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ON MAXIMAL TORSION RADICALS, II

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Let R be an associative ring with identity, and let ${}_R\mathcal{M}$ denote the category of unital left R-modules. The Walkers [**6**] raised the question of characterizing the maximal torsion radicals of ${}_R\mathcal{M}$, and showed that if R is commutative and Noetherian, then there is a one-to-one correspondence between maximal torsion radicals and minimal prime ideals of R [**6**, Theorem 1.29]. Popescu announced [**5**, Theorem 2.5] that the result remains valid for commutative rings with Gabriel dimension (in the terminology of [**2**]). Theorem 4.6 below shows that the result holds for rings (not necessarily commutative) with Krull dimension on either the left or right, extending the previous theorem for right Noetherian rings which appeared in [**1**]. This paper investigates rings with the property that the above correspondence holds, and moreover, that every torsion radical is contained in a maximal torsion radical. This property is a Morita invariant, and extends to finite direct products and (in the commutative case) polynomial rings, so that a variety of examples can be constructed which satisfy the property, since it holds for any domain or ring with Krull dimension.

The definitions and notation used will be those of [**1**]. A subfunctor σ of the identity on ${}_R\mathcal{M}$ is called a radical of ${}_R\mathcal{M}$ if $\sigma(M/\sigma(M)) = 0$ for all $M \in {}_R\mathcal{M}$, and a torsion radical if in addition $\sigma(M') = M' \cap \sigma(M)$ for all submodules $M' \subseteq M$. The module ${}_RM$ is called σ-torsionfree if $\sigma(M) = 0$, and a submodule $M' \subseteq M$ is called σ-closed if M/M' is σ-torsionfree. For ${}_RM$, the smallest torsion radical σ for which M is σ-torsionfree is $\sigma = \text{rad}_{E(M)}$, where $E(M)$ is the injective envelope of M, and for any module ${}_RX$, $\text{rad}_{E(M)}(X)$ is the intersection of kernels of R-homomorphisms from X to $E(M)$. The complete ring of quotients of R, denoted $Q_{\max}(R)$, is determined by $\text{rad}_{E(R)}$.

An ideal K of R is called a torsion ideal if $K = \sigma(R)$ for some torsion radical σ of ${}_R\mathcal{M}$. Since each torsion radical is determined by an injective module, the torsion ideals of R are just the annihilators of injective left R-modules. By [**4**, Corollary, p. 33], K is a torsion ideal if and only if K is the annihilator of $E(R/K)$, and this occurs if and only if for each $k \in K$ and each $r \in R \backslash K$ there exists $s \in R$ such that $sk = 0$ and $sr \notin K$. Thus every element in a proper torsion ideal is a zero-divisor.

If ρ and σ are radicals such that $\rho(M) \subseteq \sigma(M)$ for all $M \in {}_R\mathcal{M}$, then the notation $\rho \leqq \sigma$ will be used. A radical or torsion radical is called maximal if it is proper (not the identity functor) and maximal (among radicals or torsion radicals, respectively) with respect to the relation $\leqq$. If P and P' are prime

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ideals of a commutative ring R, then $\text{rad}_{E(R/P')} \leqq \text{rad}_{E(R/P)}$ if and only if $P \subseteq P'$. This motivates the definition of a relation $\leqq$ for ideals, with $A \leqq B$ for ideals A, B of R if $\text{rad}_{E(R/B)} \leqq \text{rad}_{E(R/A)}$. If P and P' are prime ideals, then $P \leqq P'$ implies $P \subseteq P'$, although the converse need not hold. (See [**1**].) By the remarks following Proposition 3.3 of [**1**], $A \leqq B$ if and only if for each ideal C properly containing A there exist $r \in C\backslash A$ and $s \in R\backslash B$ such that $Ar^{-1} \subseteq Bs^{-1}$, where $Ar^{-1} = \{x \in R : xr \in A\}$. If the ideal K is contained in A and B, then $A \leqq B$ (in ${}_R\mathscr{M}$) if and only if $A/K \leqq B/K$ (in ${}_{R/K}\mathscr{M}$).

