G-variety of type λ, and consider the following statements.*

(i) *The quotient $\pi: X \to X /\!\!/ G$ is a G-vector bundle with fiber V_λ.*
(ii) *$\mathbb{K}^*$ acts faithfully on X^U, the quotient $X^U \to X^U /\!\!/ \mathbb{K}^*$ is a line bundle, and $V_\lambda = \overline{O_\lambda}$.*
(iii) *The quotient $X^U \setminus X^G \to X^U /\!\!/ \mathbb{K}^*$ is a principal $\mathbb{K}^*$-bundle, and $V_\lambda = \overline{O_\lambda}$.*
(iv) *The closures of the minimal orbits of X are smooth and pairwise disjoint.*
(v) *The quotient morphism $\pi: X \to X /\!\!/ G$ is smooth.*

Then the assertions (i) and (ii) are equivalent and imply (iii)–(v). If X (or X^U) is normal, all assertions are equivalent.

Furthermore, X is smooth if and only if $X /\!\!/ G$ is smooth and $\pi: X \to X /\!\!/ G$ is a G-vector bundle.

In order to see that small-dimensional G-varieties are small (see Theorem 1.2), we have to compute the minimal dimension d_G of a nonminimal quasi-affine G-orbit.

G	$\dim G$	m_G	d_G	r_G	H	$\overline{O}$
A_1	3	2	2	2	T_1	$\mathbb{K}^2$
A_2	8	3	4	4	$\mathsf{A}_1 \times \mathsf{T}_1$	$\mathbb{K}^3, (\mathbb{K}^3)^\vee$
A_3	15	4	5	5	B_2	$\mathbb{K}^4, (\mathbb{K}^4)^\vee$
$\mathsf{A}_n, n > 3$	$n(n+2)$	$n+1$	$2n$	$2n$	$\mathsf{A}_{n-1} \times \mathsf{T}_1$	$\mathbb{K}^{n+1}, (\mathbb{K}^{n+1})^\vee$
B_2	10	4	4	4	$\mathsf{A}_1 \times \mathsf{A}_1$	$\mathcal{N}_{V_{\omega_1}}, V_{\omega_2} = \mathbb{K}^4$
$\mathsf{B}_n, n > 2$	$n(2n+1)$	$2n$	$2n$	$2n$	D_n	$\mathcal{N}_{V_{\omega_1}}$
$\mathsf{C}_n, n \geq 3$	$n(2n+1)$	$2n$	$4n-4$	$4n-4$	$\mathsf{C}_{n-1} \times \mathsf{A}_1$	$\mathbb{K}^{2n}$
D_4	14	7	7	7	B_6	$\mathcal{N}_{V_{\omega_1}}, \mathcal{N}_{V_{\omega_3}}, \mathcal{N}_{V_{\omega_4}}$
$\mathsf{D}_n, n \geq 5$	$n(2n-1)$	$2n-1$	$2n-1$	$2n-1$	B_{n-1}	$\mathcal{N}_{V_{\omega_1}}$
E_6	78	17	26	26	F_4	$\subsetneqq \mathcal{N}_{V_{\omega_i}}, i = 1, 6$
E_7	133	28	45	54	$\mathsf{E}_6 \times \mathsf{T}_1$	$\subsetneqq \mathcal{N}_{V_{\omega_7}}$
E_8	248	58	86	112	$\mathsf{E}_7 \times \mathsf{A}_1$	$\subsetneqq \mathcal{N}_{\mathrm{Lie}\,\mathsf{E}_8}$
F_4	52	16	16	16	B_4	$\subsetneqq \mathcal{N}_{V_{\omega_i}}, i = 1, 4$
G_2	14	6	6	6	A_2	$\mathcal{N}_{V_{\omega_1}}, \subsetneqq \mathcal{N}_{\mathrm{Lie}\,\mathsf{G}_2}$

Table 1. The invariants m_G, r_G, and d_G for the simple groups, the orbit closures realizing m_G, and the reductive subgroups $H \subsetneqq G$ realizing r_G.

In fact, if the dimension of the affine G-variety X is less than d_G, then every orbit in X is either minimal or a fixed point; hence, X is small.

We define the following invariants for a semisimple group G.

$$m_G := \min\{\dim O \mid O \text{ a minimal } G\text{-orbit}\},$$
$$d_G := \min\{\dim O \mid O \text{ a nonminimal quasi-affine nontrivial } G\text{-orbit}\},$$
$$r_G := \min\{\operatorname{codim} H \mid H \subsetneqq G \text{ reductive subgroup}\}.$$

The following theorem lists m_G, d_G, and r_G for the simply connected simple groups, and also gives the closure $\overline{O}$ of a minimal orbit realizing m_G and a reductive subgroup H of G realizing r_G. In the last column, the null cone $\mathcal{N}_V$ appears only if $\mathcal{N}_V \subsetneqq V$.

Theorem 1.6 *Let G be a simply connected simple group. Then the invariants m_G, r_G, and d_G are given by Table 1. In particular, $d_G = r_G$ except for E_7 and E_8.*

The third and last columns of Table 1 will be provided by Lemma 5.3, the fourth column by Proposition 5.8, and the fifth and sixth columns by Lemma 5.6. Note also that Theorem 1.2 is a consequence of Theorems 1.1 and 1.6 because X is a small G-variety in case $\dim X < d_G$.

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2 Minimal *G*-orbits

In this paragraph, we introduce and study *minimal orbits* of a semisimple group G. We will use the standard notation below and refer to the literature for details (see, for instance, [2, 9, 14–16, 19, 26]).

Let G be a semisimple group. We fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$, and denote by $U := B_u$ the unipotent radical of B.

2.1 Highest weight orbits

Let $\Lambda_G \subset X(T) := \mathrm{Hom}(T, \mathbb{K}^*)$ be the monoid of *dominant weights* of G. A simple G-module V is determined by its highest weight $\lambda \in \Lambda_G$, which is the weight of the one-dimensional subspace V^U, and we write $V = V_\lambda$. The dual module of a G-module W will be denoted by $W^\vee$, and for the highest weight of the dual module $V_\lambda^\vee$, we write $\lambda^\vee$.

Remark 2.1 Define $\Lambda := \bigoplus_{i=1}^r \mathbb{N}\omega_i \subseteq \Lambda_G \otimes_\mathbb{Z} \mathbb{Q}$, where $\omega_1, \dots, \omega_r$ are the *fundamental weights*. We have $\Lambda_G \subseteq \Lambda$ with equality if and only if G is simply connected. In general, we have $\Lambda_G = X(T) \cap \Lambda$.

For an affine G-variety X, we denote by $\pi\colon X \to X/\!\!/G$ the *algebraic or categorical quotient*, i.e., the morphism defined by the inclusion $\mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$. If $X = V$ is a G-module, then the closed subset

$$\mathcal{N}_V := \pi^{-1}(\pi(0)) = \{v \in V \mid \overline{Gv} \ni 0\} \subseteq V$$

is called the *null cone* or *null fiber* of V. It is a *closed cone in* V, i.e., it is closed and contains with any v the line $\mathbb{K}v \subset V$.

Let $V = V_\lambda$ be a simple G-module of highest weight $\lambda \in \Lambda_G$, $\lambda \neq 0$. Then $\dim V^U = 1$, and we define the *highest weight orbit* to be $O_\lambda := Gv \subset V$, where $v \in V^U \setminus \{0\}$ is an arbitrary highest weight vector of V. It is a *cone*, i.e., stable under scalar multiplication. These orbits and their closures have first been studied in [30].

For a subset S of a G-variety X, the *normalizer* and the *centralizer* of S are defined in the usual way: $\mathrm{Norm}_G(S) := \{g \in G \mid gS = S\}$ and $\mathrm{Cent}_G(S) := \{g \in G \mid gs = s \text{ for all } s \in S\}$. The *stabilizer* or *isotropy group* of a point $x \in X$ is denoted by G_x, and the group of G-equivariant automorphisms of X by $\mathrm{Aut}_G(X)$.

Lemma 2.2 *Let $V = V_\lambda$ be a simple G-module of highest weight $\lambda \neq 0$, and let $v \in V^U$ be a highest weight vector. Then the following holds.*

(1) $\overline{O_\lambda} = GV^U = O_\lambda \cup \{0\}$, *and* $\overline{O_\lambda}$ *is a normal variety.*
(2) *There are isomorphisms of G-modules* $\mathcal{O}(O_\lambda) = \mathcal{O}(\overline{O_\lambda}) \simeq \bigoplus_{k \geq 0} V_{k\lambda}^\vee \simeq \bigoplus_{k\geq 0} V_{k\lambda^\vee}$. *In particular, O_λ is not affine.*
(3) *We have* $O_\lambda^U = \mathbb{K}^* v$, *and so* $G_v = \mathrm{Cent}_G(O_\lambda^U)$. *Moreover,* $V^U = \mathbb{K}v = V^{G_v} = V^{G_v^\circ}$.
(4) *The group* $P_\lambda := \mathrm{Norm}_G(O_\lambda^U) = \mathrm{Norm}_G(\mathbb{K}v) \subset G$ *is a proper parabolic subgroup. We have* $P_v = \mathrm{Norm}_G G_v = \mathrm{Norm}_G(G_v^\circ)$, *and* $\dim O_\lambda = \operatorname{codim} P_\lambda + 1$.
(5) *The scalar multiplication on V induces an isomorphism* $\mathbb{K}^* \xrightarrow{\sim} \mathrm{Aut}_G(\overline{O_\lambda}) = \mathrm{Aut}_G(O_\lambda)$.
(6) *If* $w \in \mathcal{N}_V$ *and* $w \neq 0$, *then* $\overline{Gw} \supset O_\lambda$.
(7) *The closure* $\overline{O_\lambda}$ *is nonsingular if and only if* $\overline{O_\lambda} = V_\lambda$.

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Proof (1) and (2) These two statements can be found in [30, Theorems 1 and 2].

(3) We have $O_\lambda^U \subset V^U = \mathbb{K}^* v \cup \{0\}$; hence; $O_\lambda^U \subset \mathbb{K}^* v$. They are equal because O_λ is a cone. Since $G_v = G_w$ for all $w \in \mathbb{K}^* v$, we see that $G_v = \mathrm{Cent}_G(O_\lambda^U)$ and $V^{G_v} \supseteq \mathbb{K}v$. Now, the second claim follows because $U \subseteq G_v^\circ \subseteq G_v$, and so $V^{G_v} \subseteq V^{G_v^\circ} \subseteq V^U = \mathbb{K}v$.

(4) G acts on the projective space $\mathbb{P}(V)$, and the projection $p\colon V\setminus\{0\} \to \mathbb{P}(V)$ is G-equivariant and sends closed cones to closed subsets. In particular, $p(O_\lambda) = G\,p(v)$ is closed, and so $P_\lambda := G_{p(v)} = \mathrm{Norm}_G(\mathbb{K}v) = \mathrm{Norm}_G(O_\lambda^U) \subset G$ is a parabolic subgroup normalizing G_v. If $g \in G$ normalizes G_v°, then $G_{gv}^\circ = G_v^\circ$, and so $gv \in \mathbb{K}^* v = O_\lambda^U$ by (3). Hence, $\mathrm{Norm}_G(G_v) \subseteq \mathrm{Norm}_G(G_v^\circ) \subseteq \mathrm{Norm}(O_\lambda^U) = P_\lambda \subseteq \mathrm{Norm}_G(G_v)$.

(5) By (1) and (2), we have $\mathrm{Aut}_G(\overline{O_\lambda}) = \mathrm{Aut}_G(O_\lambda)$. Since $\overline{O_\lambda}$ is a cone, we have an inclusion $\mathbb{K}^* \hookrightarrow \mathrm{Aut}_G(\overline{O_\lambda})$. Any $\sigma \in \mathrm{Aut}_G(\overline{O_\lambda})$ is U-equivariant and hence preserves $\overline{O_\lambda}^U = V^U$ as well as $\{0\} \in V^U$, and the claim follows.

(6) Let $Y := \overline{Gv} \subset \mathcal{N}_V$, which implies that $0 \in Y$. Since Y is irreducible, the fixed point set Y^U does not contain isolated points (see, e.g., [20, Section III.5, Theorem 5.8.8]), and so $Y^U \neq \{0\}$. Hence, Y contains a highest weight vector, and so $Y \supset O_\lambda$.

(7) The tangent space $T_0\overline{O_\lambda}$ is a nontrivial submodule of V_λ, hence equal to V_λ. If $\overline{O_\lambda}$ is smooth, then $\dim \overline{O_\lambda} = \dim T_0\overline{O_\lambda} = \dim V_\lambda$ and so $\overline{O_\lambda} = V_\lambda$. The other implication is clear. ∎

For any $k \geq 1$, the kth symmetric power $S^k(V_\lambda)$ contains $V_{k\lambda}$ with multiplicity 1. It is the G-submodule generated by $v_0^k \in S^k(V_\lambda)$, where $v_0 \in V_\lambda$ is a highest weight vector. Let $p\colon S^k(V_\lambda) \to V_{k\lambda}$ be the linear projection. Then the map $v \mapsto p(v^k)$ is a homogeneous G-equivariant morphism $\varphi_k\colon V_\lambda \to V_{k\lambda}$ of degree k, classically called a *covariant*.

Lemma 2.3 *Let $V = V_\lambda$ be a simple G-module of highest weight λ, and let $v \in V^U$ be a highest weight vector. For $k \geq 1$, define $\mu_k := \{\zeta \in \mathbb{K}^* \mid \zeta^k = 1\} \subset \mathbb{K}^*$.*

The covariant $\varphi_k\colon V_\lambda \to V_{k\lambda}$ is a finite morphism of degree k and induces a bijective morphism $\bar{\varphi}_k\colon V_\lambda/\mu_k \to \varphi_k(V_\lambda)$, where μ_k acts by scalar multiplication on V_λ.

In particular, the induced map $\varphi_k\colon O_\lambda \to O_{k\lambda}$ is a finite G-equivariant cyclic covering of degree k, and $\varphi_k\colon \overline{O_\lambda} \to \overline{O_{k\lambda}}$ is the quotient by the action of μ_k.

Proof Since $\varphi_k^{-1}(0) = \{0\}$, the homogeneous morphism φ_k is finite, the image $\varphi_k(V_\lambda)$ is closed, and the fibers of φ_k are the μ_k-orbits. This yields the first statement. The last statement follows from the fact that $\overline{O_{k\lambda}}$ is normal, by Lemma 2.2(1). ∎

Remark 2.4 The following remarks are direct consequences of the lemma above.

(1) For $k > 1$, we have $\varphi_k(V_\lambda) \subsetneq V_{k\lambda}$ because the quotient V_λ/μ_k is always singular in the origin. In particular, $\dim V_{k\lambda} > \dim V_k$.
(2) The image under φ_k of any nontrivial orbit $O \subset V_\lambda$ is an orbit $\varphi_k(O) \subset V_{k\lambda}$, and the induced map $\varphi_k\colon O \to \varphi_k(O)$ is a cyclic covering of degree k.
(3) For $k > 1$, we have $\dim V_{k\lambda} > \dim V_\lambda \geq \dim \overline{O_\lambda} = \dim \overline{O_{k\lambda}}$, and hence $\overline{O_{k\lambda}}$ is singular in the origin, by Lemma 2.2(7).

The following lemma states that orbits of the form O_λ are minimal among G-orbits.

Lemma 2.5 *Let W be a G-module, and let $w \in W$ be a nonzero element. If $p\colon W \twoheadrightarrow V$ is the projection onto a simple factor $V \simeq V_\lambda$ of W such that $p(w) \neq 0$, then* $\dim Gw \geq \dim O_\lambda$.

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Proof If $v := p(w) \neq 0$, then $\dim Gw \geq \dim Gv > 0$. Hence, we can assume that $W = V$ is a simple G-module and p the identity map.

Given a closed subset $Y \subset V$ of a vector space, one defines the *associated cone* $\mathcal{C}Y \subset V$ to be the zero set of the functions $\operatorname{gr} f$, $f \in I(Y) \subset \mathcal{O}(V)$, where $\operatorname{gr} f$ denotes the homogeneous term of f of maximal degree. If Y is irreducible, G-stable and belongs to a fiber $\pi^{-1}(z)$ of the quotient morphism $\pi: V \to V/\!\!/G$, then $\mathcal{C}Y \subseteq \mathcal{N}_V$, and $\mathcal{C}Y$ is G-stable and equidimensional of dimension $\dim Y$ (see [3, Section 3]). Lemma 2.2(6) now implies that the highest weight orbit $O \subset V$ belongs to $\mathcal{C}Y$, and the claim follows. ∎

Example 2.6 The simple SL_2-modules are given by the binary forms $V_m := \mathbb{K}[x, y]_m$, $m \in \mathbb{N}$. The form $y^m \in V_m$ is a highest weight vector whose stabilizer is

$$U_m := \left\{ \begin{bmatrix} \zeta & s \\ & \zeta^{-1} \end{bmatrix} \mid \zeta^m = 1, s \in \mathbb{K} \right\},$$

and hence $O_m \simeq \mathrm{SL}_2 / U_m$. If $m = 2k$ is even, then $x^k y^k \in V_m$ is fixed by the diagonal torus $T \subset \mathrm{SL}_2$, and the orbit $O = \mathrm{SL}_2\, x^k y^k$ is closed and isomorphic to SL_2 / T for odd k and to SL_2 / N for even k where $N \subset \mathrm{SL}_2$ is the normalizer of T. It is easy to see that in both cases the associated cone $\mathcal{C}O$ is equal to $\overline{O_m}$.

2.2 Stabilizer of a highest weight vector and coverings

Let $O_\lambda = Gv \subset V_\lambda$ be a highest weight orbit where $v \in V_\lambda^U$. We have seen in Lemma 2.2(4) that

$$P_\lambda := \mathrm{Norm}_G(O_\lambda^U) = \mathrm{Norm}_G(\mathbb{K}v) = \mathrm{Norm}_G\, G_v \subset G$$

is a parabolic subgroup. It follows that the weight λ extends to a character of P_λ defining the action of P_λ on $\mathbb{K}v$:

$$pv' = \lambda(p) \cdot v' \text{ for } v' \in \mathbb{K}v \text{ and } p \in P_\lambda.$$

Note that $G_v = \ker \lambda$, and so $P_\lambda / G_v \xrightarrow{\sim} \mathbb{K}^*$.

A dominant weight $\lambda \in \Lambda_G$ is called *indivisible* if λ is not an integral multiple of some $\lambda' \in \Lambda_G$, $\lambda' \neq \lambda$. For an affine algebraic group H, we denote by H° its connected component.

Lemma 2.7

(1) *Let $\lambda \in \Lambda_G$ be a dominant weight of G. If $\lambda_0 \in \mathbb{Q}\lambda \cap \Lambda_G$ is an indivisible element, then $\mathbb{Q}\lambda \cap \Lambda_G = \mathbb{N}\lambda_0$.*

(2) *Let $v \in V_\lambda$ and $v_0 \in V_{\lambda_0}$ be highest weight vectors, and let $k \geq 1$ be the integer such that $\lambda = k\lambda_0$. Then:*
   (a) *$P_\lambda = P_{\lambda_0}$.*
   (b) *$G_v^\circ = G_{v_0}$ and G_v / G_v° is finite and cyclic of order k.*
   (c) *G_v is connected if and only if λ is indivisible.*
   (d) *If $\overline{O_\lambda}$ is smooth, then λ is indivisible.*

(3) *If O is an orbit and $\varphi: O \to O_\lambda$ a finite G-equivariant covering, then $O \simeq O_\mu$ where $\lambda = \ell\mu$ for an integer $\ell \geq 1$, and φ is cyclic of degree ℓ.*

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Proof (1) It is a standard fact that the intersection of a lattice with a line is a sublattice of rank 1 generated by any of the two indivisible elements.

(2a) Consider the covariant $\varphi_k\colon V_{\lambda_0} \to V_\lambda$. We have

$$\varphi_k^{-1}(\mathbb{K}v) = \varphi_k^{-1}(V_\lambda^U) = V_{\lambda_0}^U = \mathbb{K}v_0,$$

so by Lemma 2.2(4), we obtain that

$$P_\lambda = \mathrm{Norm}_G(\mathbb{K}v) = \mathrm{Norm}_G(\mathbb{K}v_0) = P_{\lambda_0}.$$

(2b) and (2c) Since $P_\lambda/G_\lambda^\circ \to P_\lambda/G_\lambda \simeq \mathbb{K}^*$ is a finite connected cover of $\mathbb{K}^*$, we have $G_\lambda^\circ = \ker(\lambda_1)$ for some character $\lambda_1\colon P_\lambda \to \mathbb{K}^*$, and $\lambda = l\lambda_1$, where $l = |G_\lambda/G_\lambda^\circ|$. Furthermore, λ_1 is a dominant weight because λ is a dominant weight. Since G_λ° has no finite index subgroup, it follows that λ_1 is indivisible, and so $\lambda_1 = \lambda_0$. This yields (2b) and also implies (2c).

(2d) follows from Lemma 2.2(7) and Remark 2.4(3).

(3) For $w \in O$ and $v = \varphi(w) \in O_\lambda$, we get a finite covering $G/G_w = O \to O_\lambda = G/G_v$, and hence $G_v^\circ \subseteq G_w \subseteq G_v$. By (2b), we have $G/G_v^\circ = O_{\lambda_0}$, where $\lambda = k\lambda_0$ for an integer $k \geq 1$, and the composition $G/G_{\lambda_0} = O_{\lambda_0} \to G/G_w = O \to G/G_v = O_\lambda$ is a cyclic covering of degree k. Therefore, $O_{\lambda_0} \to O$ and $O \to O_\lambda$ are both cyclic, of degree m and ℓ, respectively, and $k = \ell m$. Hence, $O \simeq O_{m\lambda_0}$ and $\ell(m\lambda_0) = \lambda$. ∎

2.3 Minimal orbits

In this subsection, we define the central notion of *minimal orbits* and prove some remarkable properties.

