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SOME QUESTIONS ABOUT ROTUNDITY AND RENORMINGS IN BANACH SPACES

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Abstract

In this paper, we show some results involving classical geometric concepts. For example, we characterize rotundity and Efimov-Stechkin property by mean of faces of the unit ball. Also, we prove that every almost locally uniformly rotund Banach space is locally uniformly rotund if its norm is Fréchet differentiable. Finally, we also provide some theorems in which we characterize the (strongly) exposed points of the unit ball using renormings.

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1. Introduction

For a subset C of a bounded closed convex subset H of a T_2 topological vector space X, it is well known that C is

(i) a *face* of H if it is closed convex and for every $x, y \in H$ and every $\alpha \in (0, 1)$ such that $\alpha x + (1-\alpha)y \in C$, then $x, y \in C$;

(ii) an *exposed face* of H if there exists f in X^* such that $C = \{x \in H : f(x) = \sup(f(H))\}$;

(iii) a *strongly exposed face* of H if there exists f in X^* satisfying $C = \{x \in H : f(x) = \sup(f(H))\}$ and for every open subset U of H with $C \subseteq U$, there exists $\delta > 0$ such that $\operatorname{slc}(H, f, \delta) \subseteq U$ (where $\operatorname{slc}(H, f, \delta) = \{h \in H : f(h) \geq \sup(f(H)) - \delta\}$ is the slice of H determined by f and δ).

If c is an element of H, then c is

(i) an *extreme point* (EXT) of H if $\{c\}$ is a face of H;

(ii) an *exposed point* (EXP) of H if $\{c\}$ is an exposed face of H;

(iii) a *strongly exposed point* (SEXP) of H if $\{c\}$ is a strongly exposed face of H.

It is not difficult to check that c is

(i) an EXT of H if and only if $H \backslash \{c\}$ is convex;

(ii) an EXP of H if and only if there exists f in X^* satisfying $f(y) < f(c)$ for every $y \in H \backslash \{c\}$;

(iii) an SEXP of H if and only if there exists f in X^* such that for every sequence $(y_n)_{n \in \mathbb{N}}$ in H with $(f(y_n))_{n \in \mathbb{N}}$ convergent to $\sup(H)$, $(y_n)_{n \in \mathbb{N}}$ converges to c.

It is well known (see [2–4]) that a point x of the unit sphere of a Banach space X is

(i) a *rotund point* (R) of B_X if every y in S_X, such that $\|(x+y)/2\| = 1$, satisfies $x = y$;

(ii) a *midpoint locally uniformly rotund point* (MLUR) of B_X if for every pair of sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in S_X such that $((x_n + y_n)/2)_{n \in \mathbb{N}}$ converges to x, both $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ converge to x;

(iii) an SEXP of B_X if it is an SEXP of B_X for the vector topology given by the norm of X;

(iv) an *almost locally uniformly rotund point* (ALUR) of B_X if for every pair of sequences $(x_n)_{n \in \mathbb{N}}$ in S_X and $(f_m)_{m \in \mathbb{N}}$ in S_{X^*} such that $\lim_m(\lim_n(f_m((x_n+x)/2))) = 1$, $(x_n)_{n \in \mathbb{N}}$ converges to x;

(v) a *locally uniformly rotund point* (LUR) of B_X if every sequence $(y_n)_{n \in \mathbb{N}}$ in S_X such that $(\|(x+y_n)/2\|)_{n \in \mathbb{N}}$ converges to 1, converges to x.

Here we have the following diagram of implications:

$$
\begin{array}{ccccccc}
 & & \mathrm{R} & \Longrightarrow & \mathrm{EXP} & \Longrightarrow & \mathrm{EXT} \\
 & & \Uparrow & & \Uparrow & & \Uparrow \\
\mathrm{LUR} & \Longrightarrow & \mathrm{ALUR} & \Longrightarrow & \mathrm{SEXP} & \Longrightarrow & \mathrm{MLUR}.
\end{array}
$$

It is said that a Banach space is respectively R, MLUR, SEXP, ALUR or LUR if every point of its unit sphere is an R, MLUR, SEXP, ALUR or LUR point of its unit ball.

It is well known (see [5, Chapter 5.3]) that a point x of the unit sphere of a Banach space X is

(i) a *smooth point* of B_X if every sequence $(f_n)_{n \in \mathbb{N}}$ in S_{X^*} such that $(f_n(x))_{n \in \mathbb{N}}$ converges to 1, verifies that $(f_n)_{n \in \mathbb{N}}$ is ω^*-convergent;

(ii) a *strongly smooth point* of B_X if every sequence $(f_n)_{n \in \mathbb{N}}$ in S_{X^*} such that $(f_n(x))_{n \in \mathbb{N}}$ converges to 1, verifies that $(f_n)_{n \in \mathbb{N}}$ is convergent.