4. Rings with enough minimal prime ideals.

4.1 THEOREM. *The following conditions are equivalent for a ring R.*

(a) (i) *Every proper torsion radical of ${}_R\mathscr{M}$ is contained in a maximal torsion radical.*

(ii) *Maximal torsion radicals of ${}_R\mathscr{M}$ are precisely those which are determined by $E(R/P)$ for a minimal prime ideal P of R.*

(b) (i) *Every proper torsion radical of ${}_R\mathscr{M}$ is contained in a maximal radical.*

(ii) *For each prime ideal P' of R there exists a minimal prime ideal P with $P \leqq P'$.*

(c) *Every nonzero injective left R-module contains a submodule whose annihilator is a minimal prime ideal.*

(d) *For each proper torsion ideal K of R there exists a minimal prime ideal P of R with $P \leqq K$.*

Proof. (a) $\Rightarrow$ (b). Let σ be a proper torsion radical of ${}_R\mathscr{M}$. By assumption σ is contained in a maximal torsion radical of the form $\text{rad}_{E(R/P)}$, for a prime ideal P. Then $\sigma \leqq \text{rad}_{E(R/P)} \leqq \text{rad}_{R/P}$, and by [**1**, Theorem 1.3], $\text{rad}_{R/P}$ is a maximal radical. Similarly, if P' is a prime ideal, then $\text{rad}_{E(R/P')} \leqq \text{rad}_{E(R/P)}$ for a minimal prime P, and thus $P \leqq P'$.

(b) $\Rightarrow$ (c). Let M be a nonzero injective left R-module. Then by assumption rad_M is contained in a maximal radical, which must be of the form $\text{rad}_{R/P'}$ for some prime ideal P'. Since rad_M is a torsion radical and $\text{rad}_M \leqq \text{rad}_{R/P'}$, it follows that $\text{rad}_M \leqq \text{rad}_{E(R/P')}$, and by (ii), $\text{rad}_{E(R/P')} \leqq \text{rad}_{E(R/P)}$ for a minimal prime ideal P. Then R/P is rad_M-torsionfree, and so $P = \text{Ann}(\{m \in M : Pm = 0\})$.

(c) $\Rightarrow$ (d). If K is a proper torsion ideal, then $E(R/K)$ is nonzero and hence contains a submodule whose annihilator is a minimal prime P, so R/P is $\text{rad}_{E(R/K)}$-torsionfree, and therefore $P \leqq K$.

(d) $\Rightarrow$ (a). Let P be a minimal prime ideal of R. If $\text{rad}_{E(R/P)} \leqq \sigma$ for a proper torsion radical σ, then $\sigma(R) \neq R$, and so by assumption there exists a minimal prime ideal P' with $P' \leqq \sigma(R)$. Then

$$\text{rad}_{E(R/P)} \leqq \sigma \leqq \text{rad}_{E(R/\sigma(R))} \leqq \text{rad}_{E(R/P')},$$

and so $P' \leqq P$, which implies $P' = P$ and $\sigma = \text{rad}_{E(R/P)}$. Thus $\text{rad}_{E(R/P)}$ is a maximal torsion radical. It is then clear from the assumption that every proper torsion radical is contained in a maximal torsion radical, and furthermore, that every maximal torsion radical is of the form $\mathrm{rad}_{E(R/P)}$ for a minimal prime ideal P.

4.2 COROLLARY. *Let R be a prime ring. Then the conditions of Theorem* 4.1 *hold $\Leftrightarrow$ every nonzero injective left R-module is faithful.*

Proof. ($\Rightarrow$) If R satisfies condition (c) of Theorem 4.1, then every nonzero injective left R-module contains a submodule whose annihilator is the zero ideal, and so the module is faithful.

($\Leftarrow$) If every nonzero injective left R-module is faithful, then since the torsion ideals of R are just the annihilators of injective left R-modules, R has no nontrivial torsion ideals, and so condition (d) of Theorem 4.1 must hold.