Definition 2.1 An orbit O in an affine G-variety X isomorphic to a highest weight orbit O_λ will be called *a minimal orbit*. This name is motivated by Lemma 2.5. The *type* of a minimal orbit $O \simeq O_\lambda$ is defined to be the indivisible element $\lambda_0 \in \mathbb{Q}\lambda \cap \Lambda_G \simeq \mathbb{N}\lambda_0$ from Lemma 2.7.

We denote by $\overline{O}^n$ the normalization of $\overline{O} \subset X$ and call it the *normal closure of* O. Clearly, $\overline{O}^n$ is an affine G-variety, and the normalization $\eta\colon \overline{O}^n \to \overline{O}$ is finite, birational, and G-equivariant.

Lemma 2.8 *The normalization $\eta\colon \overline{O}^n \to \overline{O}$ is bijective. In particular, $O \subset \overline{O}^n$ in a natural way and $\overline{O}^n \setminus O$ is a fixed point, as well as $\overline{O}\setminus O$. Moreover, $\mathcal{O}(\overline{O}^n) = \mathcal{O}(O)$.*

Proof Choose an isomorphism $\nu\colon O_\lambda \simeq O$. Since $\mathcal{O}(O_\lambda) = \mathcal{O}(\overline{O_\lambda})$ by Lemma 2.2(2), the morphism ν extends to a G-equivariant morphism $\bar\nu\colon \overline{O_\lambda} \to \overline{O}^n$. We claim that $\bar\nu$ is an isomorphism. Then the lemma follows from Lemma 2.2(1) and (2).

It remains to see that $\mathcal{O}(\overline{O}^n) = \mathcal{O}(O)$. By Lemma 2.2(2), we have $\mathcal{O}(O) \simeq \bigoplus_{k\geq 0} V_{k\lambda^\vee}$, and so the G-stable subalgebra $\mathcal{O}(\overline{O}^n) \subseteq \mathcal{O}(O)$ is a direct sum of some of the $V_{k\lambda^\vee}$. This implies that a power of every element from $V_{\lambda^\vee}$ belongs to $\mathcal{O}(\overline{O}^n)$. Hence, $V_{\lambda^\vee} \subset \mathcal{O}(\overline{O}^n)$ and the claim follows. ∎

Remark 2.9

(1) Two minimal orbits $O_1 \simeq O_{\lambda_1}$ and $O_2 \simeq O_{\lambda_2}$ are of the same type if and only if $\mathbb{Q}\lambda_1 = \mathbb{Q}\lambda_2$ (Lemma 2.7). This is the case if and only if for $v_i \in O_i$ the groups

$G^\circ_{v_1}$ and $G^\circ_{v_2}$ are conjugate (Lemma 2.7(2b)), and this implies that the parabolic subgroups $P_1 := \mathrm{Norm}_G\, G_{v_1}$ and $P_2 := \mathrm{Norm}_G\, G_{v_2}$ are conjugate.

(2) Let O be a minimal orbit of type λ_0, $O \simeq O_{k\lambda_0}$ for an integer $k \geq 1$. Then there is a finite cyclic G-equivariant covering $O_{\lambda_0} \to O$ of degree k (Lemma 2.3). Moreover, $O_{\lambda_0} \simeq G/H$, where H is connected (Lemma 2.7(2c)). In particular, if G is simply connected, then O_{λ_0} is simply connected and $O_{\lambda_0} \to O$ is the universal covering.

(3) If V is a simple G-module and $O \subset V$ a minimal orbit, then O is the highest weight orbit. In fact, O^U is nonempty; hence, O contains a highest weight vector of V.

In general, the closure of a minimal orbit needs not to be normal, as shown by the following example.

Example 2.10 Let $V_{\omega_1} = \mathbb{K}^n$ be the standard representation of SL_n. For any $k \geq 1$, the minimal orbit $O_{k\omega_1} \subset V_{k\omega_1} = S^k\mathbb{K}^n$ is the orbit of e_1^k where $e_1 = (1, 0, \dots, 0)$, and $O_{\omega_1} = \mathbb{K}^n \setminus \{0\} \to O_{k\omega_1}$ is the universal covering which is cyclic of degree k and extends to a finite morphism $\mathbb{K}^n \to \overline{O_{k\omega_1}}$, $v \mapsto v^k$.

Now, consider the SL_n-module $W := \bigoplus_{i=1}^m V_{k_i\omega_1}$, where $k_1, \dots, k_m$ are coprime and all $k_i > 1$. For $w = (e_1^{k_1}, \dots, e_1^{k_m}) \in W$, we have an SL_n-equivariant isomorphism $O_{\omega_1} \xrightarrow{\sim} O := \mathrm{SL}_n\, w$ which extends to a bijective morphism $\varphi\colon V_{\omega_1} \to \overline{O}$. However, φ is not an isomorphism because $T_0\overline{O}$ is a submodule of W, and hence cannot be isomorphic to V_{ω_1}. In particular, $\overline{O}$ is not normal. The fixed point set $\overline{O}^U$ is the cuspidal curve given by the image of the bijective morphism $\mathbb{K} \to \mathbb{K}^m$, $c \mapsto (c^{k_1}, \dots, c^{k_m})$, which shows again that $\overline{O}$ is not normal by Proposition 4.3(3).

The following result collects some important properties of minimal orbits.

Proposition 2.11 *Let X, Y be affine G-varieties, and let $O \subset X$ be a G-orbit.*

(1) *The orbit O is minimal if and only if $\overline{O} \setminus O$ is a single point (which is a fixed point of G).*

(2) *If O is minimal and $\varphi\colon O \to Y$ a nonconstant G-equivariant morphism, then $\varphi(O)$ is minimal of the same type as O, and φ extends to a finite morphism $\bar{\varphi}\colon \overline{O}^n \to \overline{\varphi(O)}$.*

(3) *Suppose that O is minimal. Let Z be a connected quasi-affine G-variety, and let $\delta\colon Z \to O$ be a finite G-equivariant covering. Then Z is a minimal orbit of the same type as O and δ is a cyclic covering.*

(4) *If $O \subset X$ is minimal, then $\overline{O} \subseteq X$ is smooth if and only if $\overline{O}$ is G-isomorphic to a simple G-module V_λ. In that case, λ is indivisible.*

For the proof, we will use the following lemma.

Lemma 2.12 *Let X, Z be affine G-varieties, and let $O \subset Z$ be a G-orbit. Assume that $\overline{O} \setminus O$ is a fixed point in Z^G, and denote by $\eta\colon Y \to \overline{O}$ the normalization.*

(1) *The morphism η induces an isomorphism $\eta^{-1}(O) \xrightarrow{\sim} O$, $Y \setminus \eta^{-1}(O)$ is a fixed point, and $\mathcal{O}(O) \xrightarrow{\sim} \mathcal{O}(\eta^{-1}(O)) = \mathcal{O}(Y)$.*

(2) *Every G-equivariant nonconstant morphism* $\varphi\colon O \to X$ *induces a finite G-equivariant morphism* $\tilde{\varphi}\colon Y \to X$

$$\begin{array}{ccc} \eta^{-1}(O) & \xrightarrow{\simeq} & O \\ \downarrow{\subseteq} & & \downarrow{\varphi} \\ Y & \xrightarrow{\tilde{\varphi}} & X \end{array}$$

and $\overline{\varphi(O)}\setminus\varphi(O)$ *is a fixed point in* X^G. *Moreover, the orbit O is a minimal orbit, as well as its image* $\varphi(O) \subset X$ *for any G-equivariant nonconstant morphism* $\varphi\colon O \to X$, *and both have the same type.*

Proof (1) Let $\overline{O} = O \cup \{x\}$ for some fixed point $x \in Z$. If $\eta\colon Y \to \overline{O}$ is the normalization, then $\eta^{-1}(O) \to O$ is an isomorphism because O is normal. Since $\eta^{-1}(x)$ is finite and G-stable, it must be a single fixed point $y \in Y$. Moreover, $Y\setminus\eta^{-1}(O) = \{y\}$ has codimension ≥ 2 in Y because a semisimple group does not have one-dimensional quasi-affine orbits. (In fact, the only simple groups having one-dimensional orbits are SL_2 and PSL_2 [7], and their orbits are projective.) It follows that $\mathcal{O}(Y) = \mathcal{O}(O)$.

(2) Since $\mathcal{O}(O) \xrightarrow{\sim} \mathcal{O}(Y)$ by (1) and X is affine, the G-equivariant morphism $\varphi\colon O \to X$ induces a G-equivariant morphism $\tilde{\varphi}\colon Y \to X$. There is a closed G-equivariant embedding of X into a G-module W, $X \hookrightarrow W$, and a linear projection $\mathrm{pr}_{V_\lambda}\colon W \to V_\lambda$ onto a simple G-module V_λ such that $\varphi(O)$ is not in the kernel of pr_{V_λ}. Set $\psi := \mathrm{pr}_{V_\lambda} \circ \tilde{\varphi}\colon Y \to V_\lambda$. Since a unipotent group U does not have isolated fixed points on an irreducible affine U-variety (see, e.g., [20, Theorem 5.8.8]), we get $O^U \neq \varnothing$, and so $\psi(O)^U \neq \varnothing$. This implies that $\psi(O) = O_\lambda$ and $\psi(Y) = \overline{O_\lambda}$. We have $\psi^{-1}(0) = \{y\}$, and so ψ is finite and surjective. In particular, O is a minimal orbit of the same type as O_λ, by Lemma 2.7(3). From the factorization

$$\psi\colon Y \xrightarrow{\tilde{\varphi}} \overline{\varphi(O)} \xrightarrow{\mathrm{pr}_V} \overline{O_\lambda},$$

we see that both maps are finite, and so $\varphi(O)$ is a minimal orbit as well, of the same type as O_λ, again by Lemma 2.7(3). ∎

Proof (of Proposition 2.11) (1) One implication follows from Lemma 2.8, and the other one from Lemma 2.12(2).

(2) This follows from (1) and Lemma 2.12(2).

(3) We can assume that $O = O_\lambda \subset V_\lambda$. Let $v_0 \in V_\lambda^U$ be a highest weight vector. Since Z is connected, it is a G-orbit, and the claim follows from Lemma 2.7(3).

(4) Any (G-equivariant) isomorphism $O \xrightarrow{\sim} O_\lambda$ extends to a (G-equivariant) isomorphism $\overline{O}^n \xrightarrow{\sim} \overline{O_\lambda}$ because $\overline{O_\lambda}$ is normal. If $\overline{O}$ is smooth, then $\overline{O}^n$ and hence $\overline{O_\lambda}$ are smooth, and so $\overline{O_\lambda} = V_\lambda$ by Lemma 2.2(7). In particular, λ is indivisible by Lemma 2.7(2d). The other implication is obvious. ∎

2.4 The canonical $\mathbb{K}^*$-action on minimal orbits

In this subsection, we show that there exists a unique $\mathbb{K}^*$-action on every minimal orbit O with the following properties.

(a) Every G-equivariant morphism $\eta: O \to O'$ between minimal orbits is also $\mathbb{K}^*$-equivariant.
(b) If $O \subset X$ is a minimal orbit in an affine G-variety X, then the $\mathbb{K}^*$-action on O extends to the closure $\overline{O}$.
(c) If $O \subset X$ is as in (b), then the limit $\lim_{t\to 0} ty$ exists for all $y \in O$ and is equal to the unique fixed point $x_0 \in \overline{O}$.
(d) If $O = O_\lambda$, where λ is indivisible, then the canonical action is the scalar multiplication.

Let $O \simeq O_\lambda$ be a minimal orbit of type λ_0, i.e., λ_0 is indivisible and $\lambda = \ell\lambda_0$ for some $\ell \in \mathbb{N}$ (see Definition 2.1). Since $\mathrm{Aut}_G(O) \simeq \mathbb{K}^*$ by Lemma 2.2(5), there are two faithful $\mathbb{K}^*$-actions on O commuting with the G-action, given by the multiplication with t and t^{-1}. Both extend to the normal closure $\overline{O}^n$, and for one of them, we have that $\lim_{t\to 0} ty$ exists for all $y \in O$ and is equal to the unique fixed point in $\overline{O}^n$. This action corresponds to the scalar multiplication in case $O = O_\lambda \subset V_\lambda$. We call it *the action by scalar multiplication* and denote it by $(t, y) \mapsto t \cdot y$.

Lemma 2.13 *Let O, O' be minimal orbits, and let $\eta: O \to O'$ be a G-equivariant morphism.*

(1) *O and O' are of the same type, and η extends to a finite G-equivariant morphism $\tilde{\eta}: \overline{O}^n \to \overline{O'}^n$.*
(2) *The G-equivariant morphisms η and $\tilde{\eta}$ are unique, up to scalar multiplication.*
(3) *For the scalar multiplication, we have $\eta(t \cdot y) = t^k \cdot \eta(y)$ for all $y \in O$, where $k := \deg \eta$.*
(4) *If $O \simeq O_\lambda$ and $O' \simeq O_{\lambda'}$, then $\lambda' = k\lambda$, where $k = \deg \eta$, and $\eta: O \to O'$ is a cyclic covering of degree k.*
(5) *The action by scalar multiplication on O_λ corresponds to the representation of $\mathbb{K}^*$ on $\mathcal{O}(O_\lambda)$, which has weight $-n$ on the isotypic component $\mathcal{O}(O_\lambda)_{n\lambda^\vee}$:*

$$tf = t^{-n} \cdot f \text{ for } t \in \mathbb{K}^*,\ f \in \mathcal{O}(O_\lambda)_{n\lambda^\vee}.$$

Proof (1) This follows from Proposition 2.11(2) and the fact that $\mathcal{O}(\overline{O}^n) = \mathcal{O}(O)$.

(2) If $\nu: O \to O'$ is another G-equivariant morphism, then, for a given $v \in O^U$, we have $\nu(v) = t_0 \cdot \eta(v)$ for a suitable $t_0 \in \mathbb{K}^*$. Since the G-action commutes with the scalar multiplication, we get $\nu(gv) = g\nu(v) = g(t_0 \cdot \eta(v)) = t_0 \cdot g\eta(v) = t_0 \cdot \eta(gv)$, and the claim follows.

(3) Choose $v \in O^U$ and set $v' := \eta(v) \in O'^U$. With respect to the scalar multiplication, we have $O^U = \mathbb{K}^* \cdot v$ and $O'^U = \mathbb{K}^* \cdot v'$. Since $\eta(O^U) = O'^U$, this implies that $\eta(t \cdot v) = t^k \cdot v' = t^k \cdot \eta(v)$ for a suitable $k \in \mathbb{Z}$. By (1), η extends to $\overline{O}$; hence, $k \geq 1$ by the definition of the scalar multiplication. Since the G-action commutes with the scalar multiplication, the formula $\eta(t \cdot v) = t^k \cdot \eta(v)$ holds for any $v \in O$, and $k = \deg \eta$.

(4) For $s \in T$, $v \in O_\lambda^U$, and $v' \in O_{\lambda'}^U$, we have $sv = \lambda(s) \cdot v$ and $sv' = \lambda'(s) \cdot v'$. By (2) and the G-equivariance of η, we get

$$\eta(sv) = \eta(\lambda(s) \cdot v) = \lambda(s)^k \cdot \eta(v) = s\eta(v),$$

where $k = \deg \eta$, and so $\lambda' = k\lambda$. The last statement follows from Proposition 2.11(3).

(5) This is clear from (3) and (4): the scalar multiplication on V_λ induces the multiplication by t^{-n} on the homogeneous component of $\mathcal{O}(V_\lambda)$ of degree n. ∎

Using this result, we can now define the *canonical* $\mathbb{K}^*$*-action* on minimal orbits.

Definition 2.2 Let $O \simeq O_\lambda$ be a minimal orbit of type λ_0, where $\lambda = \ell\lambda_0$. The *canonical* $\mathbb{K}^*$*-action on* O is defined by

$$(t, y) \mapsto t^\ell \cdot y \text{ for } t \in \mathbb{K}^* \text{ and } y \in O.$$

It follows that this $\mathbb{K}^*$-action extends to $\overline{O}^n$ such that the limits $\lim_{t\to 0} t^\ell \cdot y$ exist in $\overline{O}^n$. If λ is indivisible, then the canonical action on O_λ coincides with the scalar multiplication, but it is not faithful if λ is not indivisible.

Proposition 2.14 *Let $O \simeq O_\lambda$ be a minimal orbit of type λ_0 where $\lambda = \ell\lambda_0$.*

(1) *The canonical $\mathbb{K}^*$-action on O corresponds to the representation on $\mathcal{O}(O)$, which has weight $-n$ on the isotypic component $\mathcal{O}(O)_{n\lambda_0^\vee}$. In particular, it commutes with the G-action.*
(2) *If $\eta\colon O \to O'$ is a G-equivariant morphism of minimal orbits, then η is equivariant with respect to the canonical $\mathbb{K}^*$-action.*

Assume that O is embedded in an affine G-variety X and that $\overline{O} = O \cup \{x_0\} \subseteq X$.

(3) *The canonical $\mathbb{K}^*$-action on O extends to $\overline{O}$.*
(4) *For any $x \in O$, the limit $\lim_{t\to 0} t^\ell \cdot x$ exists in $\overline{O}$ and is equal to x_0. In particular, the canonical $\mathbb{K}^*$-action on $\overline{O}$ extends to an action of the multiplicative semigroup $(\mathbb{K}, \cdot)$.*
(5) *We have $\mathrm{Norm}_G(O^U) = \mathrm{Norm}_G(\overline{O}^U) = P_\lambda$, and the action of P_λ on $\overline{O}^U$ is given by $px = \lambda(p) \cdot x = \lambda_0(p)^\ell \cdot x$, i.e., it factors through the canonical $\mathbb{K}^*$-action.*

Proof (1) The first claim follows from Lemma 2.13(5) and obviously implies the second.

(2) This is an immediate consequence of Lemma 2.13, statements (4) and (3).

(3) Since $\mathcal{O}(O) = \mathcal{O}(\overline{O}^n)$, the claim holds if the closure $\overline{O}$ is normal. By (1), the canonical $\mathbb{K}^*$-action on $\overline{O}^n$ corresponds to the grading of the coordinate ring $\mathcal{O}(\overline{O}^n) \simeq \bigoplus_{k\geq 0} V_{k\lambda^\vee}$. In the general case, $\mathcal{O}(\overline{O})$ is a G-stable subalgebra of $\mathcal{O}(\overline{O}^n)$. Since the homogeneous components $V_{k\lambda^\vee}$ are simple and pairwise nonisomorphic G-modules, we see that $\mathcal{O}(\overline{O})$ is a graded subalgebra, hence stable under the canonical $\mathbb{K}^*$-action.

(4) This obviously holds for the scalar multiplication on $O_\lambda \subset V_\lambda$, hence in the case where $\overline{O}$ is normal. By (3), it is true in general.

(5) We have $\mathrm{Norm}_G(O_\lambda^U) = P_\lambda$ and $px = \lambda(p) \cdot x$ for $p \in P_\lambda$, $x \in O_\lambda^U$ (cf. Lemma 2.2(4)). This shows that the action of P_λ on O^U is given by the canonical $\mathbb{K}^*$-action. Since $\overline{O^U} \setminus O^U$ is the unique fixed point of $\overline{O}$ under G, we have $\mathrm{Norm}_G(\overline{O}^U) = \mathrm{Norm}_G(O^U)$. ∎

3 Isotypically graded G-algebras

Let G be a semisimple group. An affine G-variety whose nontrivial G-orbits are minimal orbits is called a *small G-variety*. We will show that the coordinate ring of a small G-variety is an *isotypically graded G-algebra*, a structure that we introduce and discuss in this paragraph.

As in the previous section, we fix a Borel subgroup $B \subset G$, a maximal torus $T \subset B$, and denote by $U := B_u$ the unipotent radical of B, which is a maximal unipotent subgroup of G.

3.1 G-algebras and isotypically graded G-algebras

Definition 3.1 A finitely generated commutative $\mathbb{K}$-algebra R with a unit $1 = 1_R$, equipped with a locally finite and rational action of G by $\mathbb{K}$-algebra automorphisms, is called a *G-algebra*.

If $\lambda_0 \in \Lambda_G$ is an indivisible dominant weight, we say that the G-algebra R is *of type* λ_0 if the highest weight of any simple G-submodule of R is a multiple of λ_0.

For any G-algebra R, we have the isotypic decomposition $R = \bigoplus_{\lambda\in\Lambda_G} R_\lambda$. If this is a grading, i.e., if $R_\lambda \cdot R_\mu \subseteq R_{\lambda+\mu}$ for all $\lambda, \mu \in \Lambda_G$, then R is called an *isotypically graded G-algebra*.

Example 3.1 Let V be a simple G-module of highest weight λ, and let $O_\lambda \subset V$ be the highest weight orbit. Assume that O_λ is of type λ_0, i.e., λ_0 is indivisible and $\lambda = k\lambda_0$ for a positive integer k. Then

$$\mathcal{O}(O_\lambda) = \mathcal{O}(\overline{O_\lambda}) = \bigoplus_{j\geq 0} V_{j\lambda^\vee} = \bigoplus_{j\geq 0} V_{jk\lambda_0^\vee}$$

by Lemma 2.2(2), and so it is an isotypically graded G-algebra of type $\lambda_0^\vee$. Note that, by Definition 2.2, this grading is induced by the canonical $\mathbb{K}^*$-action $(t, v) \mapsto t^k \cdot v$ on O_λ.

Definition 3.2 Let H be a group, and let W be an H-module. Define

$$W_H := W/\langle hw - w \mid h \in H, w \in W\rangle,$$

and denote by $\pi_H\colon W \to W_H$ the projection. Then π_H has the *universal property* that every H-invariant linear map $\varphi\colon W \to V$ factors uniquely through π_H. We call $\pi_H\colon W \to W_H$ the *universal H-projection* or simply the *H-projection*.

