ON TOUCHARD POLYNOMIALS

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1. Introduction. Wyman and Moser **(5)** have recently given an explicit form for some polynomials of Touchard **(4)**. Equation (6) of this note will give a simpler expression for these polynomials. Subsequently, generating functions (9) and (12) will be obtained.

2. A new expression. The expression given by Wyman and Moser in **(5)** is

$$Q_n(x) = 2^n\, n! \binom{2n}{n}^{-1} \sum_{r=0}^{[\frac{1}{2}n]} \binom{2x+n-2r}{n-2r}\binom{x}{r}^2. \tag{1}$$

If (1) is written in generalized hypergeometric form it becomes

$$Q_n(x) = \frac{n!\,(2x+1)_n}{(\frac{1}{2})_n\, 2^n}\, {}_4F_3\left[\begin{matrix} -\frac{1}{2}n,\ \frac{1}{2}(-n+1),\ -x,\ -x; \\ 1,\ -x-\frac{1}{2}n,\ -x+\frac{1}{2}-\frac{1}{2}n; \end{matrix}\ 1\right]. \tag{2}$$

For the time being, restrict x to the values

$$x = m, \qquad m = 0, 1, 2, \ldots, \tag{3}$$

and use the transformation formula from Bailey (**1**, p. 85, eqn. (3)):

$$\begin{aligned} {}_3F_2&\left[\begin{matrix} -n,\ -n,\ -m; \\ 1,\ -n-m; \end{matrix}\ 1\right] \\ &= \frac{(1+\frac{1}{2}n)_m\,(\frac{1}{2}+\frac{1}{2}n)_m}{(\frac{1}{2})_m\,(1+n)_m}\, {}_4F_3\left[\begin{matrix} -\frac{1}{2}n,\ \frac{1}{2}(-n+1),\ -m,\ -m; \\ 1,\ -m-\frac{1}{2}n,\ -m+\frac{1}{2}-\frac{1}{2}n; \end{matrix}\ 1\right]. \end{aligned} \tag{4}$$

Then

$$\begin{aligned} Q_n(m) &= \frac{n!\,(2m+1)_n\,(\frac{1}{2})_m\,(1+n)_m}{(\frac{1}{2})_n\, 2^n (1+\frac{1}{2}n)_m\,(\frac{1}{2}+\frac{1}{2}n)_m}\, {}_3F_2\left[\begin{matrix} -n,\ -n,\ -m; \\ 1,\ -n-m; \end{matrix}\ 1\right] \\ &= \frac{(1+m)_n\, n!}{(\frac{1}{2})_n\, 2^n}\, {}_3F_2\left[\begin{matrix} -n,\ -n,\ -m; \\ 1,\ -n-m; \end{matrix}\ 1\right]. \end{aligned} \tag{5}$$

Thus it has been shown that the $Q_n(x)$ of (2) reduces to

$$Q_n(x) = \frac{(1+x)_n\, n!}{(\frac{1}{2})_n\, 2^n}\, {}_3F_2\left[\begin{matrix} -n,\ -n,\ -x; \\ 1,\ -n-x; \end{matrix}\ 1\right] \tag{6}$$

for the special case of x any non-negative integer. But then, for fixed n, (1) and (6) (in the form (7) below) give two polynomials of degree n in x which coincide at an infinite number of values. Hence (2) and (6) are identical and

Received August 15, 1956.

(6) is a valid representation for general x (except for negative integers, in which case (6) is indeterminate).

By the many transformations possible on a ${}_3F_2$ of unit argument, equation (6) can be converted to other forms.

Equation (6) here is equivalent to the result obtained independently by Carlitz in equation (10) of the immediately preceding work **(2)**. This equivalence may be demonstrated by the formula given by Bailey **(1**, p. 22**)**:

$$\Gamma(\alpha_{123})\ \Gamma(\alpha_{124})\ \Gamma(\alpha_{125})\ Fp(0;45) = (-1)^m\ \Gamma(\alpha_{125})\ \Gamma(\alpha_{025})\ \Gamma(\alpha_{015})\ Fn(5;02).$$

Carlitz has shown that the $Q_n(x)$ are essentially the Bateman polynomials $F_n(2x+1)$. Hence the results to be obtained below may also be interpreted as results on the F_n.