It can be checked that

(i) x is a smooth point of B_X if and only if the norm of X is Gâteaux differentiable at x;

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(ii) x is a strongly smooth point of B_X if and only if the norm of X is Fréchet differentiable at x.

It is said that a Banach space is *(strongly) smooth* if every point of its unit sphere is a (strongly) smooth point of its unit ball.

2. Rotundity and smoothness

REMARK 2.1. Let X be a Banach space. We say that

(i) X is *strongly rotund* if for every sequence $(x_n)_{n\in\mathbb{N}}$ in S_X and for every f in S_{X^*} such that $(f(x_n))_{n\in\mathbb{N}}$ converges to 1, $(x_n)_{n\in\mathbb{N}}$ is convergent (see [5, pages 467–472]);

(ii) X has the *Efimov-Stechkin property* if for every sequence $(x_n)_{n\in\mathbb{N}}$ in S_X and for every f in S_{X^*} such that $(f(x_n))_{n\in\mathbb{N}}$ converges to 1, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence (see [5, pages 478–479]);

(iii) X is *almost-rotund* if all closed convex subsets of S_X are compact [1].

Ivan Singer proved in 1964 (see [6]) the following:

(a) A Banach space is strongly rotund if and only if it has the Efimov-Stechkin property and is rotund.

(b) A Banach space has the Efimov-Stechkin property if and only if it is reflexive and has the Radon-Riesz property.

THEOREM 2.2. *Let X be a Banach space. The following assertions are equivalent*:

(i) *X has the Efimov-Stechkin property.*

(ii) *X is reflexive, almost-rotund and every exposed face of B_X is a strongly exposed face of B_X.*

PROOF. Assume that (i) holds. It is easy to see that X is reflexive and almost-rotund. Let C be an exposed face of B_X. Let $f \in \mathsf{S}_{X^*}$ be such that $C = f^{-1}(\{1\}) \cap \mathsf{B}_X$. Let U be an open subset of B_X such that $C \subseteq U$. Suppose that there does not exist $\delta > 0$ such that $\operatorname{slc}(\mathsf{B}_X, f, \delta) \subseteq U$. Then, for every $n \in \mathbb{N}$, there exists $x_n \in \operatorname{slc}(\mathsf{B}_X, f, 1/n)\backslash U$. It is clear that the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to 1. Therefore, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence to some element of C. However, the last assertion is in contradiction with the fact that $x_n \notin U$ for every $n \in \mathbb{N}$.

Assume that (ii) holds. Let $g \in \mathsf{S}_{X^*}$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in S_X such that $(g(x_n))_{n\in\mathbb{N}}$ converges to 1. Since $g^{-1}(\{1\}) \cap \mathsf{B}_X$ is an exposed face of B_X, there exists a functional f in S_{X^*} that characterizes it as a strongly exposed face of B_X. Check that $(f(x_n))_{n\in\mathbb{N}}$ converges to 1. Since X is reflexive, we can suppose, without loss of generality, that $(x_n)_{n\in\mathbb{N}}$ ω-converges to some element in $g^{-1}(\{1\}) \cap \mathsf{B}_X$. Since

$g^{-1}(\{1\}) \cap \mathsf{B}_X = f^{-1}(\{1\}) \cap \mathsf{B}_X$, we deduce that $(f(x_n))_{n\in\mathbb{N}}$ converges to 1. Now we will make a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that

$$\operatorname{dist}\big(\{x_{n_k} : k \in \mathbb{N}\}, f^{-1}(\{1\}) \cap \mathsf{B}_X\big) = 0.$$

For every $k \in \mathbb{N}$, let U_k be an open subset of B_X such that $f^{-1}(\{1\}) \cap \mathsf{B}_X \subset U_k$ and $\operatorname{dist}(\{u\}, f^{-1}(\{1\}) \cap \mathsf{B}_X) < 1/k$ for every $u \in U_k$. For every $k \in \mathbb{N}$, let δ_k be a positive real number such that $\operatorname{slc}(\mathsf{B}_X, f, \delta_k) \subseteq U_k$. For every $k \in \mathbb{N}$, let n_k be a natural number such that $n_k > n_{k-1}$ and $x_{n_k} \in \operatorname{slc}(\mathsf{B}_X, f, \delta_k)$. The compacity of $f^{-1}(\{1\}) \cap \mathsf{B}_X$ allows us to deduce the result. □