If another group N acts linearly on W commuting with H, then N acts linearly on W_H, and π_H is N-equivariant. Note that if W is finite-dimensional, then π_H is the dual map to the inclusion $(W^\vee)^H \hookrightarrow W^\vee$.

Example 3.2 Let V be a simple G-module of highest weight λ and consider the universal U-projection $\pi_U\colon V \to V_U$ with respect to the action of the maximal unipotent group $U \subset G$. Since T normalizes U, we see that π_U is T-equivariant and that the kernel is the direct sum of all weight spaces of weight different from the lowest weight $-\lambda^\vee$. If $U^- \subset G$ denotes the maximal unipotent subgroup opposite to U, then V^{U^-} is the lowest weight space and thus the composition $V^{U^-} \hookrightarrow V \twoheadrightarrow V_U$ is a T-equivariant isomorphism.

Lemma 3.3 *Let R be an isotypically graded G-algebra. Then the kernel of the universal U-projection $\pi_U\colon R \to R_U$ is a graded ideal, and the composition $R^{U^-} \hookrightarrow R \overset{\pi_U}{\to} R_U$ is a T-equivariant isomorphism of $\mathbb{K}$-algebras.*

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Proof For the isotypic component R_λ of R, denote by $R'_\lambda \subset R_\lambda$ the direct sum of all weight spaces of weight different from the lowest weight. Then $R_\lambda = (R_\lambda)^{U^-} \oplus R'_\lambda$. Since $R_\lambda \cdot R_\mu \subseteq R_{\lambda+\mu}$, we get $R_\lambda \cdot R'_\mu \subset R'_{\lambda+\mu}$ because the lowest weight of $R_{\lambda+\mu}$ is equal to the sum of the lowest weights of R_λ and R_μ. It follow that $\bigoplus_\mu R'_\mu = \ker \pi_U \subset R$ is an ideal and that the induced linear isomorphism $R^{U^-} \xrightarrow{\sim} R_U$ is an isomorphism of $\mathbb{K}$-algebras. ∎

Remark 3.4 Let X be an affine G-variety, and assume that $\mathcal{O}(X)$ is an isotypically graded G-algebra. Then $\mathcal{O}(X^U) = \mathcal{O}(X)_U$ and the quotient map $X \to X/\!\!/U^-$ induces an isomorphism $X^U \xrightarrow{\sim} X/\!\!/U^-$.

In fact, we have $\mathcal{O}(X^U) = \mathcal{O}(X)/\sqrt{I}$, where I is the ideal generated by the linear span $\langle gf - f \mid g \in U, f \in \mathcal{O}(X)\rangle = \ker(\mathcal{O}(X) \to \mathcal{O}(X)_U)$. Now, Lemma 3.3 implies that this kernel is an ideal, and hence $\langle gf - f \mid g \in U, f \in \mathcal{O}(X)\rangle = I$, and since $\mathcal{O}(X)/I \simeq \mathcal{O}(X)^{U^-} \subseteq \mathcal{O}(X)$, we finally get $I = \sqrt{I}$.

It follows that the restriction map $\rho: \mathcal{O}(X) \to \mathcal{O}(X^U)$ can be identified with the universal U-projection $\pi: \mathcal{O}(X) \to \mathcal{O}(X)_U$, and thus, by Lemma 3.3, the composition $\mathcal{O}(X)^{U^-} \hookrightarrow \mathcal{O}(X) \xrightarrow{\rho} \mathcal{O}(X^U)$ is an isomorphism. In particular, the quotient $X \to X/\!\!/U^-$ induces an isomorphism $X^U \xrightarrow{\sim} X/\!\!/U^-$.

Lemma 3.5 *Let $\varphi: R \to S$ be a G-equivariant linear map between G-modules. If the induced linear map $\varphi^U: R^U \to S^U$ or $\varphi_U: R_U \to S_U$ is injective or surjective, then so is φ. In particular:*

(1) *If φ_U or φ^U is an isomorphism, then so is φ.*
(2) *If $\psi: R \to S$ is another G-equivariant linear map such that $\varphi_U = \psi_U$ or $\varphi^U = \psi^U$, then $\varphi = \psi$.*

Proof Let $V \subset R$ be a simple submodule. Then either $\varphi(V) = (0)$ or $\varphi|_V : V \to \varphi(V)$ is an isomorphism. If φ^U or φ_U is injective, then we are in the second case and so φ is injective. If $W \subset S$ is a simple submodule which is not contained in the image of φ, then $W \cap \varphi(R) = (0)$ and so W^U and W_U are not in the image of R^U (resp. R_U). This proves the first part of the lemma and (1). As for (2), we simply remark that φ is zero in case φ^U (or φ_U) is zero. ∎

Now, consider the action of $G \times G$ on G by left and right multiplication, i.e.,

$$(g, h) \cdot x := gxh^{-1}.$$

With respect to this action, one has the following well-known isotypic decomposition:

$$\mathcal{O}(G) \simeq \bigoplus_{\lambda \in \Lambda_G} V_\lambda \otimes V_\lambda^\vee.$$

This means that the only simple $G \times G$-modules occurring in $\mathcal{O}(G)$ are of the form $V \otimes V^\vee$, and they occur with multiplicity 1. The embedding $V \otimes V^\vee \hookrightarrow \mathcal{O}(G)$ is obtained as follows. The G-module structure on V corresponds to a representation $\rho_V: G \to \mathrm{GL}(V) \subset \mathrm{End}(V) \simeq V^\vee \otimes V$, and the comorphism ρ_V^* induces a $G \times G$-equivariant embedding $V \otimes V^\vee \xrightarrow{\sim} \mathrm{End}(V)^\vee \hookrightarrow \mathcal{O}(G)$. (The first map is defined by $(v \otimes \sigma)(\varphi) = \sigma(\varphi(v))$ for $v \in V$, $\sigma \in V^\vee$, and $\varphi \in \mathrm{End}(V)$.)

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The action of $U \subset G$ on G by right multiplication induces a G-equivariant isomorphism $\mathcal{O}(G/U) \simeq \mathcal{O}(G)^U$ with respect to the left multiplication of G on G/U and on G, and we obtain the following isomorphisms of G-modules:

$$(*) \qquad \mathcal{O}(G/U) \simeq \mathcal{O}(G)^U \simeq \bigoplus_{\lambda \in \Lambda_G} V_\lambda \otimes (V_\lambda{}^\vee)^U \simeq \bigoplus_{\lambda \in \Lambda_G} V_\lambda,$$

giving the isotypic decomposition of $\mathcal{O}(G/U) = \mathcal{O}(G)^U$. Thus, $\mathcal{O}(G/U)$ contains every simple G-module with multiplicity 1.

Since the torus T normalizes U, there is also an action of T on $\mathcal{O}(G)^U$ induced by the action of G by right multiplication, and this T-action commutes with the G-action. Thus, we have a $G \times T$-action on $\mathcal{O}(G/U) = \mathcal{O}(G)^U$.

Remark 3.6

(1) The isomorphism (*) above is $G \times T$-equivariant where T acts on $\mathcal{O}(G/U)_\lambda \simeq V_\lambda$ by scalar multiplication with the character $\lambda^\vee$. Thus, the T-action on $\mathcal{O}(G/U)$ corresponds to the grading given by the isotypic decomposition. In particular, $\mathcal{O}(G/U)$ is an isotypically graded G-algebra.

(2) The universal U-projection $\pi_U : \mathcal{O}(G/U) \to \mathcal{O}(G/U)_U$ is equivariant with respect to the $T \times T$-action. On the one-dimensional subspace $(\mathcal{O}(G/U)_\lambda)_U \subset \mathcal{O}(G/U)_U$ the action of $(s,t) \in T \times T$ is given by multiplication with $\lambda^\vee(s)^{-1}\lambda^\vee(t)$.

Let $\varepsilon : \mathcal{O}(G/U) \to \mathbb{K}$ denote the evaluation map $f \mapsto f(eU)$. This is the comorphism of the inclusion $\iota : \{eU\} \hookrightarrow G/U$.

Lemma 3.7 *The induced linear map* $\varepsilon_\lambda : \mathcal{O}(G/U)_\lambda \to \mathbb{K}$ *is the universal U-projection* $\pi_U : \mathcal{O}(G/U)_\lambda \to (\mathcal{O}(G/U)_\lambda)_U$*, and it induces an isomorphism* $\bar{\varepsilon}_\lambda : \mathcal{O}(G/U)_\lambda^{U^-} \xrightarrow{\sim} \mathbb{K}$.

Proof We first consider the evaluation map $\tilde{\varepsilon} : \mathcal{O}(G) \to \mathbb{K}$, $f \mapsto f(e)$, which is the comorphism of the inclusion $\tilde{\iota} : \{e\} \hookrightarrow G$. We claim that on the isotypic components $V_\lambda \otimes V_\lambda^\vee$ of $\mathcal{O}(G)$, the map $\tilde{\varepsilon}$ is given by the formula $\tilde{\varepsilon}(v \otimes \sigma) = \sigma(v)$. Indeed, let $\rho_\lambda : G \to \mathrm{GL}(V_\lambda) \subset \mathrm{End}(V_\lambda)$ denote the representation on V_λ. Then the composition $\rho_\lambda \circ \tilde{\iota}$ sends e to id_{V_λ}; hence, the comorphism $\mathrm{End}(V_\lambda)^\vee \to \mathbb{K}$ is given by $\ell \mapsto \ell(\mathrm{id}_{V_\lambda})$. We have mentioned above that the isomorphism $V \otimes V^\vee \xrightarrow{\sim} \mathrm{End}(V)^\vee$ is defined by $(v \otimes \sigma)(\varphi) := \sigma(\varphi(v))$. This implies that $\tilde{\varepsilon} : V_\lambda \otimes V_\lambda^\vee \xrightarrow{\sim} \mathrm{End}(V)^\vee \to \mathbb{K}$ is given by $v \otimes \sigma \mapsto \sigma(v)$ as claimed.

For the restriction ε of $\tilde{\varepsilon}$ to $\mathcal{O}(G/U) = \mathcal{O}(G)^U$, we thus find for $v \in V_\lambda \simeq \mathcal{O}(G/U)_\lambda$ that $\varepsilon(v) = \sigma_0(v)$, where σ_0 is a highest weight vector in $V_\lambda^\vee$. As a consequence, $\varepsilon(v) \neq 0$ if v has weight $-\lambda^\vee$, i.e., if $v \in \mathcal{O}(G/H)^{U^-}$. Now, the claims follow from Example 3.2. ∎

One can use the isomorphisms $\bar{\varepsilon}_\lambda$ to define elements $f_\lambda := \bar{\varepsilon}_\lambda^{-1}(1) \in \mathcal{O}(G/U)^{U^-}$ with the following properties: $f_\lambda \cdot f_\mu = f_{\lambda+\mu}$ and $f_0 = 1$. This means that they form a multiplicative submonoid of $\mathcal{O}(G/U)^{U^-}$ isomorphic to Λ_G. In fact, there is a canonical isomorphism $\mathbb{K}[\Lambda_G] \xrightarrow{\sim} \mathcal{O}(G/U)^{U^-}$, $x_\lambda \mapsto f_\lambda$.

3.2 The structure of an isotypically graded G-algebra

It is a basic fact from highest weight theory that the structure of a G-module M is completely determined by the T-module structure of M^U. In this subsection, we show that the structure of an isotypically graded G-algebra R is completely determined by the structure of R_U or of R^{U^-} as a T-algebra.

Theorem 3.8 *Let R be a G-module. Then there are two canonical G-equivariant isomorphisms*

$$\Psi\colon (\mathcal{O}(G/U)\otimes R_U)^T \xrightarrow{\sim} R \quad \text{and} \quad \Psi'\colon (\mathcal{O}(G/U)\otimes R^{U^-})^T \xrightarrow{\sim} R,$$

where the T-action on $\mathcal{O}(G/U)$ is by right multiplication and on R_U, R^{U^-} induced by the G-action on R. If R is an isotypically graded G-algebra, then Ψ and Ψ' are isomorphisms of $\mathbb{K}$-algebras.

For the proof, we introduce an intermediate T-module A_R. If R is a G-module, then, for every simple G-module V of highest weight λ, there is a canonical G-equivariant isomorphism

$$V \otimes \operatorname{Hom}_G(V, R) \xrightarrow{\sim} R_\lambda, \text{ given by } v\otimes\rho \mapsto \rho(v).$$

In particular, we have isomorphisms $\mathcal{O}(G/U)_\lambda \otimes \operatorname{Hom}_G(\mathcal{O}(G/U)_\lambda, R) \xrightarrow{\sim} R_\lambda$ for any dominant weight λ. Recall that we have a T-action on $\mathcal{O}(G/U)$ by scalar-multiplication with the character $\lambda^\vee$ on $\mathcal{O}(G/U)_\lambda$ (see Remark 3.6(1)).

Lemma 3.9 *There is a canonical G-equivariant isomorphism*

$$\Big(\mathcal{O}(G/U)\otimes \bigoplus_{\lambda\in\Lambda_G} \operatorname{Hom}_G(\mathcal{O}(G/U)_\lambda, R)\Big)^T \xrightarrow{\sim} R.$$

Proof The action of T on $\mathcal{O}(G/U)_\mu \otimes \operatorname{Hom}_G(\mathcal{O}(G/U)_\lambda, R)$ is by scalar multiplication with the character $\mu^\vee - \lambda^\vee$; hence, $(\mathcal{O}(G/U)_\mu \otimes \operatorname{Hom}_G(\mathcal{O}(G/U)_\lambda, R))^T = 0$ unless $\mu = \lambda$. For $\mu = \lambda$, the torus T acts trivially and so $(\mathcal{O}(G/U)_\lambda \otimes \operatorname{Hom}_G(\mathcal{O}(G/U)_\lambda, R))^T \xrightarrow{\sim} R_\lambda$ as we have seen above. Thus, the left-hand side is $\bigoplus_{\lambda\in\Lambda_G} \mathcal{O}(G/U)_\lambda \otimes \operatorname{Hom}_G(\mathcal{O}(G/U)_\lambda, R)^T$, which is canonically isomorphic to $\bigoplus_{\lambda\in\Lambda_G} R_\lambda = R$. ∎

Recall that we have natural T-actions on R_U and R^{U^-} and a T-equivariant isomorphism $R^{U^-} \xrightarrow{\sim} R_U$ (Lemma 3.3).

Proposition 3.10 *Define the T-module $A_R := \bigoplus_{\lambda\in\Lambda_G} \operatorname{Hom}_G(\mathcal{O}(G/U)_\lambda, R)$ where T acts by right multiplication on $\mathcal{O}(G/U)$. Then there are canonical T-equivariant isomorphisms*

$$\varphi\colon A_R \xrightarrow{\sim} R_U \quad \text{and} \quad \psi\colon A_R \xrightarrow{\sim} R^{U^-}.$$

Proof (1) We first show that for every dominant weight λ, there is a canonical isomorphism $\varphi_\lambda\colon \operatorname{Hom}_G(\mathcal{O}(G/U)_\lambda, R) \xrightarrow{\sim} (R_\lambda)_U$. For $\rho \in \operatorname{Hom}_G(\mathcal{O}(G/U)_\lambda, R)$, consider the composition $\pi_{U,\lambda}\circ\rho\colon \mathcal{O}(G/U)_\lambda \to R_\lambda \to (R_\lambda)_U$, where $\pi_{U,\lambda}\colon R_\lambda \to (R_\lambda)_U$

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is the universal U-projection (see Remark 3.2). From the universal property of $\varepsilon_\lambda : \mathcal{O}(G/U)_\lambda \to \mathbb{K}$ (Lemma 3.7), we obtain a unique factorization

$$\begin{array}{ccc} \mathcal{O}(G/U)_\lambda & \xrightarrow{\rho} & R_\lambda \\ \downarrow{\scriptstyle \varepsilon_\lambda} & & \downarrow{\scriptstyle \pi_{U,\lambda}} \\ \mathbb{K} & \xrightarrow{\bar\rho} & (R_\lambda)_U . \end{array}$$

It is easy to see that the map $\varphi_\lambda : \mathrm{Hom}_G(\mathcal{O}(G/U)_\lambda, R) \to (R_\lambda)_U$ defined by $\rho \mapsto \bar\rho(1)$ has the required properties.

(2) Next, we show that for every dominant weight λ, there is a canonical isomorphism $\psi_\lambda : \mathrm{Hom}_G(\mathcal{O}(G/U)_\lambda, R) \xrightarrow{\sim} (R_\lambda)^{U^-}$. Here, we use the elements $f_\lambda := \bar\varepsilon_\lambda^{-1}(1)$ defined after Lemma 3.7, and set $\psi_\lambda(\rho) := \rho(f_\lambda)$. Now, the claim follows from (1) because $\varepsilon_\lambda(f_\lambda) = 1$ and so $\pi_{U,\lambda}(\rho(f_\lambda)) = \bar\rho(1)$, i.e., $\bar\pi_{U,\lambda} \circ \psi_\lambda = \varphi_\lambda$, where $\bar\pi_{U,\lambda} : R_\lambda^{U^-} \xrightarrow{\sim} (R_\lambda)_U$ is the T-equivariant isomorphism induced by $\pi_{U,\lambda}$ (see Lemma 3.7). ■

Proof (of Theorem 3.8) From Lemma 3.9, we get an isomorphism $(\mathcal{O}(G/U) \otimes A_R)^T \xrightarrow{\sim} R$ of G-modules. Now, the first part of the theorem follows from Proposition 3.10.

For the last claim, we have to work out the multiplication $*$ on $A = A_R$ given by the isomorphism $\psi : A_R \xrightarrow{\sim} R^{U^-}$. If $\rho \in A_\mu$ and $\sigma \in A_\lambda$, then $\rho * \sigma \in A_{\mu+\lambda}$ is uniquely defined by $(\rho * \sigma)(f_{\mu+\lambda}) = \rho(f_\mu) \cdot \sigma(f_\lambda) \in R_\mu \cdot R_\lambda \subset R_{\mu+\lambda}$. The claim follows if we show that

$$(**) \qquad (\rho * \sigma)(p \cdot q) = \rho(p) \cdot \sigma(q) \text{ for } p \in \mathcal{O}(G/U)_\mu \text{ and } q \in \mathcal{O}(G/U)_\lambda.$$

Since $\mathcal{O}(G/U)_\mu \otimes \mathcal{O}(G/U)_\lambda \xrightarrow{\rho\otimes\sigma} R_\mu \otimes R_\lambda \xrightarrow{\mathrm{mult}} R_{\mu+\lambda}$ is a G-equivariant linear map, it factors uniquely through the multiplication map $\mathcal{O}(G/U)_\mu \otimes \mathcal{O}(G/U)_\lambda \to \mathcal{O}(G/U)_{\mu+\lambda}$:

$$\begin{array}{ccc} \mathcal{O}(G/U)_\mu \otimes \mathcal{O}(G/U)_\lambda & \xrightarrow{\rho\otimes\sigma} & R_\mu \otimes R_\lambda \\ \downarrow{\scriptstyle \mathrm{mult}} & & \downarrow{\scriptstyle \mathrm{mult}} \\ \mathcal{O}(G/U)_{\mu+\lambda} & \xrightarrow{\tau} & R_{\mu+\lambda} . \end{array}$$

By construction, τ is G-equivariant and has the property that $\tau(p \cdot q) = \rho(p) \cdot \sigma(q)$ for $p \in \mathcal{O}(G/U)_\mu, q \in \mathcal{O}(G/U)_\lambda$. In particular, $\tau(f_{\mu+\lambda}) = \tau(f_\mu \cdot f_\lambda) = \rho(f_\mu) \cdot \sigma(f_\lambda) = (\rho * \sigma)(f_{\mu+\lambda})$, and hence $\tau = \rho * \sigma$ by uniqueness, and so equation (**) follows. ■

Remark 3.11 We will later need an explicit description of the isomorphism Ψ from Theorem 3.8. Let $f \in \mathcal{O}(G/U)_\lambda \setminus \ker \pi_U$ and $h \in (R_\lambda)_U$. Proposition 3.10 shows that there is a unique G-equivariant homomorphism $\rho : \mathcal{O}(G/U)_\lambda \to R_\lambda$ such that

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$\pi_\lambda(\rho(f)) = h$, and then $\Psi(f \otimes h) = \rho(f)$ by Lemma 3.9:

$$\begin{array}{ccc} \mathcal{O}(G/U)_\lambda & \xrightarrow{\rho} & R_\lambda \\ \downarrow{\scriptstyle \varepsilon_\lambda} & & \downarrow{\scriptstyle \pi_\lambda} \\ \mathbb{K} & \xrightarrow{\bar\rho} & (R_\lambda)_U . \end{array}$$

Since $\varepsilon_\lambda(f_\lambda) = 1$, we get $\bar\rho(1) = h$ and so $\pi_\lambda(\Psi(f \otimes h)) = \pi_\lambda(\rho(f)) = \bar\rho(\varepsilon_\lambda(f)) = \varepsilon_\lambda(f)h$. This shows that the diagram

$$\begin{array}{ccc} (\mathcal{O}(G/U) \otimes R_U)^T & \xrightarrow{\Psi} & R \\ \downarrow{\scriptstyle \varepsilon\otimes\mathrm{id}} & & \downarrow{\scriptstyle \pi_U} \\ R_U & = & R_U \end{array}$$

commutes.

3.3 Deformation of *G*-algebras

In 1980, the first author wrote a letter to Michel Brion [18] in connection with his theses [5], explaining him a general method how to "reconstruct" a G-variety X from its U-invariants where G is a connected reductive group and $U \subset G$ a maximal unipotent subgroup. This method allows to show that certain properties of the U-invariants also hold for X (Proposition 3.16). At that time, Brion was interested in *rational singularities*, and he gave the proofs for this special case in his thesis, attributing them to Kraft.

In 1986, Popov reproved these results in [25] and added a statement about properties inherited by the U-invariants (see Remark 3.17). Later on, similar results appeared in the literature, e.g., in [11, 29] where in both cases they were wrongly attributed to Popov.