3. Some more generating functions. The form (6) makes available generating functions in addition to the one given by Wyman and Moser in **(5)**. Convert (6) into:

$$Q_n(x) = \frac{n!\,n!\,n!}{(\frac{1}{2})_n\,2^n}\sum_{k=0}^{n}\frac{(1+x)_{n-k}\,(-x)_k\,(-1)^k}{(n-k)!\,(n-k)!\,k!\,k!}. \tag{7}$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{Q_n(x)\,(\frac{1}{2})_n\,(2t)^n}{n!\,n!\,n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(1+x)_{n-k}\,(-x)_k\,(-1)^k\,t^n}{(n-k)!\,(n-k)!\,k!\,k!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(1+x)_n\,(-x)_k\,(-1)^k\,t^{n+k}}{n!\,n!\,k!\,k!}\\ &= {}_1F_1(1+x;1;t)\cdot{}_1F_1(-x;1;-t).\end{aligned} \tag{8}$$

The last line may be converted by Kummer's first formula to get the generating function:

$$e^t[{}_1F_1(-x;1;-t)]^2 = \sum_{n=0}^{\infty}\frac{Q_n(x)\,(\frac{1}{2})_n\,(2t)^n}{n!\,n!\,n!}. \tag{9}$$

It should be noted that for x a non-negative integer, the ${}_1F_1$ on the left of (9) becomes a Laguerre polynomial, so that (9) is:

$$e^t[L_x(-t)]^2 = \sum_{n=0}^{\infty}\frac{Q_n(x)\,(\frac{1}{2})_n\,(2t)^n}{n!\,n!\,n!}. \tag{10}$$

For the next generating function, sum (7) as indicated below to get:

$$\sum_{n=0}^{\infty}\frac{Q_n(x)\,(\frac{1}{2})_n\,(2t)^n}{n!\,n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(1+x)_{n-k}\,(-x)_k\,(-1)^k\,n!\,t^n}{(n-k)!\,(n-k)!\,k!\,k!} \tag{11}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(1+x)_n (-x)_k (-1)^k (n+k)!\, t^{n+k}}{n!\, n!\, k!\, k!}$$
$$= \sum_{n=0}^{\infty} \frac{(1+x)_n t^n}{n!} \sum_{k=0}^{\infty} \frac{(-x)_k (-t)^k (1+n)_k}{k!\, k!}$$
$$= \sum_{n=0}^{\infty} \frac{(1+x)_n t^n}{n!} {}_2F_1\left[\begin{matrix} 1+n, -x; \\ 1; \end{matrix} -t\right]$$
$$= (1+t)^x \sum_{n=0}^{\infty} \frac{(1+x)_n t^n}{n!} {}_2F_1\left[\begin{matrix} -n, -x; \\ 1; \end{matrix} \frac{t}{t+1}\right].$$

By a well-known formula due to Chaundy (**3**, p. 62) the last line above sums to give the generating function:

(12)
$$(1+t)^x (1-t)^{-1-x} {}_2F_1\left[\begin{matrix} -x, 1+x; \\ 1; \end{matrix} \frac{-t^2}{1-t^2}\right] = \sum_{n=0}^{\infty} \frac{Q_n(x)\, (\frac{1}{2})_n\, (2t)^n}{n!\, n!}.$$

The ${}_2F_1$ in (12) is essentially a Legendre polynomial if x is a nonnegative integer, in which case (12) becomes

(13)
$$(1+t)^x (1-t)^{-1-x} P_x\left(\frac{1+t^2}{1-t^2}\right) = \sum_{n=0}^{\infty} \frac{Q_n(x)\, (\frac{1}{2})_n\, (2t)^n}{n!\, n!}.$$

References

1. W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge Tract No. 32 (1935).

2. L. Carlitz, *Some polynomials of Touchard connected with the Bernoulli numbers*, Can. J. Math., *9* (1957), 188–190.

3. T. W. Chaundy, *An extension of hypergeometric functions*, Quart. J. Math., *14* (1943), 55–78.

4. Jacques Touchard, *Nombres exponentiels et nombres de Bernoulli*, Can. J. Math., *8* (1956), 305–320.

5. Max Wyman and Leo Moser, *On some polynomials of Touchard*, Can. J. Math., *8* (1956), 321–322.

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https://doi.org/10.4153/CJM-1957-022-6 Published online by Cambridge University Press