COROLLARY 2.3. *Let X be a Banach space. The following assertions are equivalent*:

(i) *X is strongly rotund.*
(ii) *X is reflexive and strongly exposed.*

THEOREM 2.4. *Let X be a Banach space and let $x \in \mathsf{S}_X$. The following assertions are equivalent*:

(i) *x is a rotund point of B_X.*
(ii) *For every $y \in \mathsf{S}_X \backslash \{x\}$, $\lim_{t\to 0^+} \big((\|x + ty\| - \|x\|)/t\big) < 1$.*

PROOF. Assume that (i) holds. Let $y \in \mathsf{S}_X\backslash\{x\}$. Then $\|(x+y)/2\| < 1$, therefore

$$\frac{\|x + 1y\| - \|x\|}{1} < 1.$$

Since the mapping

$$\begin{aligned} \mathbb{R}\backslash\{0\} &\longrightarrow \mathbb{R} \\ t &\longmapsto \big(\|x + ty\| - \|x\|\big)/t \end{aligned}$$

is increasing, we deduce that (ii) holds.

Assume that (ii) holds. Let $y \in \mathsf{S}_X$ be such that $\|(x+y)/2\| = 1$. Take a functional $f \in \mathsf{S}_{X^*}$ such that $\{x, y\} \subseteq f^{-1}(\{1\}) \cap \mathsf{B}_X$. For every $t > 0$, we have

$$1 = f(y) = \frac{f(x+ty) - f(x)}{t} \leq \frac{\|x + ty\| - \|x\|}{t}.$$

Therefore, if we suppose that $y \neq x$, then

$$1 \leq \lim_{t\to 0^+} \left(\frac{\|x + ty\| - \|x\|}{t}\right) < 1$$

which is a contradiction. □

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THEOREM 2.5. *Let X be a Banach space. The following assertions are equivalent:*

(i) *X is rotund.*

(ii) *If C is a closed convex subset of S_X such that $\mathsf{B}_X \backslash C$ is convex, then C is a face of B_X.*

PROOF. It is clear that (i) implies (ii). Assume that (ii) holds and X is not rotund. There exists a proper face D of B_X with $\operatorname{diam}(D) > 0$. Let $f \in \mathsf{S}_{X^*}$ be such that $f(D) \cap (0, \infty) \neq \varnothing$ and $f(D) \cap (-\infty, 0) \neq \varnothing$. Consider the set $C = D \cap f^{-1}([0, \infty))$. It is clear that C is closed and convex.

We check that C is not a face of B_X. Fix c in C and d in D with $f(c) > 0$ and $f(d) < 0$. Since

$$\lim_{t \to 1^-} \left(\frac{1-t}{t}(-f(d)) \right) = 0,$$

there exists $t \in (0, 1)$ such that $((1-t)/t)(-f(d)) < f(c)$. We deduce that $f(tc + (1-t)d) > 0$. Therefore, C is not a face of B_X.

Next we check that $\mathsf{B}_X \backslash C$ is convex. Let x and y be in $\mathsf{B}_X \backslash C$. Suppose that there exists $t \in (0, 1)$ such that $tx + (1-t)y$ is in C. Since D is a face of B_X, we deduce that x and y are both in D. Since x and y are not in C, we obtain that $f(x) < 0$ and $f(y) < 0$. Therefore, $f(tx + (1-t)y) < 0$, a contradiction with the fact that $tx + (1-t)y$ is in C. □

LEMMA 2.6. *Let X be a Banach space. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in B_X such that $\left(\|(x_n + y_n)/2\|\right)_{n\in\mathbb{N}}$ converges to 1. For every $n \in \mathbb{N}$, let λ_n be in $[0, 1]$. Then the sequence $\left(\|\lambda_n x_n + (1-\lambda_n) y_n\|\right)_{n\in\mathbb{N}}$ is convergent to 1.*