We believe that our proofs are shorter and more transparent, and so we give them here, as an application of the methods developed above. The results are interesting in their own, but they will not be used in the remaining parts of the paper. We keep the assumption that G is semisimple, although it is not difficult to see that the results carry over to connected reductive groups.

Let R be a G-algebra with isotypic decomposition $R = \bigoplus_{\lambda\in\Lambda_G} R_\lambda$. We define an *isotypically graded G-algebra* $\operatorname{gr} R$ in the following way. As a G-module, we set $\operatorname{gr} R := R = \bigoplus_{\lambda\in\Lambda_G} R_\lambda$, and the multiplication is defined by the symmetric bilinear map

$$R_\lambda \times R_\mu \xrightarrow{\text{mult}} R \xrightarrow{\text{pr}} R_{\lambda+\mu}.$$

It is not difficult to see that this multiplication is associative, and hence defines a $\mathbb{K}$-algebra structure on $\operatorname{gr} R$ such that $\operatorname{gr} R$ becomes an isotypically graded G-algebra. We now generalize Theorem 3.8 to general G-algebras.

Proposition 3.12 *For any G-algebra R, there is a canonical G-equivariant isomorphism of $\mathbb{K}$-algebras*

$$(\mathcal{O}(G/U) \otimes R^{U^-})^T \xrightarrow{\sim} \operatorname{gr} R.$$

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Proof The definition of the multiplication on $\operatorname{gr} R$ implies that the subalgebra $(\operatorname{gr} R)^{U^-} \subset \operatorname{gr} R$ is equal to the subalgebra $R^{U^-} \subset R$ since one has $R_\mu^{U^-} \cdot R_\lambda^{U^-} \subseteq R_{\mu+\lambda}^{U^-}$. Applying Theorem 3.8 to the isotypically graded G-algebra $\operatorname{gr} R$, we get

$$(\mathcal{O}(G/U) \otimes R^{U^-})^T = (\mathcal{O}(G/U) \otimes (\operatorname{gr} R)^{U^-})^T \xrightarrow{\sim} \operatorname{gr} R,$$

hence the claim. ∎

The following *Deformation Lemma* shows that there exists *a flat deformation of* $\operatorname{gr} R$ whose general fiber is R.

Lemma 3.13 *Let R be a G-algebra. There exists a $\mathbb{K}[s]$-algebra $\tilde{R}$ with the following properties:*

(1) *$\tilde{R}$ is a free $\mathbb{K}[s]$-module and, in particular, flat over $\mathbb{K}[s]$.*
(2) *There is an isomorphism $\tilde{R}/s\tilde{R} \simeq \operatorname{gr} R$ as G-algebras.*
(3) *$\tilde{R}_s := \mathbb{K}[s, s^{-1}] \otimes_{\mathbb{K}[s]} \tilde{R} \simeq \mathbb{K}[s, s^{-1}] \otimes_{\mathbb{K}} R$.*

Proof On Λ_G, we have a partial ordering

$$\mu \leq \lambda :\Longleftrightarrow \lambda - \mu \text{ is a sum of positive roots},$$

which has the following property: if V_λ, V_μ are simple G-modules of highest weight λ and μ, then $V_\lambda \otimes V_\mu = V_{\lambda+\mu} \oplus W$ where the simple summands of W have highest weights $< \lambda + \mu$.

The cone $\mathbb{Q}_{\geq 0} \cdot \Phi_G^+ \subset \mathbb{Q} \otimes \Lambda_G$ generated by the positive roots Φ_G^+ is pointed and contains $\mathbb{Q}_{\geq 0} \cdot \Lambda_G$. Therefore, we can find a $\mathbb{Q}$-linear function $p\colon \mathbb{Q} \otimes \Lambda_G \to \mathbb{Q}$ such that the following holds:

(1) $p(\lambda) \in \mathbb{N}$ for all dominant weights $\lambda \in \Lambda_G$.
(2) $p(\alpha) \in \mathbb{N}_{>0}$ for all positive roots $\alpha \in \Phi_G^+$.

In particular, if $\mu < \lambda$, then $p(\mu) < p(\lambda)$. Setting $R_n := \bigoplus_{p(\lambda) \leq n} R_\lambda$ for $n \geq 0$, we get $R_n \cdot R_m \subseteq R_{n+m}$. It follows that the G-stable subspace

$$\tilde{R} := \bigoplus_{n \geq 0} \mathbb{K} s^n \otimes R_n \subseteq \mathbb{K}[s] \otimes R$$

is a subalgebra containing $\mathbb{K}[s]$. Since the isotypic component $\tilde{R}_\lambda$ is given by

$$\tilde{R}_\lambda = \bigoplus_{n \geq p(\lambda)} \mathbb{K} s^n \otimes R_\lambda,$$

we see that $\tilde{R}$ is a free $\mathbb{K}[s]$-module, proving (1). Moreover,

$$s(\mathbb{K} s^n \otimes R_n) = \mathbb{K} s^{n+1} \otimes R_n \subseteq \mathbb{K} s^{n+1} \otimes R_{n+1},$$

and hence $\tilde{R}/s\tilde{R} = \bigoplus_{n \geq 0} R_n/R_{n-1}$ where we set $R_{-1} = (0)$. From the canonical decomposition $R_n = (\bigoplus_{p(\lambda)=n} R_\lambda) \oplus R_{n-1}$, we see that $\tilde{R}/s\tilde{R} = R = \operatorname{gr} R$ as a G-module, and the multiplication of $R_\lambda \subset R_n/R_{n-1}$ with $R_\mu \subset R_m/R_{m-1}$ is given by the product in R followed by the projection onto R_{n+m}/R_{n+m-1}. We have $R_\lambda \cdot R_\mu = V \oplus W$, where $V \subseteq R_{\lambda+\mu}$ and all summands of W have highest weights $\rho < \lambda + \mu$. This implies that $p(\rho) < p(\lambda + \mu) = n + m$ and so $W \subseteq R_{n+m-1}$. Hence, the product of R_λ and R_μ in $\tilde{R}/s\tilde{R}$ coincides with the product in $\operatorname{gr} R$, proving (2).

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Finally, the subalgebra $\tilde{R}_s \subseteq \mathbb{K}[s, s^{-1}] \otimes R$ contains $\mathbb{K}s^\ell \otimes R_n$ for all $\ell \in \mathbb{Z}$ and all $n \in \mathbb{N}$, and hence is equal to $\mathbb{K}[s, s^{-1}] \otimes R$, proving the last claim (3). ■

Remark 3.14 Let Z be a variety. For simplicity, we assume that Z is affine. Then *a flat family* $(A_z)_{z\in Z}$ *of finitely generated* $\mathbb{K}$*-algebras* is a finitely generated and flat $\mathcal{O}(Z)$-algebra A such that $A_z = A/\mathfrak{m}_z A$ where $\mathfrak{m}_z$ is the maximal ideal of $z \in Z$.

We say that a property $\mathcal{P}$ for finitely generated $\mathbb{K}$-algebras is *open* if for any flat family $(A_z)_{z\in Z}$ of finitely generated $\mathbb{K}$-algebras the subset $\{z \in Z \mid A_z \text{ has property } \mathcal{P}\}$ is open in Z.

The Deformation Lemma 3.13 tells us that for a given G-algebra R, there is a flat family $(R_z)_{z\in\mathbb{A}^1}$ of finitely generated G-algebras over the affine line $\mathbb{A}^1$ such that $R_0 \simeq \operatorname{gr} R$ and $R_z \simeq R$ for all $z \in \mathbb{A}^1 \setminus \{0\}$. Together with Proposition 3.12, this allows to show that certain properties of the U-invariants R^U also hold for R.

Example 3.15 The following result is due to Vust [31, Section 1, Théorème 1]: *if R is a finitely generated G-algebra such that R^U is normal, then R is normal.* In fact, since $R^{U^-} \simeq R^U$ and $\mathcal{O}(G/U)$ are both normal, we see that $(\mathcal{O}(G/U) \otimes R^{U^-})^T$ is normal, and hence $\operatorname{gr} R$ is normal, by Proposition 3.12. Moreover, normality is an open property (see [12, Corollaire 12.1.7(v)]). Since $\operatorname{gr} R \simeq R_0$ is normal, the Deformation Lemma implies that R_x is normal for all x in an open neighborhood W of $0 \in \mathbb{A}^1$; hence, R is normal.

The argument from this example can be formalized in the following way.

Proposition 3.16 *Let $\mathcal{P}$ be a property for finitely generated $\mathbb{K}$-algebras which satisfies the following conditions.*

(i) *$\mathcal{P}$ is open.*
(ii) *$\mathcal{O}(G/U)$ has property $\mathcal{P}$.*
(iii) *If R and S have property $\mathcal{P}$, then so does $R \otimes S$.*
(iv) *If R is a T-algebra with property $\mathcal{P}$, then R^T has property $\mathcal{P}$.*

Then a finitely generated G-algebra R has property $\mathcal{P}$ if R^U has property $\mathcal{P}$.

Proof If R^U has property $\mathcal{P}$, then so does R^{U^-}. Hence, assumptions (ii)–(vi) imply that $(\mathcal{O}(G/U) \otimes R^{U^-})^T$ has property $\mathcal{P}$. In particular, $\operatorname{gr} R$ has property $\mathcal{P}$ by Proposition 3.12. Now, (i) implies that R has property $\mathcal{P}$ as well. ■

A very interesting property satisfying the assumption of the proposition above is that of *rational singularities* (see [4]).

Remark 3.17 Let X be a G-variety, and consider the action of $G \times U$ on $G \times X$ given by $h(g, x) := (hg, hx)$ and $u(g, x) := (gu^{-1}, x)$. Then

$$(G \times X)/\!\!/(G \times U) \simeq G/\!\!/U \star^G X \simeq X/\!\!/U.$$

In particular, $(\mathcal{O}(G/U) \otimes \mathcal{O}(X))^G \simeq \mathcal{O}(X)^U$. In fact, the isomorphism $G \times X \xrightarrow{\sim} G \times X$, $(g, x) \mapsto (g, g^{-1}x)$ is $G \times U$-equivariant for the action $u \cdot (g, x) := (gu^{-1}, ux)$ and $h \cdot (g, x) := (hg, x)$ on the right-hand side, and the claim follows.

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This formula gives the following result (cf. [25]). *Assume that a property $\mathcal{P}$ for finitely generated $\mathbb{K}$-algebra satisfies the following conditions.*

(i) $\mathcal{O}(G/U)$ *has property* $\mathcal{P}$.
(ii) *If R and S have property* $\mathcal{P}$, *then so does* $R \otimes S$.
(iii) *If R is a G-algebra with property* $\mathcal{P}$, *then* R^G *has property* $\mathcal{P}$.

Then R^U *has property* $\mathcal{P}$ *if R does.*

4 Small *G*-varieties

Recall that an affine G-variety is *small* if every nontrivial orbit is a minimal orbit. We will show that the coordinate ring of a small G-variety is an isotypically graded G-algebra and then use the results of the previous section to obtain important properties of small G-varieties and a classification.

Remark 4.1 The G-action on a small G-variety X is *fix-pointed*, which means that the closed orbits are fixed points. This has some interesting consequences. For example, it is not difficult to see that for a fix-pointed action of a reductive group on an affine variety X, the algebraic quotient $\pi: X \to X /\!\!/ G$ induces an isomorphism $X^G \xrightarrow{\sim} X /\!\!/ G$ (cf. [1, Section 10, p. 475]).

4.1 A geometric formulation

We first translate Theorem 3.8 into the geometric setting. By a result of Hadziev [13] (cf. [19, Lemma 3.2]), the U-invariants $\mathcal{O}(G)^U$ are finitely generated, and hence define an affine G-variety $G/\!\!/U$ with a G-equivariant quotient map $\eta: G \to G/\!\!/U$. Since $\mathcal{O}(G/U) = \mathcal{O}(G)^U = \mathcal{O}(G/\!\!/U)$, the canonical G-equivariant map $G/U \to G/\!\!/U$, $gU \mapsto \eta(g)$, is birational, hence an open immersion: $G/U = G\eta(e) \subset G/\!\!/U$. Moreover, the T-action on G/U by right multiplication extends to a T-action on $G/\!\!/U$ commuting with the G-action.

For an affine G-variety X, we have a canonical G-equivariant morphism

$$G/U \times X^U \to X, \quad (gU, x) \mapsto gx,$$

and a T-action on $G/U \times X^U$ given by $(t, (gU, x)) \mapsto (gt^{-1}U, tx)$. As $\mathcal{O}(G/U \times X^U) = \mathcal{O}(G/U) \otimes \mathcal{O}(X^U) = \mathcal{O}(G/\!\!/U) \otimes \mathcal{O}(X^U) = \mathcal{O}(G/\!\!/U \times X^U)$ (see [6, Chapter I, Section 2, Proposition 2.6]), they, respectively, extend to a morphism $\varphi: G/\!\!/U \times X^U \to X$ and a T-action on $G/\!\!/U \times X^U$. It follows that φ is constant on the T-orbits, and thus induces a G-equivariant morphism

$$\Phi: G/\!\!/U \star^T X^U := (G/\!\!/U \times X^U)/\!\!/T \to X.$$

Proposition 4.2 *Let X be an affine G-variety, and assume that* $\mathcal{O}(X)$ *is an isotypically graded G-algebra. Then the canonical morphism*

$$\Phi: G/\!\!/U \star^T X^U \to X$$

is a G-equivariant isomorphism. Its comorphism is the inverse of the isomorphism Ψ *from Theorem 3.8.*

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Proof By definition, the comorphism $\Phi^*\colon \mathcal{O}(X) \longrightarrow (\mathcal{O}(G/U)\otimes\mathcal{O}(X^U))^T$ is given as follows: if $\Phi^*(f) = \sum_j f_j \otimes h_j$, then $\Phi^*(f)(gU,x) = f(gx) = \sum_j f_j(gU)h_j(x)$. Consider the evaluation map $\varepsilon\colon \mathcal{O}(G/U) \to \mathbb{K}$, $f \mapsto f(eU)$. Then $f(x) = \sum_j \varepsilon(f_j)h_j(x)$ for all $x \in X^U$, which shows that the diagram

$$\begin{array}{ccc} \mathcal{O}(X) & \xrightarrow{\Phi^*} & (\mathcal{O}(G/U)\otimes\mathcal{O}(X^U))^T \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\varepsilon\otimes\mathrm{id}} \\ \mathcal{O}(X^U) & = & \mathcal{O}(X^U) \end{array}$$

commutes, where ρ is the restriction map, i.e., $\rho(f) = \sum_j \varepsilon(f_j)h_j = (\varepsilon\otimes\mathrm{id})(\Phi^*(f))$. Since $\mathcal{O}(X)$ is an isotypically graded G-algebra, it follows from Remark 3.4 that the restriction map ρ is equal to the universal U-projection $\pi_U\colon \mathcal{O}(X) \to \mathcal{O}(X)_U$. If we show that $\varepsilon\otimes\mathrm{id}$ is also equal to the U-projection $\pi_U\colon (\mathcal{O}(G/U)\otimes\mathcal{O}(X^U))^T \to \left((\mathcal{O}(G/U)\otimes\mathcal{O}(X^U))^T\right)_U$, then Φ^* is an isomorphism by Lemma 3.5. We have

$$(\mathcal{O}(G/U)\otimes\mathcal{O}(X^U))^T = \bigoplus_{\lambda\in\Lambda_G} \mathcal{O}(G/U)_\lambda \otimes \mathcal{O}(X^U)_{[-\lambda^\vee]},$$

where $\mathcal{O}(X^U)_{[\mu]}$ is the T-weight space of $\mathcal{O}(X^U)$ of weight μ. Since the evaluation map $\varepsilon_\lambda\colon \mathcal{O}(G/U)_\lambda \to \mathbb{K}$, $f \mapsto f(eU)$, is the universal U-projection (Lemma 3.7), we see that the linear map $\mathcal{O}(G/U)_\lambda \otimes \mathcal{O}(X^U)_{[-\lambda^\vee]} \to \mathcal{O}(X^U)_{[-\lambda^\vee]}$, $\sum_j f_j\otimes h_j \mapsto \sum_j \varepsilon(f_j)h_j$, is the U-projection as well, and the claim follows.

It remains to see that Φ^* is equal to the inverse of Ψ from Theorem 3.8. Using again Lemma 3.5, it suffices to show that the diagram

$$\begin{array}{ccc} (\mathcal{O}(G/U)\otimes\mathcal{O}(X^U))^T & \xrightarrow{\Psi} & \mathcal{O}(X) \\ \downarrow{\scriptstyle\varepsilon\otimes\mathrm{id}} & & \downarrow{\scriptstyle\rho} \\ \mathcal{O}(X^U) & = & \mathcal{O}(X^U) \end{array}$$

commutes. This is stated in Remark 3.11. ∎

4.2 The structure of small *G*-varieties

Proposition 4.3 *Let X be an irreducible small G-variety. Then the following holds.*

(1) *The G-action is fix-pointed, and all minimal orbits in X have the same type λ.*
(2) *$\mathcal{O}(X)$ is an isotypically graded G-algebra of type $\lambda^\vee$.*
(3) *The quotient $X \to X/\!\!/U^-$ restricts to an isomorphism $X^U \xrightarrow{\sim} X/\!\!/U^-$. Moreover,*

$$X \text{ normal} \iff X^U \text{ normal} \iff X/\!\!/U \text{ normal}.$$

We call such a variety X *a small G-variety of type* λ.

Proof (1) By hypothesis, any nontrivial orbit $O \subset X$ is minimal, so $\overline{O} = O \cup \{x_0\}$, where $x_0 \in X^G$ by Proposition 2.11(1). In particular, the G-action is fix-pointed.

We can assume that X is a closed G-stable subvariety of a G-module W (see, for example, [20, Corollary 2.3.5]). Let $O \subset X$ be a nontrivial orbit. There is a linear projection $p\colon W \to V$ onto a simple G-module V of highest weight λ such that

$O \notin \ker p$. Proposition 2.11(2) implies that $p(O) = O_\lambda$ and that O is of the same type as O_λ. The same is true for all orbits O' from the open subset $X' := X \setminus \ker p$ of X. Since X is irreducible, all minimal orbits are of type λ. It follows from Example 3.1 that $\mathcal{O}(O)$ is of type $\lambda^\vee$.

(2) Since X is small, we have $X = G \cdot X^U$, showing that the morphism $G/U \times X^U \to X$ is surjective. Thus, we obtain a G-equivariant inclusion $\mathcal{O}(X) \hookrightarrow \mathcal{O}(G/U) \otimes \mathcal{O}(X^U)$ where the G-algebra on the right is isotypically graded (by (*) and Example 3.1). Hence, it follows from (1) that $\mathcal{O}(X)$ is isotypically graded of type $\lambda^\vee$.

(3) The first part follows from Remark 3.4. It also shows that if X is normal, then X^U is normal. The other implication follows from the isomorphism Φ in Proposition 4.2 because $G /\!\!/ U$ is normal (cf. Example 3.15). The second equivalence is clear. ∎

Proposition 4.4 *Let X be an irreducible small G-variety of type λ.*

(1) *There is a unique $\mathbb{K}^*$-action on X which induces the canonical $\mathbb{K}^*$-action on each minimal orbit and commutes with the G-action. This action on X^U is fix-pointed, and we get isomorphisms $X^U /\!\!/ \mathbb{K}^* \xrightarrow{\sim} X /\!\!/ G \xleftarrow{\sim} X^G$.*

(2) *The morphism $G \times X^U \to X$, $(g, x) \mapsto gx$, induces a G-equivariant isomorphism*

$$\Phi \colon \overline{O_\lambda} \star^{\mathbb{K}^*} X^U \xrightarrow{\sim} X,$$

where $\mathbb{K}^*$ acts on $\overline{O_\lambda}$ by the inverse of the scalar multiplication: $(t, x) \mapsto t^{-1} \cdot x$.

(3) *We have* $\mathrm{Norm}_G(X^U) = P_\lambda$, *and the G-equivariant morphism*

$$\Theta \colon G \times^{P_\lambda} X^U \to X, \quad [g, x] \mapsto gx,$$

is proper, surjective, and birational, and it induces an isomorphism between the algebras of regular functions.

Proof (1) By Proposition 4.3(2), $\mathcal{O}(X)$ is an isotypically graded G-algebra of type $\lambda^\vee$. If we define the $\mathbb{K}^*$-action on $\mathcal{O}(X)$ such that the isotypic component of type $n\lambda^\vee$ has weight $-n$, then this action is fix-pointed and restricts to the canonical $\mathbb{K}^*$-action on the closure of each minimal orbit (Proposition 2.14(3)). Since X is the union of the closures of the minimal orbits, this $\mathbb{K}^*$-action is unique and commutes with the G-action. We have $X^G = (X^U)^{\mathbb{K}^*}$ and the $\mathbb{K}^*$-action on X^U is fix-pointed, since this holds for the closure of a minimal orbit. This implies that $X^G = (X^U)^{\mathbb{K}^*} \xrightarrow{\sim} X^U /\!\!/ \mathbb{K}^*$, and $X^G \xrightarrow{\sim} X /\!\!/ G$ since the G-action is fix-pointed, which yields the remaining claims.

(2) Choose $x_0 \in O_\lambda^U$ and consider the G-equivariant morphism $\eta \colon G /\!\!/ U \to \overline{O_\lambda}$ induced by $gU \mapsto gx_0$. For $t \in T$, we get $\eta(gt^{-1}U) = gt^{-1}x_0 = \lambda(t)^{-1} \cdot gx_0$. This shows that the T-action on $G /\!\!/ U$ by right multiplication induces a T-action on $\overline{O_\lambda}$ which factors through $\lambda \colon T \to \mathbb{K}^*$, and the induced $\mathbb{K}^*$-action is the inverse of the scalar multiplication, i.e., the inverse of the canonical $\mathbb{K}^*$-action.