In a von Neumann regular ring, every ideal is a torsion ideal [**4**, Proposition 2.3], and so a prime von Neumann regular ring satisfying Theorem 4.1 must be simple. Thus prime rings which do not satisfy Theorem 4.1 can easily be found. On the other hand, since a nontrivial torsion ideal contains zero-divisors, any domain has only trivial torsion ideals and thus as in the proof of Corollary 4.2 it must satisfy the conditions of Theorem 4.1. Furthermore, this implies that if $P \subseteq P'$ are prime ideals of a ring R such that R/P is a domain, then $P \leqq P'$ since $0 \leqq P'/P$ in ${}_{R/P}\mathscr{M}$.

If a ring R has no nilpotent elements modulo its prime radical, then it follows (from unpublished class notes of Kaplansky) that R/P is a domain for every minimal prime ideal P of R. In this case condition (b)(ii) of Theorem 4.1 must hold. In general, condition (b)(ii) holds if R/P satisfies the conditions of Theorem 4.1 for all minimal prime ideals P. As noted in [**1**, Proposition 3.2], condition (b)(i) holds if R satisfies the maximum condition for two-sided ideals.

An ideal A of R is rad_M-closed if and only if A is the annihilator of a submodule of M, and so an equivalent formulation of condition (c) of Theorem 4.1 is the following: for every proper torsion radical σ of ${}_R\mathscr{M}$ there exists a σ-closed minimal prime ideal of R. If R is commutative and P is a prime ideal of R, then for any $r \in R\backslash P$, $Pr^{-1} = P$, and so for an ideal K, $P \leqq K$ if and only if $P \subseteq Ks^{-1}$ for some $s \in R\backslash K$. Thus if R is commutative, condition (d) can be stated as follows: for each proper torsion ideal K of R there exists a prime ideal P and an element $s \in R\backslash K$ such that $Ps \subseteq K$.

The next proposition gives a condition under which a prime ring satisfies Theorem 4.1 on both the left and right. It may be of independent interest to state the proposition as follows.

4.3 PROPOSITION. *Let R be a prime ring such that $Q_{\max}(R)$ is simple Artinian. Then every nonzero injective* (left or right) *R-module is faithful.*

Proof. Since R is a prime ring, every nonzero two-sided ideal of R is essential as a left ideal. By [**3**, Proposition 3, p. 110], every essential left ideal of R

contains an element which is not a zero-divisor, since $Q_{\max}(R)$ is simple Artinian. This implies that R has no nontrivial torsion ideals on either the left or right, and so every nonzero injective R-module is faithful.

For convenience, the following definition will be used.

4.4 *Definition.* The ring R has enough minimal prime ideals (on the left) if for each proper torsion ideal K of R there exists a minimal prime ideal P of R with $P \leqq K$ (in ${}_R\mathcal{M}$).

4.5 LEMMA. *The ring R has enough minimal prime ideals if*

(i) *every ideal of R contains a product of prime ideals, and*

(ii) *R/P has enough minimal prime ideals for each minimal prime ideal P of R.*

Proof. Let M be a nonzero injective left R-module. Then $\operatorname{Ann}(M)$ contains a product $P_1 \cdots P_n$ of prime ideals P_i, $i = 1, \cdots, n$. Since $P_1 \cdots P_n \cdot M = 0$, either $P_2 \cdots P_n \cdot M$ is nonzero and annihilated by P_1, or else $P_2 \cdots P_n \cdot M = 0$. Continuing this process, it follows that for some i, P_i annihilates a nonzero submodule of M. Then P_i contains a minimal prime ideal P of R, and since M is an injective R-module, it can be shown that $M' = \{m \in M : Pm = 0\}$ is a (nonzero) injective R/P-module. It follows from condition (ii) and Corollary 4.2 that M' is faithful as an R/P-module, and so $P = \operatorname{Ann}(M')$. This shows by Theorem 4.1 (c) that R has enough minimal prime ideals.