PROOF. We assume, without loss of generality, that $(\lambda_n)_{n\in\mathbb{N}}$ is convergent to some λ in $[0, 1]$. It will be enough to prove that the sequence $\left(\|\lambda x_n + (1-\lambda) y_n\|\right)_{n\in\mathbb{N}}$ is convergent to 1. Assume that λ is in $[0, 1/2]$. If λ is in $[1/2, 1]$, then we can reason similarly. For every $n \in \mathbb{N}$, set $z_\lambda^n = \lambda x_n + (1-\lambda) y_n$. Then

$$\begin{aligned}
\left\| \frac{x_n + y_n}{2} \right\| &= \left\| \frac{1}{2} x_n + \frac{1}{2} \left(\frac{1}{1-\lambda} z_\lambda^n - \frac{\lambda}{1-\lambda} x_n \right) \right\| = \left\| \frac{1-2\lambda}{2-2\lambda} x_n + \frac{1}{2-2\lambda} z_\lambda^n \right\| \\
&\leq \frac{1-2\lambda}{2-2\lambda} \|x_n\| + \frac{1}{2-2\lambda} \|z_\lambda^n\| \leq \frac{1-2\lambda}{2-2\lambda} + \frac{1}{2-2\lambda} \|z_\lambda^n\| \\
&\leq \frac{1-2\lambda}{2-2\lambda} + \frac{1}{2-2\lambda} = 1.
\end{aligned}$$

Therefore, the sequence

$$\left(\frac{1-2\lambda}{2-2\lambda} + \frac{1}{2-2\lambda} \|z_\lambda^n\| \right)_{n\in\mathbb{N}}$$

converges to 1. So, the sequence $(\|z_\lambda^n\|)_{n\in\mathbb{N}}$ converges to 1. □

THEOREM 2.7. *Let X be a Banach space and let $x \in \mathsf{S}_X$. The following assertions are equivalent*:

(i) *x is a locally uniformly rotund point of B_X.*

(ii) *If $(C_n)_{n\in\mathbb{N}}$ is a sequence of convex subsets of B_X such that the sequence $(\operatorname{dist}(\{0\}, C_n))_{n\in\mathbb{N}}$ converges to 1, and x is a diametral point of C_n for every $n \in \mathbb{N}$, then the sequence $(\operatorname{diam}(C_n))_{n\in\mathbb{N}}$ converges to 0.*

PROOF. Assume that (i) holds. Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of convex subsets of B_X such that the sequence $(\operatorname{dist}(\{0\}, C_n))_{n\in\mathbb{N}}$ converges to 1 and, for every $n \in \mathbb{N}$, x is a diametral point of C_n. Since, for every $n \in \mathbb{N}$, $\operatorname{diam}(C_n) = \sup(\{\|x - c\| : c \in C_n\})$, there exists $y_n \in C_n$ such that $\operatorname{diam}(C_n) < \|x - y_n\| + 1/n$. Since $(x + y_n)/2$ is in C_n, we deduce that

$$\operatorname{dist}(\{0\}, C_n) \leq \left\| \frac{x + y_n}{2} \right\| \leq 1.$$

Since x is a locally uniformly rotund point of B_X, we obtain that $(\operatorname{diam}(C_n))_{n\in\mathbb{N}}$ converges to 0.

Assume that (ii) holds. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in S_X such that the sequence $(\|(x+y_n)/2\|)_{n\in\mathbb{N}}$ converges to 1. For every $n \in \mathbb{N}$, let C_n be the segment of extremes y_n and x. It is clear that x is a diametral point of C_n. In addition, according to Lemma 2.6, it is not difficult to see that $(\operatorname{dist}(\{0\}, C_n))_{n\in\mathbb{N}}$ converges to 1. Therefore, the sequence $(\operatorname{diam}(C_n))_{n\in\mathbb{N}}$ converges to 0. In consequence, $(y_n)_{n\in\mathbb{N}}$ converges to x. □

REMARK 2.8. It is known that a Banach space is uniformly rotund (see [5, pages 441–459]) if and only if for every pair of sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in S_X such that $(\|(x_n + y_n)/2\|)_{n\in\mathbb{N}}$ converges to 1, $(x_n - y_n)_{n\in\mathbb{N}}$ converges to 0. It is also well known (see [5, page 460]) that every uniformly rotund Banach space is a locally uniformly rotund Banach space. Moreover, this implication is not an equivalence (see [5, pages 462–463]).

A little modification of proof of Theorem 2.7 yields the equivalence of these assertions for a Banach space X:

(a) X is uniformly rotund.

(b) If $(C_n)_{n\in\mathbb{N}}$ is a sequence of convex subsets of B_X such that the sequence $(\operatorname{dist}(\{0\}, C_n))_{n\in\mathbb{N}}$ converges to 1, then the sequence $(\operatorname{diam}(C_n))_{n\in\mathbb{N}}$ converges to 0.