Define $D := \ker \lambda \subset T$. We claim that η is the algebraic quotient under the action of D. In fact, the action of $t \in T$ on $\mathcal{O}(G/U)_\mu$ is by scalar multiplication with $\mu^\vee(t)$ (Remark 3.6(1)). Hence, the action of D is trivial if and only if μ is a multiple of $\lambda^\vee$. This implies that

$$\mathcal{O}(G/U)^D = \bigoplus_{\mu \in \Lambda_G} \mathcal{O}(G/U)^D_\mu = \bigoplus_{k \geq 0} V_{k\lambda^\vee} \simeq \mathcal{O}(\overline{O_\lambda})$$

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(see Lemma 2.2(2)). Since X is of type λ, it follows that $D = \ker\lambda$ acts trivially on X^U. Therefore, $(G/\!\!/U \times X^U)/\!\!/D = \overline{O_\lambda} \times X^U$, and so

$$G/\!\!/U \star^T X^U = (G/\!\!/U \star^D X^U)/\!\!/T = (\overline{O_\lambda} \times X^U)/\!\!/T.$$

By construction, the T-action on $\overline{O_\lambda} \times X^U$ is given by $t(v,x) = (\lambda(t)^{-1} \cdot v, tx)$, i.e., by the inverse of the canonical $\mathbb{K}^*$-action on $\overline{O_\lambda}$ and the given action on X^U. Hence, $(\overline{O_\lambda} \times X^U)/\!\!/T = \overline{O_\lambda} \star^{\mathbb{K}^*} X^U$, and the claim follows from Proposition 4.2.

(3) Consider the action of P_λ on $G \times X^U$ given by $p(g,x) = (gp^{-1}, px)$. Then the action map $G \times X^U \to X$, $(g,x) \mapsto gx$, factors through the geometric quotient

$$G \times^{P_\lambda} X^U := (G \times X^U)/P_\lambda.$$

For Θ, we have the following factorization:

$$\Theta\colon G \times^{P_\lambda} X^U \xrightarrow{\subseteq} G \times^{P_\lambda} X \xrightarrow[\simeq]{[g,x]\mapsto(gP_\lambda,gx)} G/P_\lambda \times X \xrightarrow{\mathrm{pr}_X} X,$$

where the first map is a closed immersion and the second an isomorphism. Since G/P_λ is complete, it follows that Θ is proper. Moreover, Θ is surjective because every G-orbit meets X^U. We claim that Θ induces a bijection $G \times^{P_\lambda} (X^U \setminus X^G) \to X \setminus X^G$, which implies that Θ is birational. Indeed, if $x \in X^U \setminus X^G$, then $x \in O^U$ for a minimal orbit $O \subset X$. If $gx = g'x'$ for some $x' \in X^U$, $g' \in G$, then $x' \in O^U$, and hence $x' = qx$ for some $q \in P_\lambda$ because the action of P_λ on O^U is transitive. It follows that $g^{-1}g'q \in G_x \subset P_\lambda$, and hence $p := g^{-1}g' \in P_\lambda$. Thus, $[g',x'] = [gp,x'] = [g,px'] = [g,x]$.

It remains to see that the comorphism of Θ is an isomorphism on the global functions. Let K_λ be the kernel of the character $\lambda\colon P_\lambda \to \mathbb{K}^*$. Then $G/K_\lambda \simeq O_\lambda$, and the action of P_λ on G by right multiplication induces an action of $\mathbb{K}^* = P_\lambda/K_\lambda$ on G/K_λ by right multiplication corresponding to the canonical action on O_λ. This gives the G-equivariant isomorphisms

$$G \times^{P_\lambda} X^U \xrightarrow{\sim} G/K_\lambda \star^{\mathbb{K}^*} X^U \xrightarrow{\sim} O_\lambda \star^{\mathbb{K}^*} X^U,$$

and the claim follows from (2). ∎

Example 4.5 Let $X := \overline{O_\mu} \subset V_\mu$ be the closure of the minimal orbit in V_μ, and let $\mu = \ell\lambda$, where λ is indivisible. Then $X^U = \mathbb{K}$, and from Proposition 4.4(2), we get an isomorphism

$$\overline{O_\lambda} \star^{\mathbb{K}^*} \mathbb{K} \simeq X = \overline{O_\mu},$$

where $\mathbb{K}^*$ acts on $\overline{O_\lambda}$ by the inverse of the canonical action, $(t,x) \mapsto t^{-1} \cdot x$, and by the canonical action on $\mathbb{K} = \overline{O_\mu}$, which is the scalar multiplication with $\mu(t)$.

The second statement of Proposition 4.4 says that a small G-variety X can be *reconstructed* from the $\mathbb{K}^*$-variety X^U. In order to give a more precise statement, we introduce the following notion. A $\mathbb{K}^*$-action on an affine variety Y is called *positively fix-pointed* if for every $y \in Y$ the limit $\lim_{t\to 0} ty$ exists and is therefore a fixed point.

For a fix-pointed $\mathbb{K}^*$-action on an irreducible affine variety Y, either the action is positively fix-pointed or the inverse action $(t,y) \mapsto t^{-1}y$ is positively fix-pointed. Indeed, for any $y \in Y$, either $\lim_{t\to 0} ty$ or $\lim_{t\to\infty} ty$ exists. Embedding Y equivariantly into a $\mathbb{K}^*$-module, one sees that the subsets $Y_+ := \{y \in Y \mid \lim_{t\to 0} ty \text{ exists}\}$ and

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$Y_- := \{y \in Y \mid \lim_{t\to\infty} ty \text{ exists}\}$ are closed. As Y is irreducible, this yields the claim. (The claim does not hold for connected $\mathbb{K}^*$-varieties, as the example of the union of the coordinate lines in the two-dimensional representation $t(x, y) := (tx, t^{-1}y)$ shows.)

Remark 4.6 A positively fix-pointed $\mathbb{K}^*$-action on Y extends to an action of the multiplicative semigroup $(\mathbb{K}, \cdot)$, and the morphism $\mathbb{K} \times Y \to Y$, $(s, y) \mapsto sy$, induces an isomorphism $\mathbb{K} \star^{\mathbb{K}^*} Y \xrightarrow{\sim} Y$. This follows from the commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow{y \mapsto (1,y)} & \mathbb{K} \times Y & \xrightarrow{(s,y)\mapsto sy} & Y \\ \Big\downarrow \mathrm{id}_Y & & \Big\downarrow \pi & & \Big\downarrow \mathrm{id}_Y \\ Y & \longrightarrow & \mathbb{K} \star^{\mathbb{K}^*} Y & \longrightarrow & Y \end{array}$$

where the compositions of the horizontal maps are the identity.

Lemma 4.7 *Let Y be a positively fix-pointed affine $\mathbb{K}^*$-variety, and let $\lambda \in \Lambda_G$ be indivisible. Consider the $\mathbb{K}^*$-action on $\overline{O_\lambda} \times Y$ given by $t(v, y) := (t^{-1} \cdot v, ty)$. Then*

$$X := \overline{O_\lambda} \star^{\mathbb{K}^*} Y = (\overline{O_\lambda} \times Y)/\!\!/\mathbb{K}^*$$

is a small G-variety of type λ where the action of G is induced by the action on $\overline{O_\lambda}$. Moreover, there is canonical $\mathbb{K}^*$-equivariant isomorphism $X^U \xrightarrow{\sim} Y$.

Proof By definition, X is an affine G-variety. For $x = [v, y] \in \overline{O_\lambda} \star^{\mathbb{K}^*} Y$, $v \neq 0$, the G-orbit $Gx \subset X$ is the image of $O_\lambda \times \{y\}$ in X, hence a minimal orbit of type λ or a point (Proposition 2.11(2)). As a consequence, X is a small G-variety of type λ. Furthermore, since the canonical $\mathbb{K}^*$-action on $\overline{O_\lambda}$ commutes with the G-action, we have

$$X^U = (\overline{O_\lambda} \star^{\mathbb{K}^*} Y)^U = \overline{O_\lambda}^U \star^{\mathbb{K}^*} Y \simeq \mathbb{K} \star^{\mathbb{K}^*} Y \xrightarrow{\sim} Y,$$

where the last morphism is given by $[t, y] \mapsto ty$ which is an isomorphism, as explained in Remark 4.6. ∎

Proof (of Corollary 1.4) This corollary follows from Proposition 4.4(2) and Lemma 4.7. ∎

4.3 Smoothness of small G-varieties

Before describing the smoothness properties of small varieties, let us look at some examples. As before, G is always a semisimple algebraic group.

Remark 4.8 Let W be a G-module whose nontrivial orbits are all minimal. We claim that W is a simple G-module and contains a single nontrivial orbit which is minimal. In particular, the highest weight of W is indivisible.

Indeed, all minimal orbits in W have the same type by Lemma 4.3(1) and therefore the same dimension $d > 1$, by Remark 2.9(2). Moreover, every minimal orbit meets W^U in a punctured line, by Lemma 2.2(3). This implies that $\dim W = \dim W^U - 1 + d$. Let $W = \bigoplus_{i=1}^m V_i$ be the decomposition into simple G-modules. Since a simple G-module contains exactly one minimal orbit (Remark 2.9(3)), we obtain $\dim W = md$, and since $\dim W^U = m$, we get $md = m - 1 + d$, and so $m = 1$.

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Remark 4.9 If a small G-variety X is smooth and contains exactly one fixed point, then X is a simple G-module V_λ containing a dense minimal orbit, and λ is indivisible. Indeed, smoothness and having exactly one fixed point imply by Luna's Slice theorem [24, Section III.1, Corollaire 2] that X is a G-module, and the rest follows from the remark above.

Example 4.10 Let $\mathbb{K}^n$ be the standard representation of SL_n, and set $W := (\mathbb{K}^n)^{\oplus m}$. Define $Y := \mathbb{K}e_1 \oplus \mathbb{K}e_1 \oplus \cdots \oplus \mathbb{K}e_1 \subset W$, where $e_1 = (1, 0, \ldots, 0)$, and set $X := \mathrm{SL}_n Y \subseteq W$. Since Y is B-stable and closed, it follows that X is a closed and SL_n-stable subvariety of W with the following properties (cf. Example 2.10).

(1) X contains a single closed SL_n-orbit, namely the fixed point $\{0\}$.
(2) Every nontrivial orbit $O \subset X$ is minimal of type ε_1, and $\overline{O} \simeq \mathbb{K}^n$ as an SL_n-variety. In particular, X is a small SL_n-variety.
(3) Since $X^U = Y$ is normal (even smooth), X is also normal, by Lemma 4.3(3).

However, by Remark 4.9 and (2), X is not smooth if $m > 1$.

Concerning the smoothness of small G-varieties, we have the following rather strong result (cf. Theorem 1.5).

Theorem 4.11 *Let X be an irreducible small G-variety of type λ, and consider the following statements.*

(i) *The quotient $\pi: X \to X/\!\!/G$ is a G-vector bundle with fiber V_λ.*
(ii) *$\mathbb{K}^*$ acts faithfully on X^U, the quotient $X^U \to X^U/\!\!/\mathbb{K}^*$ is a line bundle, and $V_\lambda = \overline{O_\lambda}$.*
(iii) *The quotient $X^U \setminus X^G \to X^U/\!\!/\mathbb{K}^*$ is a principal $\mathbb{K}^*$-bundle, and $V_\lambda = \overline{O_\lambda}$.*
(iv) *The closures of the minimal orbits of X are smooth and pairwise disjoint.*
(v) *The quotient morphism $\pi: X \to X/\!\!/G$ is smooth.*

Then the assertions (i) *and* (ii) *are equivalent and imply* (iii)–(v). *If X (or X^U) is normal, all assertions are equivalent.*

Furthermore, X is smooth if and only if $X/\!\!/G$ is smooth and $\pi: X \to X/\!\!/G$ is a G-vector bundle.

We will prove Theorem 4.11 just after the following example.

Example 4.12 This example of a normal small G-variety X illustrates what might go wrong in the different statements of Theorem 4.11 if X is not smooth. Let $W := \mathbb{K}^3$ be the $\mathbb{K}^*$-module with weights $(2, 1, 0)$, i.e., $t(x, y, z) := (t^2 \cdot x, t \cdot y, z)$. The homogeneous function $f := xz - y^2$ defines a normal $\mathbb{K}^*$-stable closed subvariety $Y = \mathcal{V}(f) \subset \mathbb{K}^3$ with an isolated singularity at 0. The invariant z defines the quotient $\pi = z: Y \to \mathbb{K} = Y/\!\!/\mathbb{K}^*$. The (reduced) fibers of π are isomorphic to $\mathbb{K}$, but π is not a line bundle, because the zero fiber is not reduced. The action of $\mathbb{K}^*$ is given by $(t, s) \mapsto t \cdot s$ on the fibers over $\mathbb{K} \setminus \{0\}$ and by $(t, s) \mapsto t^2 \cdot s$ on the zero fiber. In fact, the zero fiber contains the point $(1, 0, 0)$, which is fixed by $\{\pm 1\}$, but not by $\mathbb{K}^*$.

By Lemma 4.7, $X := \overline{O_{\varepsilon_1}} \star^{\mathbb{K}^*} Y$ is a small G-variety and $X^U \simeq Y$, and hence X is normal (Proposition 4.3(3)). Moreover, $X/\!\!/G \simeq Y/\!\!/\mathbb{K}^* = \mathbb{K}$ by Proposition 4.4(1). All fibers of the quotient map $\pi: X \to X/\!\!/G = \mathbb{K}$ different from the zero fiber are isomorphic to $\mathbb{K}^3 = \overline{O_{\varepsilon_1}}$, but $\pi^{-1}(0) \simeq \overline{O_{2\varepsilon_1}}$.

Proof (of Theorem 4.11) (i) ⇒ (ii): If $X \to X/\!\!/G$ is a G-vector bundle with fiber V_λ, then the induced morphism $X^U \to X/\!\!/G = X^U/\!\!/\mathbb{K}^*$ is a subbundle with fiber $V_\lambda^U \simeq \mathbb{K}$, hence a line bundle.

(ii) ⇒ (i): Since $\overline{O_\lambda} = V_\lambda$, we have a canonical isomorphism $V_\lambda \star^{\mathbb{K}^*} X^U \xrightarrow{\sim} X$, where $\mathbb{K}^*$ acts by the inverse of the scalar multiplication on V_λ (see Proposition 4.4(2)). If $X^U \to X^U/\!\!/\mathbb{K}^*$ is a line bundle, then it looks locally like $\mathbb{K} \times W \xrightarrow{\mathrm{pr}_W} W$, and $\mathbb{K}^*$ acts by scalar multiplication on $\mathbb{K}$. Hence, $V_\lambda \star^{\mathbb{K}^*} X^U$ looks locally like

$$V_\lambda \star^{\mathbb{K}^*} (\mathbb{K} \times W) = (V_\lambda \star^{\mathbb{K}^*} \mathbb{K}) \times W \simeq V_\lambda \times W,$$

where we use the canonical isomorphism $V_\lambda \star^{\mathbb{K}^*} \mathbb{K} \xrightarrow{\sim} V_\lambda$, $[v,s] \mapsto s \cdot v$ (see Remark 4.6). This shows that $V_\lambda \star^{\mathbb{K}^*} X^U \xrightarrow{\sim} X$ is a G-vector bundle over $X^U/\!\!/\mathbb{K}^* = X/\!\!/G$.

(i) ⇒ (v): This is obvious.

(v) ⇒ (iv): The (reduced) fibers of $\pi\colon X \to X/\!\!/G$ are small G-varieties with a unique fixed point. If such a fiber F is smooth, then $F \simeq V_\lambda$ and $V_\lambda = \overline{O_\lambda}$ by Remark 4.9.

(iv) ⇒ (iii): If the closure of a minimal orbit O is smooth, then $O \simeq O_\lambda$ and $\overline{O_\lambda} = V_\lambda$, again by Remark 4.9. It follows that the action of $\mathbb{K}^*$ on $X^U \setminus X^G$ is free and so $P := X^U \setminus X^G \to X^U/\!\!/\mathbb{K}^*$ is a principal $\mathbb{K}^*$-bundle.

(iii) ⇒ (ii) assuming X^U normal: If $P := X^U \setminus X^G \to X^U/\!\!/\mathbb{K}^*$ is a principal $\mathbb{K}^*$-bundle and $L := \mathbb{K} \star^{\mathbb{K}^*} P \to X^U/\!\!/\mathbb{K}^*$ the associated line bundle, then there is a canonical morphism (see Remark 4.6)

$$\sigma\colon L = \mathbb{K} \star^{\mathbb{K}^*} (X^U \setminus X^G) \longrightarrow \mathbb{K} \star^{\mathbb{K}^*} X^U \simeq X^U.$$

By construction, σ is bijective, hence an isomorphism, because X^U is normal.

It remains to prove the last statement where one implication is clear. Assume that X is smooth. Since the G-action is fix-pointed, it follows from [1, Theorem (10.3)] that $\pi\colon X \to X/\!\!/G$ is a G-vector bundle. ∎

5 Computations

In this paragraph, we calculate the invariants m_G, d_G, and r_G, which are defined for any semisimple algebraic group G in the following way:

$$\begin{aligned} m_G &:= \min\{\dim O \mid O \text{ a minimal orbit}\}, \\ d_G &:= \min\{\dim O \mid O \text{ a nonminimal quasi-affine nontrivial orbit}\}, \\ r_G &:= \min\{\operatorname{codim} H \mid H \subsetneq G \text{ reductive}\}. \end{aligned}$$

For any nontrivial orbit O in an affine G-variety X, Lemma 2.5 implies that $\dim O \geq m_G$. An orbit $O \simeq G/H$ with H reductive is affine and thus cannot be minimal (Lemma 2.2(2)).

If $O \subset X$ is an orbit of dimension m_G, then it is either minimal or closed. In fact, if O is not closed, then $\overline{O} \setminus O$ must be a fixed point since it cannot contain an orbit of positive dimension. This implies, by Proposition 2.11(1), that O is minimal. It follows that if $d_G = m_G$, then $d_G = r_G$. Hence, we get

(5.1) $$r_G \geq d_G \geq m_G, \text{ and } d_G > m_G \text{ in case } r_G > m_G.$$

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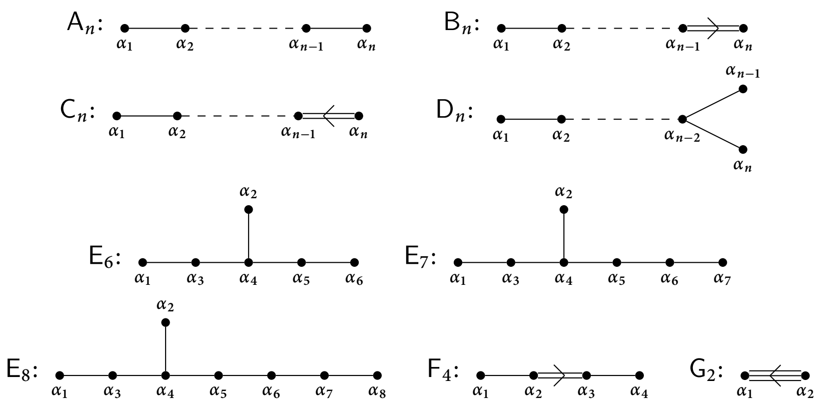
If $d_G > m_G$, then an irreducible G-variety X of dimension $< d_G$ is small and we can apply our results about small G-varieties.

For simplicity, we assume from now on that G is *simply connected*.

5.1 Notation

Let G be a simply connected simple group. As before, we fix a Borel subgroup $B \subset G$, a maximal torus $T \subset B$, and denote by $W := \mathrm{Norm}_G(T)/T$ the Weyl group. The monoid of dominant weights $\Lambda_G \subset X(T) := \mathrm{Hom}(T, \mathbb{K}^*)$ is freely generated by the fundamental weights $\omega_1, \ldots, \omega_r$, i.e., $\Lambda_G = \bigoplus_{i=1}^r \mathbb{N}\omega_i$ (see Section 2.1). We denote by $\Phi = \Phi_G \subset X(T)$ the root system of G, by $\Phi^+ = \Phi_G^+ \subset \Phi$ the set of positive roots corresponding to B and by $\Delta = \Delta_G \subset \Phi^+$ the set of simple roots. Furthermore, $\mathfrak{g} := \mathrm{Lie}\, G$, $\mathfrak{b} := \mathrm{Lie}\, B$, and $\mathfrak{h} := \mathrm{Lie}\, T$ are the Lie algebras of G, B, and T, respectively, $\mathfrak{g}_\alpha \subset \mathfrak{g}$ is the root subspace of $\alpha \in \Phi$, and $G_\alpha \subset G$ is the corresponding root subgroup of G, isomorphic to $\mathbb{K}^+$.

The nodes of the Dynkin diagram of G are the simple roots $\Delta_G = \{\alpha_1, \ldots, \alpha_r\}$. We will use the Bourbaki-labeling of the nodes:



We also have a canonical bijection between the simple roots Δ_G and the fundamental weights $\{\omega_1, \ldots, \omega_r\}$ induced by the Weyl group invariant scalar product $(\cdot,\cdot)$ on $X(T)_\mathbb{R} := X(T) \otimes_\mathbb{Z} \mathbb{R}$:

$$\langle \omega_i, \alpha_j \rangle := \frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}.$$

For any root α, we denote by σ_α the corresponding reflection of $X(T)_\mathbb{R}$:

$$\sigma_\alpha(\lambda) := \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha = \lambda - \langle \lambda, \alpha \rangle \alpha.$$

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5.2 Parabolic subgroups

We now recall some classical facts about parabolic subgroups of G (cf. [14, Sections 29 and 30]).

If $R \subset \Delta$ is a set of simple roots and $I := \Delta \backslash R$ the complement, we define $P(R) := BW_I B \subseteq G$, where $W_I \subseteq W$ is the subgroup generated by the reflections σ_i corresponding to the elements of I. Thus, $\alpha \in I$ if and only if $\mathfrak{g}_{-\alpha} \subset \operatorname{Lie} P(R)$. Any parabolic subgroup of G containing B is of the form $P(R)$, and we have $R \subseteq S$ if and only if $P(S) \subseteq P(R)$, with $R = S$ being equivalent to $P(R) = P(S)$. In particular, $P(\varnothing) = G$ and $P(\Delta) = B$, and the $P(\alpha_i) := P(\{\alpha_i\})$ are the maximal parabolic subgroups of G containing B.