The Krull dimension of a module ${}_R\mathcal{M}$ is defined by transfinite recursion as follows: if $M = 0$, then $K\text{-dim}(M) = -1$; if α is an ordinal and $K\text{-dim}(M) \not< \alpha$, then $K\text{-dim}(M) = \alpha$ provided there is no infinite descending chain $M = M_0 \supset M_1 \supset \ldots$ of submodules M_i such that for $i = 1, 2, \ldots, K\text{-dim}(M_{i-1}/M_i) \not< \alpha$. If no such ordinal α exists, then M does not have Krull dimension. The Krull dimension of R (on the left) is $K\text{-dim}({}_RR)$. A left Noetherian ring has Krull dimension, and rings with Krull dimension exhibit many of the properties of left Noetherian rings. (See the excellent exposition in [**2**].)

4.6 THEOREM. *The ring R has enough minimal prime ideals if it has Krull dimension on either the left or right.*

Proof. Suppose that R has Krull dimension. By [**2**, Theorem 7.4], every ideal of R contains a product of prime ideals. For any minimal prime ideal P of R, by [**2**, Corollary 3.4], R/P is left (right) Goldie if R has Krull dimension on the left (right). Proposition 4.3 then shows that R/P has enough minimal prime ideals, and the desired conclusion follows from Lemma 4.5.

4.7 PROPOSITION. *Having enough minimal prime ideals is a Morita invariant property.*

Proof. Suppose that rings R and S are Morita equivalent, i.e., there exists a category equivalence $F : {}_R\mathcal{M} \rightarrow {}_S\mathcal{M}$. Then the functor F preserves radicals, torsion radicals, and the relation $\leqq$ for radicals, and so condition (b)(i) of Theorem 4.1 holds for S if it holds for R. If A is a two-sided ideal of R, then the image under F of the class of R/A-modules is the class of S/B-modules for some two-sided ideal B of S. (This can be shown by using [**6**, Proposition 5.15].) The image under F of $\text{rad}_{E(R/A)}$ is $\text{rad}_{E(R/B)}$, since $\sigma = \text{rad}_{E(R/A)}$ is characterized as the largest torsion radical σ for which every R/A-torsionless module is σ-torsionfree. (A module M is R/A-torsionless if it can be embedded in a direct product of copies of R/A.) This shows that the one-to-one correspondence between two-sided ideals of R and S preserves the relation $\leqq$ for ideals, and since it also preserves prime ideals, condition (b)(ii) of Theorem 4.1 holds for S if it holds for R.

4.8 PROPOSITION. *A finite direct product of rings has enough minimal prime ideals if and only if each factor does.*

Proof. It is sufficient to prove the proposition for $R = R_1 \times R_2$. Suppose that R_1 and R_2 have enough minimal prime ideals, and let K be a proper torsion ideal of R. It can be shown that $K = K_1 \times K_2$ for torsion ideals $K_1 \subseteq R_1$ and $K_2 \subseteq R_2$, and one of these must be proper, say K_1. Then by assumption there exists a minimal prime ideal P_1 with $P_1 \leqq K_1$, and $P \leqq K$ for the minimal prime ideal $P = P_1 \times R_2$.

Conversely, suppose that $R = R_1 \times R_2$ has enough minimal prime ideals, and that K_1 is a proper torsion ideal of R_1. Then $K_1 \times R_2$ is a proper torsion ideal of R, and so there exists a minimal prime ideal P of R such that $P \leqq K_1 \times R_2$. It can be shown that P must be of the form $P = P_1 \times R_2$ for a minimal prime ideal P_1 of R_1, and then $P_1 \leqq K_1$, which shows that R_1 has enough minimal prime ideals.

4.9 PROPOSITION. *If the ring R is commutative and has enough minimal prime ideals, then so does the polynomial ring $R[X]$.*

Proof. Let M be a nonzero injective $R[X]$-module. It can be shown that M is injective as an R-module, so by assumption there exists an R-submodule of M whose annihilator in R is a prime ideal P. Then $M' = \{m \in M : P[X] \cdot m = 0\}$ is a nonzero, injective $R[X]/P[X]$-module, and since $R[X]/P[X]$ is a domain, M' is a faithful $R[X]/P[X]$-module, so $P[X] = \text{Ann}(M')$. Since $R[X]$ is commutative, this establishes condition (c) of Theorem 4.1.

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