Consider the *Levi decomposition* $P(R) = L(R) \ltimes U(R)$, where $U(R)$ is the unipotent radical of $P(R)$ and $L(R)$ the Levi part of $P(R)$ containing T, i.e., $L(G) = \operatorname{Cent}_G(Z)$ where $Z := \bigcap_{\alpha \in I} \ker \alpha \subseteq T$. In particular, $L(R)$ is reductive, and so its derived subgroup $(L(R), L(R))$ is semisimple. The connected center $\mathrm{Z}(L(R))^\circ$ of $L(R)$ is equal to Z, and hence

$$\dim \mathrm{Z}(L(R)) = \dim Z = \dim T - |I| = |R|.$$

It follows that

$$\dim(L(R), L(R)) = \dim L(R) - \dim \mathrm{Z}(L(R)) = \dim L(R) - |R|. \tag{5.2}$$

On the level of Lie algebras, we see that $\operatorname{Lie} P(R)$ contains all positive root spaces $\mathfrak{g}_\beta$, and that for a simple root $\alpha \in \Delta$, we have $\mathfrak{g}_{-\alpha} \subset \operatorname{Lie} P(R)$ if and only if $\alpha \in I = \Delta \backslash R$:

$$\mathfrak{p}(R) := \operatorname{Lie} P(R) = \mathfrak{h} \oplus \textstyle\bigoplus_{\alpha \in \Theta} \mathfrak{g}_\alpha \quad \text{where} \quad \Theta := \Phi^+ \cup (\Phi^- \cap \sum_{\alpha \in I} \mathbb{Z}\alpha).$$

If $\Phi_I \subseteq \Phi$ is the subsystem generated by I, we get

$$\mathfrak{p}(R) = \mathfrak{l}(R) \oplus \mathfrak{n}(R), \quad \mathfrak{l}(R) := \operatorname{Lie} L(R) = \mathfrak{h} \oplus \textstyle\bigoplus_{\alpha \in \Phi_I} \mathfrak{g}_\alpha,$$
$$\mathfrak{n}(R) := \operatorname{Lie} U(R) = \textstyle\bigoplus_{\alpha \in \Phi^+ \backslash \Phi_I} \mathfrak{g}_\alpha.$$

Moreover, if $R \subseteq S$, we have $\mathfrak{l}(S) \subseteq \mathfrak{l}(R)$ and $\mathfrak{n}(S) \supseteq \mathfrak{n}(R)$.

Setting $\mathfrak{n}^-(R) := \bigoplus_{\mathfrak{g}_\alpha \subset \mathfrak{n}(R)} \mathfrak{g}_{-\alpha}$, we get $\dim \mathfrak{n}^-(R) = \dim \mathfrak{n}(R)$ and $\mathfrak{g} = \mathfrak{n}^-(R) \oplus \mathfrak{p}(R) = \mathfrak{n}^-(R) \oplus \mathfrak{l}(R) \oplus \mathfrak{n}(R)$, and hence

$$\dim \mathfrak{g} - \dim \mathfrak{l}(R) = 2 \dim \mathfrak{n}(R) = 2 \dim(\mathfrak{g}/\mathfrak{p}(R)). \tag{5.3}$$

Furthermore, (5.2) and (5.3) yield

$$\dim \mathfrak{n}(R) = \frac{1}{2}(\dim \mathfrak{g} - \dim[\mathfrak{l}(R), \mathfrak{l}(R)] - |R|). \tag{5.4}$$

Remark 5.1 The following facts will be important in our calculations of the invariants m_G and d_G. From the Dynkin diagram of G, we can read off the type of semisimple group $S(R) := (L(R), L(R))$ by simply removing the nodes corresponding to the roots in R. Moreover, we have $\mathfrak{g}_\alpha \subset \mathfrak{n}(R)$ for any $\alpha \in R$, and one can determine the irreducible representation $V(\alpha) \subseteq \mathfrak{n}(R)$ of $L(R)$ generated by $\mathfrak{g}_\alpha$, because $\mathfrak{g}_\alpha \subset V(\alpha)$ is the lowest weight space.

In the special case $R = \{\alpha_i\}$, the Cartan numbers $\langle \alpha_i, \alpha_j \rangle$ are the coefficients in the decomposition of $\alpha_i|_{S(\alpha_i)}$ with respect to the fundamental weights of $S(\alpha_i)$.

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It is also easy to see that the Lie subalgebra generated by $V(\alpha)$ consists of all root spaces $\mathfrak{g}_\beta$ where β is a positive root containing α. In the special case $R = \{\alpha_i\}$, this implies that $\mathfrak{n}(\alpha_i)$ is equal to the Lie subalgebra generated by $V(\alpha_i)$.

5.3 The parabolic subgroup P_λ

Recall that for a simple G-module $V = V_\lambda$ with highest weight λ, the subgroup

$$P_\lambda := \operatorname{Norm}_G(V_\lambda^U) = \operatorname{Norm}_G(O_\lambda^U)$$

is a parabolic subgroup of G, and λ induces a character $\lambda: P_\lambda \to \mathbb{K}^*$. For $v \in V_\lambda^U$, $v \neq 0$, we have

$$O_\lambda = Gv \text{ and } G_v = \ker(\lambda: P_\lambda \to \mathbb{K}^*).$$

In particular, $\dim O_\lambda = \operatorname{codim}_G P_\lambda + 1$. As above, there is a well-defined Levi decomposition $P_\lambda = L_\lambda \ltimes U_\lambda$ where $T \subseteq L_\lambda$, which carries over to the Lie algebra:

$$\mathfrak{p}_\lambda := \operatorname{Lie} P_\lambda = \mathfrak{l}_\lambda \oplus \mathfrak{n}_\lambda, \quad \mathfrak{l}_\lambda := \operatorname{Lie} L_\lambda, \ \mathfrak{n}_\lambda := \operatorname{Lie} U_\lambda.$$

Since P_λ contains B, it is of the form $P(R)$ where the subset $R \subseteq \Delta_G$ has the following description.

Lemma 5.2

(1) *If $\lambda = \sum_{i=1}^r m_i\omega_i$, then $P_\lambda = P(R)$ where $R := \{\alpha_i \in \Delta_G \mid m_i \neq 0\}$.*
(2) *We have $P_\lambda = P_{\lambda'}$ if the same ω_i appear in λ and λ'. More generally, if every ω_i appearing in λ' also appears in λ, then $P_\lambda \subseteq P_{\lambda'}$, $L_\lambda \subseteq L_{\lambda'}$ and $U_\lambda \supseteq U_{\lambda'}$.*
(3) *$P_{k\omega_i} = P(\alpha_i)$ for all $k > 0$, and these are the maximal parabolic subgroups of G containing B.*

Proof (1) For a positive root α, the α-string through λ, i.e., the set of weights $\{\lambda - i\alpha \mid i \geq 0\}$ of V, has length $\langle \lambda, \alpha\rangle + 1$ (cf. [15, Proposition 21.3]). Thus, $\mathfrak{g}_{-\alpha}V^U = (0)$ if and only if $\langle\lambda, \alpha\rangle = 0$. If $\alpha = \alpha_j$ is a simple root, this is equivalent to the condition that ω_j does not occur in λ, showing that $P_\lambda = P(R)$.

(2) follows from (1) and (3) from (2). ■

5.4 The invariant m_G

It follows from (5.3) that

$$m_G = \min_{\lambda\in\Lambda_G} \dim O_\lambda = \min_{\lambda\in\Lambda_G} \operatorname{codim}_G P_\lambda + 1.$$

So, it suffices to calculate

(5.5) $$p_G := \min\{\dim G/P \mid P \subsetneq G \text{ a parabolic subgroup}\} = m_G - 1.$$

For this, it is sufficient to consider the maximal parabolic subgroups $P_{\omega_i} = P(\alpha_i)$.

Lemma 5.3 *The following table lists the invariants m_G and p_G for the simple groups G, the corresponding maximal parabolic subgroups P_ω, and the dimensions of the fundamental representations V_ω. The last column gives some indication about $\overline{O_\omega}$ for ω as in the fourth column where the null cone $\mathcal{N}_V$ appears only if $\mathcal{N}_V \subsetneq V$.*

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G	m_G	p_G	maximal P_ω	$\dim V_\omega$	$\overline{O_\omega}$
$\mathsf{A}_n, n \geq 1$	$n+1$	n	$P_{\omega_1},\ P_{\omega_n}$	$n+1, n+1$	$\mathbb{K}^{n+1}, (\mathbb{K}^{n+1})^\vee$
B_2	4	3	$P_{\omega_1}, P_{\omega_2}$	5, 4	$\mathcal{N}_{V_{\omega_1}}, V_{\omega_2}$
$\mathsf{B}_n, n \geq 3$	$2n$	$2n-1$	P_{ω_1}	$2n+1$	$\mathcal{N}_{V_{\omega_1}}$
$\mathsf{C}_n, n \geq 3$	$2n$	$2n-1$	P_{ω_1}	$2n$	V_{ω_1}
D_4	7	6	$P_{\omega_1}, P_{\omega_3}, P_{\omega_4}$	8, 8, 8	$\mathcal{N}_{V_{\omega_1}}, \mathcal{N}_{V_{\omega_3}}, \mathcal{N}_{V_{\omega_4}}$
$\mathsf{D}_n, n \geq 5$	$2n-1$	$2n-2$	P_{ω_1}	$2n$	$\mathcal{N}_{V_{\omega_1}}$
E_6	17	16	$P_{\omega_1}, P_{\omega_6}$	27, 27	$\subsetneqq \mathcal{N}_{V_{\omega_i}}, i = 1, 6$
E_7	28	27	P_{ω_7}	56	$\subsetneqq \mathcal{N}_{V_{\omega_7}}$
E_8	58	57	P_{ω_8}	248	$\subsetneqq \mathcal{N}_{\mathrm{Lie}\,\mathsf{E}_8}$
F_4	16	15	$P_{\omega_1}, P_{\omega_4}$	52, 26	$\subsetneqq \mathcal{N}_{V_{\omega_i}}, i = 1, 4$
G_2	6	5	$P_{\omega_1}, P_{\omega_2}$	7, 14	$\mathcal{N}_{V_{\omega_1}}, \subsetneqq \mathcal{N}_{\mathrm{Lie}\,\mathsf{G}_2}$

Table 2. Minimal dimension of minimal orbits for the simple groups.

Remark 5.4 Table 2 lists all parabolic subgroups P_λ of codimension p_G. Therefore, if O_λ is a minimal orbit of dimension m_G, then there is finite covering $O_\lambda \to O_\omega$ for a fundamental weight ω from the table. In particular, $\lambda = k\omega$ for some $k \geq 1$, and so $\overline{O_\lambda}$ is singular if $\lambda \neq \omega$, by Remark 2.4(3).

Proof By (5.3), we have to find the maximal dimensional Levi subgroups L_{ω_i}. For this, it suffices to compute the maximum of $\dim(L_{\omega_i}, L_{\omega_i})$. A short calculation in each case will give the possible ω_i from which we will obtain columns 2–5 of Table 2. For the last column, we use that

$$O_{\omega_i} \subseteq \mathcal{N}_{V_{\omega_i}} \quad \text{where} \quad \dim O_{\omega_i} = \operatorname{codim} P_{\omega_i} + 1 = \frac{1}{2}(\dim G - \dim \mathfrak{d}_i - 1) + 1$$

(see Lemma 2.2(1) and (4) and Section 5.2).

We now apply the above strategy to each simple group G. In each case, $\dim \mathfrak{d}_i$ turns out to be quadratic in i and achieves its minimum on the interval $[1, n]$. Hence, if $\mathfrak{d}_i$ is of maximal dimension, then i is either 1 or n.

(row A_n) For $i = 1, \ldots, n$, we obtain $\mathfrak{d}_i = \mathfrak{sl}_i \oplus \mathfrak{sl}_{n-i+1}$. It is of maximal dimension for $i = 1, n$. Furthermore, $V_{\omega_1} = \mathbb{K}^{n+1}$ and $V_{\omega_n} = (\mathbb{K}^{n+1})^\vee$ are the standard representation of SL_{n+1} and its dual which yields $\operatorname{codim} P_{\omega_1} = \operatorname{codim} P_{\omega_2} = n$ and $\overline{O_{\omega_i}} = V_{\omega_i}$.

(rows $\mathsf{B}_2 = \mathsf{C}_2$ and B_n) For $i = 1, \ldots, n$, we obtain $\mathfrak{d}_i = \mathfrak{sl}_i \oplus \mathfrak{so}_{2(n-i)+1}$. It is of maximal dimension for $i = 1$ if $n \geq 3$ and for $i = 1, 2$ if $n = 2$. Furthermore, $V_{\omega_1} = \mathbb{K}^{2n+1}$ is the standard representation of SO_{2n+1}, and the quotient $V_{\omega_1} /\!\!/ \mathrm{SO}_{2n+1} \simeq \mathbb{K}$ is given by the invariant quadratic form. In particular, $\dim \mathcal{N}_{V_{\omega_1}} = 2n$, and SO_{2n+1} acts transitively

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on the isotropic vectors $\mathcal{N}_{V_{\omega_1}}\setminus\{0\}$, and hence $\overline{O_{\omega_1}} = \mathcal{N}_{V_{\omega_1}}$. This gives the row B_n, $n \geq 3$, and half of the row B_2.

If $n = 2$, then V_{ω_2} is the standard representation $\mathbb{K}^4$ of Sp_4, and hence $O_{\omega_2} = \mathbb{K}^4\setminus\{0\}$, giving the other part of the row B_2.

(row C_n) Here, we get $\mathfrak{d}_i = \mathfrak{sl}_i \oplus \mathfrak{sp}_{2(n-i)}$, which is of maximal dimension for $i = 1$. Furthermore, $V_{\omega_1} = \mathbb{K}^{2n}$ is the standard representation of Sp_{2n}, and $\overline{O_{\omega_1}} = V_{\omega_1}$, and hence $m_{\mathrm{Sp}_{2n}} = 2n$.

(rows D_4 and D_n) For $i = 1, \ldots, n-3$, we get $\mathfrak{d}_i = \mathfrak{sl}_i \oplus \mathfrak{so}_{2(n-i)}$. Moreover, $\mathfrak{d}_{n-2} = \mathfrak{sl}_{n-2} \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and $\mathfrak{d}_{n-1} = \mathfrak{d}_n = \mathfrak{sl}_n$. They are maximal dimensional for $i = 1$ if $n \geq 5$ and for $i = 1, 3$, and 4 if $n = 4$. Furthermore, $V_{\omega_1} = \mathbb{K}^{2n}$ is the standard representation of SO_{2n}, and we get the claim for D_n, $n \geq 5$ and for V_{ω_1} in case $n = 4$. In this case, V_{ω_3} and V_{ω_4} are conjugate to the standard representation $V_{\omega_1} = \mathbb{K}^8$ by an outer automorphism of D_4. For the standard representation V, we have $V /\!\!/ G = \mathbb{K}$, given by the invariant quadratic form, and the nullcone consists of two orbits, $\{0\}$ and the minimal orbit of nonzero isotropic vectors.

(row E_6) Here, we find $\mathfrak{d}_1 = \mathfrak{d}_6 = \mathfrak{so}_{10}$, $\mathfrak{d}_2 = \mathfrak{sl}_6$, $\mathfrak{d}_3 = \mathfrak{d}_5 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_5$, and $\mathfrak{d}_4 = \mathfrak{sl}_3 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_3$. The maximal dimension is reached for $i = 1, 6$, and we get $p_{\mathsf{E}_6} = 16$. The representations V_{ω_1} and V_{ω_6} of dimension 27 are dual to each other. The quotient $V_{\omega_1} /\!\!/ \mathsf{E}_6 = \mathbb{K}$ is given by the cubic invariant of V_{ω_1} (see [27, Table 5b]), and so $\dim \mathcal{N}_{V_{\omega_1}} = 26$. It follows that $\overline{O_{\omega_i}} \subsetneqq \mathcal{N}_{V_{\omega_i}}$, $i = 1, 6$.

(row E_7) We have $\mathfrak{d}_1 = \mathfrak{so}_{12}$, $\mathfrak{d}_2 = \mathfrak{sl}_7$, $\mathfrak{d}_3 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_6$, $\mathfrak{d}_4 = \mathfrak{sl}_3 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_4$, $\mathfrak{d}_5 = \mathfrak{sl}_5 \oplus \mathfrak{sl}_3$, $\mathfrak{d}_6 = \mathfrak{so}_{10} \oplus \mathfrak{sl}_2$, and $\mathfrak{d}_7 = \mathsf{E}_6$. The maximal dimension is reached for $i = 7$, and we get $p_{\mathsf{E}_7} = 27$. We have $\dim V_{\omega_7} = 56$ and $\dim V_{\omega_7} /\!\!/ \mathsf{E}_7 = 1$ (see, for instance, [27, Table 5a]), and hence $\mathcal{N}_{V_{\omega_7}} \subset V_{\omega_7}$ has codimension 1 and so $\overline{O_{\omega_7}} \subsetneqq \mathcal{N}_{V_{\omega_7}}$.

(row E_8) Here, we obtain $\mathfrak{d}_1 = \mathfrak{so}_{14}$, $\mathfrak{d}_2 = \mathfrak{sl}_8$, $\mathfrak{d}_3 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_7$, $\mathfrak{d}_4 = \mathfrak{sl}_3 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_5$, $\mathfrak{d}_5 = \mathfrak{sl}_5 \oplus \mathfrak{sl}_4$, $\mathfrak{d}_6 = \mathfrak{so}_{10} \oplus \mathfrak{sl}_3$, $\mathfrak{d}_7 = \mathsf{E}_6 \oplus \mathfrak{sl}_2$, and $\mathfrak{d}_8 = \mathsf{E}_7$. The maximal dimension is reached for $i = 8$, and we get $p_{\mathsf{E}_8} = 57$. Moreover, V_{ω_8} is the adjoint representation of dimension 248, $\dim \mathcal{N}_{V_{\omega_8}} = \dim \mathsf{E}_8 - \operatorname{rank} \mathsf{E}_8 = 240$ [22, Example 2.1], and thus $\overline{O_{\omega_8}} \subsetneqq \mathcal{N}_{V_{\omega_8}}$.

(row F_4) We have $\mathfrak{d}_1 = \mathfrak{sp}_6$, $\mathfrak{d}_2 = \mathfrak{d}_3 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_3$, $\mathfrak{d}_4 = \mathfrak{so}_7$, and the maximal dimension is reached for $i = 1, 4$. This yields $p_{\mathsf{F}_4} = 15$. Moreover, V_{ω_1} is the adjoint representation of dimension 52, and thus $\dim \mathcal{N}_{V_{\omega_1}} = \dim \mathsf{F}_4 - \operatorname{rank} \mathsf{F}_4 = 48$, and so $\overline{O_{\omega_1}} \subsetneqq \mathcal{N}_{V_{\omega_1}}$. The other representation V_{ω_4} has dimension 26, is cofree, and $\dim V_{\omega_4} /\!\!/ G = 2$ (cf. [27, Table 5a]). Hence, $\dim \mathcal{N}_{\omega_4} = 24$ and thus $\overline{O_{\omega_4}} \subsetneqq \mathcal{N}_{V_{\omega_4}}$.

(row G_2) We have $\mathfrak{d}_1 = \mathfrak{d}_2 = \mathfrak{sl}_2$, and hence $p_{\mathsf{G}_2} = 5$ and $\dim O_{\omega_i} = 6$. Furthermore, $\dim V_{\omega_1} = 7$, $\dim V_{\omega_2} = 14$, and G_2 preserves a quadratic form on V_{ω_1} (see [9, Section 22.3]), which implies that $\overline{O_{\omega_1}} = \mathcal{N}_{V_{\omega_1}}$. Moreover, V_{ω_2} is the adjoint representation, $\dim \mathcal{N}_{V_{\omega_2}} = \dim \mathsf{G}_2 - \operatorname{rank} \mathsf{G}_2 = 12$, and hence $\overline{O_{\omega_2}} \subsetneqq \mathcal{N}_{V_{\omega_2}}$. ∎

Remark 5.5 The lemma above has the following consequence. Let G be a simple group. *If $\overline{O_\lambda}$ is smooth, then we are in one of the following cases (see Table 2 in Lemma 5.3):*

(1) *$G = \mathrm{SL}_n$ and $\lambda = \omega_1$ or $\lambda = \omega_n$, i.e., $\overline{O_\lambda}$ is the standard representation or its dual.*
(2) *$G = \mathrm{Sp}_{2n}$ and $\lambda = \omega_1$, i.e., $\overline{O_\lambda}$ is the standard representation.*

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G	A_3	$\mathsf{A}_n, n \neq 3$	B_n	$\mathsf{C}_n, n \geq 3$	$\mathsf{D}_n, n \geq 4$
H	B_2	$\mathsf{A}_{n-1} \times \mathsf{T}_1$	D_n	$\mathsf{C}_{n-1} \times \mathsf{A}_1$	B_{n-1}
r_G	5	$2n$	$2n$	$4(n-1)$	$2n-1$
m_G	4	$n+1$	$2n$	$2n$	$2n-1$
G	E_6	E_7	E_8	F_4	G_2
H	F_4	$\mathsf{E}_6 \times \mathsf{T}_1$	$\mathsf{E}_7 \times \mathsf{A}_1$	B_4	A_2
r_G	26	54	112	16	6
m_G	17	28	58	16	6

Table 3. Maximal reductive subgroups of simple groups.

Indeed, if $\overline{O_\lambda}$ is smooth, then $\overline{O_\lambda} = V_\lambda$ by Lemma 2.2(7), and so $\dim V_\lambda /\!\!/ G = 0$. These irreducible representations are known (see [17]):

(a) A_n: $V_{\omega_1}, V_{\omega_n}$, (b) A_n (n even > 2): $V_{\omega_2}, V_{\omega_{n-1}}$, (c) C_n: V_{ω_1}, (d) D_5: $V_{\omega_4}, V_{\omega_5}$.

(a) and (c) correspond to (1) and (2) above, and for (b) and (d), one has $\dim O_\lambda < \dim V_\lambda$.

Now, we can prove the first theorem from the introduction.

Proof (of Theorem 1.1) Theorem 1.5 implies that $X \to X/\!\!/G$ is a G-vector bundle with fiber V_λ, where λ is the type of X, and the minimal orbits are smooth. This means that $\overline{O_\lambda} = V_\lambda$, by Lemma 2.2(7), and the claim follows from Remark 5.5. ∎

5.5 The invariant r_G

In this subsection, we compute the invariant

$$r_G = \min\{\operatorname{codim}_G H \mid H \subsetneq G \text{ reductive}\},$$

which is the minimal dimension of a nontrivial affine G-orbit. These orbits are never minimal orbits, by Lemma 2.2(2).

Lemma 5.6 *Table 3 lists the types of the proper reductive subgroups H of the simple groups G of maximal dimension, their codimension $r_G = \operatorname{codim}_G H$, and the invariant m_G from Lemma 5.3. (In the table, T_1 denotes the one-dimensional torus.)*

Proof The classification of maximal subalgebras $\mathfrak{h}$ of a simple Lie algebra $\mathfrak{g}$ is due to Dynkin (see [7, 8]). His results are reformulated in [10, Chapter 6, Sections 1 and 3].

(a) If $\mathfrak{h}$ is maximal reductive of maximal rank $\ell := \operatorname{rank} \mathfrak{g}$, then the classification is given in [10, Corollary to Theorem 1.2, p. 186] (the results are listed in Tables 5

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and 6, pp. 234–235). From these tables, one gets the following candidates for reductive subalgebras of minimal codimension.[1]

$\mathfrak{g}$	$\mathsf{A}_n, n \geq 1$	$\mathsf{B}_n, n \geq 2$	$\mathsf{C}_n, n \geq 3$	$\mathsf{D}_n, n \geq 4$	
$\mathfrak{h}$	$\mathsf{A}_{n-1} \times \mathsf{T}_1$	D_n	$\mathsf{C}_{n-1} \times \mathsf{A}_1$	$\mathsf{D}_{n-1} \times \mathsf{T}_1$	
codim	$2n$	$2n$	$4(n-1)$	$4(n-1)$	
$\mathfrak{g}$	E_6	E_7	E_8	F_4	G_2
$\mathfrak{h}$	$\mathsf{A}_5 \times \mathsf{A}_1$	$\mathsf{E}_6 \times \mathsf{T}_1$	$\mathsf{E}_7 \times \mathsf{A}_1$	B_4	A_2
codim	40	54	112	16	6

(b) If $\mathfrak{h} \subset \mathfrak{g}$ is a maximal subalgebra, then it is either semisimple or parabolic [10, Theorem 1.8]. Since the Levi parts of the parabolic subalgebras have maximal rank, the second case does not produce any new candidate. It is therefore sufficient to look at the maximal semisimple subalgebras.

For the exceptional groups G, the classification is given in [10, Theorem 3.4], and one finds one new case, namely $\mathsf{F}_4 \subset \mathsf{E}_6$, which has codimension 26. Thus, the claim is proved for the exceptional groups.

(c) From now on, G is a classical group and we can use [10, Theorems 3.1–3.3]. From the first two theorems, one finds the new candidates $\mathsf{B}_{n-1} \subset \mathsf{D}_n$ of codimension $2n-1$, including $\mathsf{B}_2 \subset \mathsf{A}_3$ of codimension 5. This gives the following table.

G	SL_4	$\mathrm{SL}_n, n \neq 4$	$\mathrm{SO}_n, n \geq 4$	$\mathrm{Sp}_n, n = 2m \geq 4$
H	Sp_4	GL_{n-1}	SO_{n-1}	$\mathrm{Sp}_{n-2} \times \mathrm{SL}_2$
$\dim H$	10	$(n-1)^2$	$\frac{n^2-3n}{2}+1$	$\frac{n^2-3n}{2}+4$
$c_G := \mathrm{codim}_G H$	5	$2n-2$	$n-1$	$2n-4$

Our claim is that $c_G = r_G$, i.e., that we have found the minimal codimensions of reductive subgroups of the classical groups. In order to prove this, we have to show that [10, Theorem 3.3] does not give any reductive subgroup of smaller codimension:

If $H \subsetneq G$ *is an irreducible simple subgroup of a classical group* $G = \mathrm{SL}_n, \mathrm{SO}_n, \mathrm{Sp}_n$, *then* $\mathrm{codim}_G H \geq c_G$ (irreducible means that the representation of $H \hookrightarrow \mathrm{GL}_n$ is irreducible).

Now, the table above implies the following. Assume $n \geq 4$. If $H \subsetneq G \subset \mathrm{GL}_n$ is an irreducible subgroup of a classical group $G = \mathrm{SL}_n, \mathrm{SO}_n, \mathrm{Sp}_n$ and $\dim H < d(n) := \frac{n^2-3n}{2}+1$, then $\mathrm{codim}_G H > c_G$, and so H can be omitted.

[1] One has to be careful since the tables contain several errors.

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The following table contains the minimal dimensions of irreducible representations of the simply connected exceptional groups. They have been calculated using [9, Exercise 24.9], which says that one has only to consider the fundamental representations.

H	E_6	E_7	E_8	F_4	G_2
$\dim H$	78	133	248	52	14
λ	ω_1, ω_6	ω_7	ω_8	ω_4	ω_1
$n = \dim V_\lambda$	27	56	248	26	7

In all cases, we have $\dim H < d(n) = \frac{n^2-3n}{2} + 1$, so that $\operatorname{codim}_G H > c_G$ for an exceptional group H.

(d) It remains to consider the simple subgroups $H \subsetneq G$ of classical type where $G = \mathrm{SL}_n, \mathrm{SO}_n, \mathrm{Sp}_n$.

(d_1) The irreducible representations $H \to \mathrm{SL}_n$ of minimal dimension of a group H of classical type are given by the following table. It is obtained by using again the fact that one has only to consider the fundamental representations (see [9, Exercise 24.9]).

H	A_ℓ	B_2	$\mathsf{B}_\ell, \ell \geq 3$	$\mathsf{C}_\ell, \ell \geq 3$	D_4	$\mathsf{D}_\ell, \ell \geq 5$
$\dim H$	$\ell(\ell+2)$	10	$\ell(2\ell+1)$	$\ell(2\ell+1)$	28	$\ell(2\ell-1)$
λ	ω_1, ω_ℓ	ω_2	ω_1	ω_1	$\omega_1, \omega_3, \omega_4$	ω_1
$n = \dim V_\lambda$	$\ell+1$	4	$2\ell+1$	2ℓ	8	2ℓ

They correspond to the standard representations $\mathrm{SL}_n \subset \mathrm{GL}_n$, $\mathrm{SO}_n \subset \mathrm{GL}_n$, and $\mathrm{Sp}_n \subset \mathrm{GL}_n$, except for $\mathsf{B}_2 = \mathsf{C}_2$ where it is $\mathrm{Sp}_4 \subset \mathrm{GL}_4$. If H is not of type A, we have $\operatorname{codim}_{\mathrm{SL}_n} H > c_{\mathrm{SL}_n} = 2n-2$ except for type B_2 where $\operatorname{codim}_{\mathrm{SL}_4} \mathrm{Sp}_4 = 5 = c_{\mathrm{SL}_4}$. Moreover, if $\mathrm{SL}_k \to \mathrm{SL}_n$ is not an isomorphism, then $k < n$ and $\operatorname{codim}_{\mathrm{SL}_n} \mathrm{SL}_k > c_{\mathrm{SL}_n}$.

(d_2) Next, we consider irreducible orthogonal representations $\rho\colon H \to \mathrm{SO}_n$ for H of classical type where $n \geq 5$. If H is a candidate not already in (a), then $\operatorname{rank} H < \operatorname{rank} \mathrm{SO}_n$, and one calculates straight forwardly that $\operatorname{codim}_{\mathrm{SO}_n} H > c_{\mathrm{SO}_n} = n-1$.

(d_3) Finally, we consider irreducible symplectic representations $\rho\colon H \to \mathrm{Sp}_{2m}$ for H of classical type where $m \geq 2$. As above, if H is a candidate not already in (a), then $\operatorname{rank} H < \operatorname{rank} \mathrm{Sp}_{2m} = m$. Again, an easy calculation shows that $\operatorname{codim}_{\mathrm{Sp}_{2m}} H > c_{\mathrm{Sp}_{2m}} = 4m-4$. ∎

5.6 The invariant d_G

In this subsection, we compute the invariant

$$d_G = \min\{\dim O \mid O \text{ nonminimal quasi-affine nontrivial orbit}\}.$$

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G	A_1	A_2	A_3	$\mathsf{A}_n, n \geq 4$	B_n	$\mathsf{C}_n, n \geq 3$	$\mathsf{D}_n, n \geq 4$
r_G	2	4	5	$2n$	$2n$	$4(n-1)$	$2n-1$
d_G	2	4	5	$2n$	$2n$	$4(n-1)$	$2n-1$
m_G	2	3	4	$n+1$	$2n$	$2n$	$2n-1$

G	E_6	E_7	E_8	F_4	G_2
r_G	26	54	112	16	6
d_G	26	45	86	16	6
m_G	17	28	58	16	6

Table 4. The invariants r_G, d_G, and m_G for the simple groups.

Formula (5.1) shows that $r_G \geq d_G \geq m_G$ and that $d_G > m_G$ in case $r_G > m_G$. Comparing the values of r_G and m_G in Table 3 of Lemma 5.6, we get the following result.

Lemma 5.7 *Let G be simple and simply connected. If $r_G = d_G = m_G$, then we are in one of the following cases.*

(1) *G is of type A_1 and $d_G = 2$.*
(2) *G is of type B_n and $d_G = 2n$.*
(3) *G is of type D_n, $n \geq 4$, and $d_G = 2n-1$.*
(4) *G is of type F_4 and $d_G = 16$.*
(5) *G is of type G_2 and $d_G = 6$.*

In all other cases, we have $r_G \geq d_G > m_G$.

Proposition 5.8 *Table 4 lists the invariants r_G, d_G, and m_G for the simply connected simple algebraic groups G.*

The first and last rows of Table 4 are rows from Table 3. We have seen above that for $r_G \leq m_G + 1$, we have $d_G = r_G$ because $r_G > m_G$ implies that $d_G > m_G$. Thus, the only cases to be considered are A_n for $n \geq 4$, C_n for $n \geq 3$, E_6, E_7, and E_8.

We have seen in Section 5.3 that for a dominant weight $\lambda \in \Lambda_G$, the corresponding parabolic subgroup $P_\lambda \subset G$ and its Lie algebra $\mathfrak{p}_\lambda$ have well-defined Levi decompositions $P_\lambda = L_\lambda \ltimes U_\lambda$ where $T \subseteq L_\lambda$ and $\mathfrak{p}_\lambda := \operatorname{Lie} P_\lambda = \mathfrak{l}_\lambda \oplus \mathfrak{n}_\lambda$. In addition, we define the closed subgroup $P_{(\lambda)} := \ker(\lambda\colon P_\lambda \to \mathbb{K}^*)$, which has the Levi decomposition $P_{(\lambda)} = L_{(\lambda)} \ltimes U_\lambda$, $L_{(\lambda)} := \ker(\lambda\colon L_\lambda \to \mathbb{K}^*)$, and its Lie algebra

$$\mathfrak{p}_{(\lambda)} := \operatorname{Lie} P_{(\lambda)} = \mathfrak{l}_{(\lambda)} \oplus \mathfrak{n}_\lambda, \quad \mathfrak{l}_{(\lambda)} := \operatorname{Lie} L_{(\lambda)} = \ker(d\lambda\colon \mathfrak{l}_\lambda \to \mathbb{K}).$$

By construction, the semisimple Lie algebra $[\mathfrak{l}_\lambda, \mathfrak{l}_\lambda]$ is contained in $\mathfrak{l}_{(\lambda)}$, and they are equal in case λ is a fundamental weight ω_i. Note also that $P_{(\lambda)} = G_v$ for $v \in V_\lambda^U$, $v \neq 0$ (see Section 2.2). For an affine G-variety X and $x \in X$, we set $\mathfrak{g}_x := \operatorname{Lie} G_x$ and denote by $\mathfrak{n}_x \subseteq \mathfrak{g}_x$ the nilradical of $\mathfrak{g}_x$.

The method for proving Proposition 5.8 was communicated to us by Oksana Yakimova, who also worked out the result for the symplectic groups and for E_6.

It is based on the following lemma, which is a translation of a fundamental result of Sukhanov (see [28, Theorem 1]).

Lemma 5.9 *Let O be a nontrivial quasi-affine G-orbit. Then there exist $\lambda \in \Lambda_G$ and $x \in O$ such that $\mathfrak{g}_x \subseteq \mathfrak{p}_{(\lambda)}$ and $\mathfrak{n}_x \subseteq \mathfrak{n}_\lambda$. In particular, we get an embedding $\mathfrak{l}_x := \mathfrak{g}_x/\mathfrak{n}_x \hookrightarrow \mathfrak{l}_{(\lambda)} = \mathfrak{p}_{(\lambda)}/\mathfrak{n}_\lambda$. If O is not a minimal orbit, then* $\dim O \geq \dim \mathfrak{n}_\lambda + 2$.

Proof In Sukhanov's paper, a subgroup $L \subset G$ is called *observable* if G/L is quasi-affine. Now, [28, Theorem 1] implies that such an L is *subparabolic*, which means that there is an embedding $L \hookrightarrow Q$ such that $L_u \hookrightarrow Q_u$ where Q is the isotropy group of a highest weight vector. Translating this into the language of Lie algebras, we get the first part of the lemma.

For the second part, we note that $\mathfrak{g}_x \subsetneq \mathfrak{p}_{(\lambda)}$, so that

$$\dim O = \operatorname{codim}_{\mathfrak{g}} \mathfrak{g}_x \geq \operatorname{codim}_{\mathfrak{g}} \mathfrak{p}_{(\lambda)} + 1 = \dim O_\lambda + 1 = \operatorname{codim}_{\mathfrak{g}} \mathfrak{p}_\lambda + 2 = \dim \mathfrak{n}_\lambda + 2,$$

and the claim follows. ∎

The strategy of the proof of Proposition 5.8 is the following. Let $O = Gx \subset X$ be a nonminimal nontrivial orbit, and consider an embedding $\mathfrak{g}_x \hookrightarrow \mathfrak{p}_{(\lambda)}$ given by the lemma above.

(1) Since O is not minimal, we have $\dim O \geq \dim \mathfrak{n}_\lambda + 2$. Thus, in order to show that $\dim O \geq d_G$, one has only to consider those $\mathfrak{p}_\lambda$ with $\dim \mathfrak{n}_\lambda + 2 < d_G$ for the numbers d_G given in Table 4. For this, one first calculates $\dim \mathfrak{n}_{\omega_i}$, $i = 1, \ldots, n$, and then uses that $\dim \mathfrak{n}_\lambda \geq \dim \mathfrak{n}_{\omega_i}$ for all i such that ω_i appears in λ (see Lemma 5.2).

It turns out that in all cases, the remaining λ are fundamental weights, and we are left to study some of the embeddings $\mathfrak{g}_x \hookrightarrow \mathfrak{p}_{(\omega_i)}$.

(2) Since O is not minimal, the embedding $\mathfrak{g}_x \hookrightarrow \mathfrak{p}_{(\omega_i)}$ is strict, and hence one of the two inclusions $\mathfrak{n}_x \subseteq \mathfrak{n}_{\omega_i}$ or $\mathfrak{l}_x \subseteq \mathfrak{l}_{(\omega_i)}$ has to be strict.

(2a) If $\mathfrak{l}_x = \mathfrak{l}_{(\omega_i)}$, then $\mathfrak{n}_x$ must be a strict $\mathfrak{l}_{(\omega_i)}$-submodule of $\mathfrak{n}_{\omega_i}$. As we have seen in Remark 5.1, $\mathfrak{n}_x$ cannot contain the simple module $V(\alpha_i)$, and hence the codimension of $\mathfrak{n}_x$ in $\mathfrak{n}_{\omega_i}$ is at least $\dim V(\alpha_i)$.

(2b) If $\mathfrak{l}_x \subsetneq \mathfrak{l}_{(\omega_i)}$, then $L_x^\circ \subsetneq L_{(\omega_i)}^\circ$ is a proper reductive subgroup of the semisimple group $L_{(\omega_i)}$, and the codimension can be estimated using the values of r_H given in our tables.

Remark 5.10 In the cases of E_7 and E_8, we will have to construct quasi-affine orbits of a given dimension. For this, we will use the following result.

Let $H \subset G$ be a closed subgroup. If the character group $X(H)$ is trivial, then there is a G-module V and a $v \in V$ such that $G_v = H$.

In fact, there is a G-module V and a line $L = \mathbb{K}v \subset V$ such that $H = \operatorname{Norm}_G(L)$ [2, Chapter II, Theorem 5.1]. Since H has no characters, it acts trivially on L and so $H = G_v$.

5.6.1 The type A_n, $n \geq 4$

Suppose that $G = \mathrm{SL}_{n+1}$ and $\mathfrak{g} = \mathfrak{sl}_{n+1}$ with $n \geq 4$, and let O be a nonminimal and nontrivial quasi-affine orbit. We have to show that $\dim O \geq 2n$. We have seen above

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that it suffices to consider those embeddings $\mathfrak{g}_x \subset \mathfrak{p}_{(\lambda)}$ where $\dim \mathfrak{n}_\lambda < 2n - 2$. We have

$$\mathfrak{p}_{(\omega_i)} = \mathfrak{sl}_i \oplus \mathfrak{sl}_{n+1-i} \oplus \mathfrak{n}_{\omega_i},$$

and so $\dim \mathfrak{n}_{\omega_i} = i(n+1-i)$, which is greater than or equal to $2n-2$ for $i \neq 1, n$. Moreover, we have $\mathfrak{p}_{\omega_1+\omega_n} = (\mathfrak{sl}_{n-1} \oplus \mathbb{K}^2) \oplus \mathfrak{n}_{\omega_1+\omega_n}$, and hence $\mathfrak{p}_{(\omega_1+\omega_2)} = (\mathfrak{sl}_{n-1} \oplus \mathbb{K}) \oplus \mathfrak{n}_{\omega_1+\omega_n} = \mathfrak{gl}_{n-1} \oplus \mathfrak{n}_{\omega_1+\omega_n}$, which implies that $\dim \mathfrak{n}_{\omega_1+\omega_n} = 2n-1$. Thus, by (5.4) and Lemma 5.2, the only cases to consider are $\lambda = \omega_1$ and $\lambda = \omega_n$.

If $\mathfrak{g}_x \subsetneqq \mathfrak{p}_{(\omega_1)} = \mathfrak{sl}_n \oplus (\mathbb{K}^n)^\vee$, then we have either $\mathfrak{n}_x = (0)$ or $\mathfrak{l}_x \subsetneqq \mathfrak{sl}_n$. In the first case, we get $\dim O = \operatorname{codim} \mathfrak{g}_x = \operatorname{codim} \mathfrak{p}_{(\omega_1)} + n = 2n+1$. In the second case, $\mathfrak{l}_x$ is a reductive Lie subalgebra of $\mathfrak{sl}_n$ and thus has codimension at least $r_{\mathsf{A}_{n-1}} = 2(n-1)$ for $n > 4$ and at least 5 for $n = 4$. Hence, $\dim O = \operatorname{codim} \mathfrak{g}_x \geq \operatorname{codim} \mathfrak{p}_{(\omega_1)} + (2(n-1)-1) = 3n-2 > 2n$.

The other case $\lambda = \omega_n$ is similar.

Remark 5.11 We have just shown that, for $n \neq 3$, any quasi-affine SL_{n+1}-orbit of dimension $<2n$ is minimal. Furthermore, we have $\dim O_\lambda = \dim \mathfrak{n}_\lambda + 1$ by (5.3) and Section 5.3. In particular, since $\dim O_{\omega_i} = i(n+1-i)+1$ (see above), we get

$$\dim O_{\omega_1} = \dim O_{\omega_n} = n+1, \quad \dim O_{\omega_2} = \dim O_{\omega_{n-1}} = 2n-1,$$

and all other minimal orbits have dimension $\geq 2n$.

Note that $r_{\mathsf{A}_n} = 2n$ appears as dimension of the affine orbit $\mathrm{SL}_{n+1}/\mathrm{GL}_n$ as well as of the minimal orbit $O_{\omega_1+\omega_2}$ (see above).

5.6.2 The type C_n, $n \geq 3$

Suppose that $G = \mathrm{Sp}_{2n}$ and $\mathfrak{g} = \mathfrak{sp}_{2n}$, where $n \geq 3$, and let O be a nonminimal and nontrivial quasi-affine orbit. We have to show that $\dim O \geq 4n-4$. We have seen above that it suffices to consider those embeddings $\mathfrak{g}_x \subset \mathfrak{p}_{(\lambda)}$ where $\dim \mathfrak{n}_\lambda < 4n-6$.

For the fundamental weights, we get $\mathfrak{p}_{(\omega_j)} = \mathfrak{sl}_j \oplus \mathfrak{sp}_{2n-2j} \oplus \mathfrak{n}_{\omega_j}$. An easy calculation shows that

$$\dim \mathfrak{n}_{\omega_j} = 2jn + \frac{j(1-3j)}{2}.$$

Thus, $\dim \mathfrak{n}_{\omega_j} + 2 \geq 4n-4$ except for $j=1$, and in this case, we have $\dim \mathfrak{n}_{\omega_1} = 2n-1$ and $\operatorname{codim} \mathfrak{p}_{(\omega_1)} = 2n$. Thus, by (5.4) and Lemma 5.2, it suffices to look at the embedding $\mathfrak{g}_x \subset \mathfrak{p}_{(\omega_1)} = \mathfrak{sp}_{2n-2} \oplus \mathfrak{n}_{\omega_1}$. As a representation of $\mathfrak{sp}_{2n-2}$, we get $\mathfrak{n}_{\omega_1} = V(\alpha_1) \oplus \mathbb{K}$, $V(\alpha_1) \simeq \mathbb{K}^{2n-2}$.

Therefore, if $\mathfrak{l}_x = \mathfrak{sp}_{2n-2}$, then the codimension of $\mathfrak{g}_x$ in $\mathfrak{p}_{(\omega_1)}$ is $\geq \dim V(\alpha_1) = 2n-2$, and so $\dim O = \operatorname{codim} \mathfrak{g}_x \geq \operatorname{codim} \mathfrak{p}_{(\omega_1)} + 2(n-1) = 4n-2 > 4n-4$.

If $\mathfrak{l}_x \subsetneqq \mathfrak{sp}_{2n-2}$, then the codimension is at least $r_{\mathsf{C}_{n-1}} = 4n-8$, and so $\dim O \geq 2n + 4n - 8 = 6n-8 > 4n-4$.

Remark 5.12 We have just shown above that any quasi-affine orbit of dimension $< 4n-4$ is minimal. Furthermore, we have $\dim O_\lambda = \dim \mathfrak{n}_\lambda + 1$ by (5.3) and Section 5.3. In particular, since $\dim O_{\omega_i} = 2jn + \frac{j(1-3j)}{2}$ (see above), we get $\dim O_{\omega_1} = 2n$, and all other minimal orbits have dimension $\geq 4n-4$.

Note that $r_{C_n} = 4n - 4$ appears as dimension of an affine orbit as well as of the minimal orbit O_{ω_2}.

5.6.3 The type E_6

Let G be simply connected of type E_6 and $\mathfrak{g} = \operatorname{Lie} G$, and let O be a nonminimal and nontrivial quasi-affine orbit. We have to show that $\dim O \geq 26$. We have seen above that it suffices to consider those embeddings $\mathfrak{g}_x \hookrightarrow \mathfrak{p}_{(\lambda)}$ where $\dim \mathfrak{n}_\lambda < 24$. For the fundamental weights λ, we find

$$\mathfrak{p}_{(\omega_1)} = 10 \oplus \mathfrak{n}_{\omega_1},\ \dim \mathfrak{n}_{\omega_1} = 16 = \dim \mathfrak{n}_{\omega_6},$$
$$\mathfrak{p}_{(\omega_2)} = \mathfrak{sl}_6 \oplus \mathfrak{n}_{\omega_2},\ \dim \mathfrak{n}_{\omega_2} = 21,$$
$$\mathfrak{p}_{(\omega_3)} = (\mathfrak{sl}_2 \oplus \mathfrak{sl}_5) \oplus \mathfrak{n}_{\omega_3},\ \dim \mathfrak{n}_{\omega_3} = 25 = \dim \mathfrak{n}_{\omega_5},$$
$$\mathfrak{p}_{(\omega_4)} = (\mathfrak{sl}_3 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_3) \oplus \mathfrak{n}_{\omega_4},\ \dim \mathfrak{n}_{\omega_4} = 29.$$

Since $\dim \mathfrak{n}_{\omega_1+\omega_2} = \dim \mathfrak{n}_{\omega_2+\omega_6} = \frac{1}{2}(\dim \mathsf{E}_6 - \dim \mathsf{A}_4 - 2) = 26$ and $\dim \mathfrak{n}_{\omega_1+\omega_6} = \frac{1}{2}(\dim \mathsf{E}_6 - \dim \mathsf{D}_4 - 2) = 24$, we have only to consider the cases $\lambda \in \{\omega_1, \omega_2, \omega_6\}$.

(1) We have $\mathfrak{p}_{(\omega_1)} = 10 \oplus \mathfrak{n}_{\omega_1}$, and $\mathfrak{n}_{\omega_1} = V(\alpha_1)$ is the irreducible representation V_{ω_4} of 10 of dimension 16. Since $16 > r_{\mathsf{D}_5} = 9$, we see that the codimension of $\mathfrak{g}_x$ in $\mathfrak{p}_{(\omega_1)}$ is at least 9. Thus, $\dim O = \operatorname{codim} \mathfrak{g}_x \geq \operatorname{codim} \mathfrak{p}_{(\omega_1)} + 9 = 17 + 9 = 26$.

(2) We have $\mathfrak{p}_{(\omega_2)} = \mathfrak{sl}_6 \oplus \mathfrak{n}_{\omega_2}$, $\mathfrak{n}_{\omega_2} = V(\alpha_2) \oplus \mathbb{K}$, and $V(\alpha_2)$ is the irreducible representation $V_{\omega_3} = \bigwedge^3 \mathbb{K}^6$ of $\mathfrak{sl}_6$ of dimension 20. Since $20 > r_{\mathrm{SL}_6} = 10$, we see that the codimension of $\mathfrak{g}_x$ in $\mathfrak{p}_{(\omega_2)}$ is at least 10. Thus, $\dim O = \operatorname{codim} \mathfrak{g}_x \geq \operatorname{codim} \mathfrak{p}_{(\omega_2)} + 10 = 22 + 10 = 32$.

(3) The case $\mathfrak{p}_{(\omega_6)}$ is similar to $\mathfrak{p}_{(\omega_1)}$ from (1).

Remark 5.13 We have just shown that any quasi-affine orbit of dimension <26 is minimal. Furthermore, $\dim O_\lambda = \dim \mathfrak{n}_\lambda + 1$ by (5.3) and Section 5.3. From above, we get

$$\dim O_{\omega_1} = \dim O_{\omega_6} = 17, \quad \dim O_{\omega_2} = 22, \quad \dim O_{\omega_1+\omega_6} = 25,$$

and all other minimal orbits are of dimension ≥26 by (5.4). Moreover, $r_{\mathsf{E}_6} = 26$ appears as dimension of an affine orbit as well as of the minimal orbits O_{ω_3} and O_{ω_5}.

5.6.4 The type E_7

Let G be simply connected of type E_7 and $\mathfrak{g} = \operatorname{Lie} G$, and let O be a nonminimal and nontrivial quasi-affine orbit. We have to show that $\dim O \geq 45$. We have seen above that it suffices to consider those embeddings $\mathfrak{g}_x \subset \mathfrak{p}_{(\lambda)}$ where $\dim \mathfrak{n}_\lambda < 43$.

If λ is a fundamental weight, then we find

$$\mathfrak{p}_{(\omega_1)} = 12 \oplus \mathfrak{n}_{\omega_1},\ \dim \mathfrak{n}_{\omega_1} = 33,$$
$$\mathfrak{p}_{(\omega_2)} = \mathfrak{sl}_7 \oplus \mathfrak{n}_{\omega_2},\ \dim \mathfrak{n}_{\omega_2} = 42,$$
$$\mathfrak{p}_{(\omega_3)} = (\mathfrak{sl}_2 \oplus \mathfrak{sl}_6) \oplus \mathfrak{n}_{\omega_3},\ \dim \mathfrak{n}_{\omega_3} = 47,$$
$$\mathfrak{p}_{(\omega_4)} = (\mathfrak{sl}_3 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_4) \oplus \mathfrak{n}_{\omega_4},\ \dim \mathfrak{n}_{\omega_4} = 53,$$
$$\mathfrak{p}_{(\omega_5)} = (\mathfrak{sl}_5 \oplus \mathfrak{sl}_3) \oplus \mathfrak{n}_{\omega_5},\ \dim \mathfrak{n}_{\omega_5} = 50,$$

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$$\mathfrak{p}_{(\omega_6)} = (\mathsf{10} \oplus \mathfrak{sl}_2) \oplus \mathfrak{n}_{\omega_6},\ \dim \mathfrak{n}_{\omega_6} = 42,$$
$$\mathfrak{p}_{(\omega_7)} = \mathsf{E}_6 \oplus \mathfrak{n}_{\omega_7},\ \dim \mathfrak{n}_{\omega_7} = 27.$$

Since $\dim \mathfrak{n}_{\omega_1+\omega_2} = \frac{1}{2}(\dim \mathsf{E}_7 - \dim \mathsf{A}_5 - 2) = 48$, $\dim \mathfrak{n}_{\omega_1+\omega_6} = \frac{1}{2}(\dim \mathsf{E}_7 - \dim \mathsf{D}_4 - \dim \mathsf{A}_1 - 2) = 50$, $\dim \mathfrak{n}_{\omega_1+\omega_7} = \frac{1}{2}(\dim \mathsf{E}_7 - \dim \mathsf{D}_5 - 2) = 43$, $\dim \mathfrak{n}_{\omega_2+\omega_6} = \frac{1}{2}(\dim \mathsf{E}_7 - \dim \mathsf{A}_4 - \dim \mathsf{A}_1 - 2) = 52$, $\dim \mathfrak{n}_{\omega_2+\omega_7} = \frac{1}{2}(\dim \mathsf{E}_7 - \dim \mathsf{A}_5 - 2) = 48$, and $\dim \mathfrak{n}_{\omega_6+\omega_7} = \frac{1}{2}(\dim \mathsf{E}_7 - \dim \mathsf{D}_5 - 2) = 43$, we have only to consider the cases $\lambda \in \{\omega_1, \omega_2, \omega_6, \omega_7\}$.

(1) We have $\mathfrak{p}_{(\omega_1)} = \mathsf{12} \oplus \mathfrak{n}_{\omega_1}$, $\mathfrak{n}_{\omega_1} = V(\alpha_1) \oplus \mathbb{K}$, and $V(\alpha_1)$ is the irreducible representation V_{ω_5} of $\mathsf{12}$ of dimension $32 > r_{\mathsf{D}_6} = 11$. Thus, the codimension of $\mathfrak{g}_x$ in $\mathfrak{p}_{(\omega_1)}$ is at least 11, and so $\dim O = \operatorname{codim} \mathfrak{g}_x \geq \operatorname{codim} \mathfrak{p}_{(\omega_1)} + 11 = 34 + 11 = 45$. Moreover, the subalgebra $\mathfrak{h} := \mathsf{11} \oplus \mathfrak{n}_{\omega_1} \subset \mathfrak{g}$ is the Lie algebra of a subgroup H of codimension $34 + 11 = 45$ which has no characters. By Remark 5.10, we see that G/H is a quasi-affine orbit of dimension 45, and so $d_{\mathsf{E}_7} \leq 45$.

(2) We have $\mathfrak{p}_{(\omega_2)} = \mathfrak{sl}_7 \oplus \mathfrak{n}_{\omega_2}$, $\mathfrak{n}_{\omega_2} = V(\alpha_2) \oplus \mathbb{K}^7$, and $V(\alpha_2)$ is the irreducible representation $V_{\omega_4} = \bigwedge^4 \mathbb{K}^7$ of $\mathfrak{sl}_7$ of dimension $35 > r_{\mathsf{SL}_7} = 12$. Thus, the codimension of $\mathfrak{g}_x$ in $\mathfrak{p}_{(\omega_2)}$ is at least 12, and so $\dim O = \operatorname{codim} \mathfrak{g}_x \geq \operatorname{codim} \mathfrak{p}_{(\omega_2)} + 12 = 43 + 12 = 55 > 45$.

(3) We have $\mathfrak{p}_{(\omega_6)} = (\mathsf{10} \oplus \mathfrak{sl}_2) \oplus \mathfrak{n}_{\omega_6}$, $\mathfrak{n}_{\omega_6} = V(\alpha_6) \oplus \mathbb{K}^{10}$, and $V(\alpha_6)$ is the irreducible representation $V_{\omega_5} \otimes \mathbb{K}^2$ of $\mathsf{10} \oplus \mathfrak{sl}_2$ of dimension $2 \times 16 = 32 > r_{\mathsf{D}_5 \times \mathsf{A}_1} = 2$. Thus, the codimension of $\mathfrak{g}_x$ in $\mathfrak{p}_{(\omega_6)}$ is at least 2, and so $\dim O = \operatorname{codim} \mathfrak{g}_x \geq \operatorname{codim} \mathfrak{p}_{(\omega_6)} + 2 = 43 + 2 = 45$.

(4) We have $\mathfrak{p}_{(\omega_7)} = \mathsf{E}_6 \oplus \mathfrak{n}_{\omega_7}$, and $V(\alpha_7) = \mathfrak{n}_{\omega_7}$ is the irreducible representation V_{ω_6} of E_6 of dimension $27 > r_{\mathsf{E}_6} = 26$. Thus, the codimension of $\mathfrak{g}_x$ in $\mathfrak{p}_{(\omega_7)}$ is at least 26, and so $\dim O = \operatorname{codim} \mathfrak{g}_x \geq \operatorname{codim} \mathfrak{p}_{(\omega_7)} + 26 = 28 + 26 = 54 > 45$.

5.6.5 The type E_8

Let G be simply connected of type E_8 and $\mathfrak{g} = \operatorname{Lie} G$, and let O be a nonminimal and nontrivial quasi-affine orbit. We have to show that $\dim O \geq 86$. We have seen above that it suffices to consider those embeddings $\mathfrak{g}_x \subset \mathfrak{p}_{(\lambda)}$ where $\dim \mathfrak{n}_\lambda < 84$.

If λ is a fundamental weight, then we find

$$\mathfrak{p}_{(\omega_1)} = \mathsf{14} \oplus \mathfrak{n}_{\omega_1},\ \dim \mathfrak{n}_{\omega_1} = 78,$$
$$\mathfrak{p}_{(\omega_2)} = \mathfrak{sl}_8 \oplus \mathfrak{n}_{\omega_2},\ \dim \mathfrak{n}_{\omega_2} = 92,$$
$$\mathfrak{p}_{(\omega_3)} = (\mathfrak{sl}_2 \oplus \mathfrak{sl}_7) \oplus \mathfrak{n}_{\omega_3},\ \dim \mathfrak{n}_{\omega_3} = 98,$$
$$\mathfrak{p}_{(\omega_4)} = (\mathfrak{sl}_3 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_5) \oplus \mathfrak{n}_{\omega_4},\ \dim \mathfrak{n}_{\omega_4} = 106,$$
$$\mathfrak{p}_{(\omega_5)} = (\mathfrak{sl}_5 \oplus \mathfrak{sl}_4) \oplus \mathfrak{n}_{\omega_5},\ \dim \mathfrak{n}_{\omega_5} = 104,$$
$$\mathfrak{p}_{(\omega_6)} = (\mathsf{10} \oplus \mathfrak{sl}_3) \oplus \mathfrak{n}_{\omega_6},\ \dim \mathfrak{n}_{\omega_6} = 97,$$
$$\mathfrak{p}_{(\omega_7)} = (\mathsf{E}_6 \oplus \mathfrak{sl}_2) \oplus \mathfrak{n}_{\omega_7},\ \dim \mathfrak{n}_{\omega_7} = 83,$$
$$\mathfrak{p}_{(\omega_8)} = \mathsf{E}_7 \oplus \mathfrak{n}_{\omega_8},\ \dim \mathfrak{n}_{\omega_8} = 57.$$

Since $\dim \mathfrak{n}_{\omega_1+\omega_7} = \frac{1}{2}(\dim \mathsf{E}_8 - \dim \mathsf{D}_5 - \dim \mathsf{A}_1 - 2) = 99$, $\dim \mathfrak{n}_{\omega_1+\omega_8} = \frac{1}{2}(\dim \mathsf{E}_8 - \dim \mathsf{D}_6 - 2) = 90$, and $\dim \mathfrak{n}_{\omega_7+\omega_8} = \frac{1}{2}(\dim \mathsf{E}_8 - \dim \mathsf{E}_6 - 2) = 84$, we have only to consider the cases $\lambda \in \{\omega_1, \omega_7, \omega_8\}$.

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(1) We have $\mathfrak{p}_{(\omega_1)} = 14 \oplus \mathfrak{n}_{\omega_1}$, $\mathfrak{n}_{\omega_1} = V(\alpha_1) \oplus \mathbb{K}^{14}$, and $V(\alpha_1)$ is the irreducible representation V_{ω_7} of 14 of dimension $64 > r_{\mathsf{D}_7} = 13$. Thus, the codimension of $\mathfrak{g}_x$ in $\mathfrak{p}_{(\omega_1)}$ is at least 13, and so $\dim O = \operatorname{codim} \mathfrak{g}_x \geq \operatorname{codim} \mathfrak{p}_{(\omega_1)} + 13 = 79 + 13 = 92 > 86$.

(2) We have $\mathfrak{p}_{(\omega_7)} = (\mathsf{E}_6 \oplus \mathfrak{sl}_2) \oplus \mathfrak{n}_{\omega_7}$, and $V(\alpha_7) \subset \mathfrak{n}_{\omega_7}$ is the irreducible representation $V_{\omega_6} \otimes \mathbb{K}^2$ of $\mathsf{E}_6 \oplus \mathfrak{sl}_2$ of dimension $2 \times 27 = 54 > r_{\mathsf{E}_6 \times \mathsf{A}_1} = 2$. Thus, the codimension of $\mathfrak{g}_x$ in $\mathfrak{p}_{(\omega_7)}$ is at least 2, and so $\dim O = \operatorname{codim} \mathfrak{g}_x \geq \operatorname{codim} \mathfrak{p}_{(\omega_7)} + 2 = 84 + 2 = 86$.

(3) We have $\mathfrak{p}_{(\omega_8)} = \mathsf{E}_7 \oplus \mathfrak{n}_{\omega_8}$, $\mathfrak{n}_{\omega_8} = V(\alpha_8) \oplus \mathbb{K}$, and $V(\alpha_8)$ is the irreducible representation V_{ω_7} of E_7 of dimension $56 > r_{\mathsf{E}_7} = 54$. Thus, the codimension of $\mathfrak{g}_x$ in $\mathfrak{p}_{(\omega_8)}$ is at least 54, and so $\dim O = \operatorname{codim} \mathfrak{g}_x \geq \operatorname{codim} \mathfrak{p}_{(\omega_8)} + 54 = 58 + 54 = 112 > 86$.

(4) The subalgebra $\mathfrak{p}_{(\omega_7+\omega_8)} = \mathsf{E}_6 \oplus \mathfrak{n}_{\omega_7+\omega_8}$ corresponds to a closed subgroup $H \subset G$ of codimension $84 + 2 = 86$ which has no characters. Thus, by Remark 5.10, G/H is a quasi-affine orbit of dimension 86, and so $d_{\mathsf{E}_8} \leq 86$.

For these computations, we used the Computer Algebra Program LiE [23] (cf. the version of our paper on the arXiv [21, Section 5.7]). It can also be done directly using the following facts (see Remark 5.1).

- The Dynkin diagram of $\mathfrak{d}_i := [\mathfrak{l}_{\omega_i}, \mathfrak{l}_{\omega_i}]$ is obtained by removing the ith node from the Dynkin diagram of G.
- The Cartan numbers $\langle \alpha_i, \alpha_j \rangle$ are the coefficients of the decomposition of the lowest weight $\alpha_i|_{\mathfrak{d}_i}$ of $V(\alpha_i)$ with respect to the fundamental weights of $\mathfrak{d}_i$.
- Using the highest root $\alpha_{\max}$ of $\mathfrak{g}$ visible from the extended Dynkin diagram, we see that $\omega_{\max}|_{\mathfrak{d}_i}$ is the highest weight of a simple $\mathfrak{d}_i$-submodule of $\mathfrak{n}_{\omega_i}$ which coincides with $V(\alpha_i)$ only if $\mathfrak{n}_{\omega_i} = V(\alpha_i)$.

Here is an example suggested by the referee.

Example 5.14 In Section 5.6.5(2), one sees from the Dynkin diagram that $\mathfrak{d}_7 = \mathsf{E}_6 \oplus \mathfrak{sl}_2$, and the nonzero Cartan numbers are $\langle \alpha_7, \alpha_6 \rangle = \langle \alpha_7, \alpha_8 \rangle = -1$. By duality, the highest weight of $V(\alpha_7)$ is the sum of two fundamental weights corresponding to the simple roots α_1 of E_6 and α_8 of $\mathfrak{sl}_2$. Hence, $V(\alpha_7)$ is the tensor product of the minimal representation of E_6 with $\mathbb{K}^2$, and is of dimension $27 \times 2 = 54$.

Moreover, the only nonzero Cartan number of $\alpha_{\max}$ is $\langle \alpha_{\max}, \alpha_8 \rangle = 1$. Hence, the respective representation of $\mathfrak{d}_7$ is $\mathbb{K}^2$. Finally, since $\langle \alpha_{\max}, \alpha_8 \rangle > 0$, $\alpha := \alpha_{\max} - \alpha_8$ is a root, and $\beta := \alpha - \alpha_7$ is a root for the same reason. As $\beta + \alpha_j$ is not a root, except for $j = 7$, $\beta|_{\mathfrak{d}_7}$ is the highest weight of another simple $\mathfrak{d}_7$-submodule of $\mathfrak{n}_{\omega_7}$, with the only nonzero Cartan number $\langle \beta, \alpha_6 \rangle = 1$. Hence, this submodule is given by the other minimal representation of E_6. By dimension count, we have found all simple summands of $\mathfrak{n}_{\omega_7}$.

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Departement Mathematik und Informatik, Universität Basel, Spiegelgasse 1, CH-4051 Basel, Switzerland
e-mail: hanspeter.kraft@unibas.ch

Institut für Mathematik, Friedrich-Schiller-Universität Jena, Ernst-Abbe-Platz 1-2,
D-07743 Jena, Germany
e-mail: andriyregeta@gmail.com

Université Angers, CNRS, LAREMA, SFR MATHSTIC, F-49045 Angers, France
e-mail: susanna.zimmermann@univ-angers.fr

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