REMARK 2.9. Recently, Bandyopadhyay and Lin [3] proved the following equivalences for a point x of the unit sphere of a Banach space X:

(i) x is a rotund point of B_X if and only if it is an exposed point of B_X for each $f \in \mathsf{S}_{X^*}$ such that $f(x) = 1$.

(ii) x is an almost locally uniformly rotund point of B_X if and only if it is a strongly exposed point of B_X for each $f \in \mathsf{S}_{X^*}$ such that $f(x) = 1$.

According to these results, it is clear that

(i) if x is an exposed point and a smooth point of B_X, then it is a rotund point of B_X;

(ii) if x is a strongly exposed point and a smooth point of B_X, then it is an almost locally uniformly rotund point of B_X.

THEOREM 2.10. *Let X be a Banach space and let $x \in \mathsf{S}_X$. If x is a strongly exposed point of B_X and a strongly smooth point of B_X, then it is a locally uniformly rotund point of B_X.*

PROOF. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in S_X such that $(\|(x + y_n)/2\|)_{n\in\mathbb{N}}$ converges to 1. Take the functional f in S_{X^*} that characterizes x as a strongly exposed point of B_X. For every $n \in \mathbb{N}$, let f_n in S_{X^*} be such that

$$f_n\left(\frac{x + y_n}{2}\right) = \left\|\frac{x + y_n}{2}\right\|.$$

Since $(f_n((x + y_n)/2))_{n\in\mathbb{N}}$ converges to 1, we assume, without loss of generality, that the sequences $(f_n(y_n))_{n\in\mathbb{N}}$ and $(f_n(x))_{n\in\mathbb{N}}$ are both convergent to 1. Since x is a strongly smooth point of B_X, we deduce that $(f_n)_{n\in\mathbb{N}}$ converges to f. For every $n \in \mathbb{N}$, we have that

$$|1 - f(y_n)| \le |1 - f_n(y_n)| + |f_n(y_n) + f(y_n)| \le |1 - f_n(y_n)| + \|f_n - f\|.$$

Therefore, $(f(y_n))_{n\in\mathbb{N}}$ converges to 1. Since x is a strongly exposed point of B_X, we deduce that the sequence $(y_n)_{n\in\mathbb{N}}$ converges to x. □

COROLLARY 2.11. *Let X be a Banach space. Then X is locally uniformly rotund if it is almost locally uniformly rotund and its norm is Fréchet differentiable in S_X.*

3. Renormings

REMARK 3.1. Let X be a Banach space. Let f be a norm-attaining functional of S_{X^*}. The mapping $\|\cdot\|_0 : X \to \mathbb{R}$, defined for every y in X by the formula

$$\|y\|_0 = \frac{1}{\sqrt{2}}\sqrt{\|y\|^2 + |f(y)|^2} \tag{1}$$

is an equivalent norm on X such that (where X_0 denotes the space X with the norm $\|\cdot\|_0$)

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(i) $\mathsf{B}_X \subseteq \mathsf{B}_{X_0} \subseteq \sqrt{2}\mathsf{B}_X$;
(ii) $\mathsf{S}_{X_0} \cap \mathsf{S}_X = C \cup -C$, where $C = f^{-1}(\{1\}) \cap \mathsf{B}_X$;
(iii) f is in $\mathsf{S}_{X_0^*}$.

THEOREM 3.2. *Let X be a Banach space. Let C be a nonempty subset of S_X. The following statements are equivalent*:

(i) *C is an exposed face of B_X.*
(ii) *There exists an equivalent norm $\|\cdot\|_0$ on X such that $\mathsf{B}_X \subseteq \mathsf{B}_{X_0} \subseteq \sqrt{2}\mathsf{B}_X$, $\mathsf{S}_{X_0} \cap \mathsf{S}_X = C \cup -C$, and C is a maximal face of B_{X_0}, where X_0 denotes the space X with the norm $\|\cdot\|_0$.*

PROOF. It is an easy exercise to see that (ii) implies (i). Assume that (i) holds. Let $f \in \mathsf{S}_{X^*}$ be the functional that characterizes C as an exposed face of B_X. Consider the equivalent norm on X given by (1). We know that $\mathsf{B}_X \subseteq \mathsf{B}_{X_0} \subseteq \sqrt{2}\mathsf{B}_X$ and $\mathsf{S}_{X_0} \cap \mathsf{S}_X = C \cup -C$. We see that C is a maximal face of B_{X_0}. Let D be a face of B_{X_0} such that $C \subset D$ and consider an element d in $D \backslash C$. Since $\|d\|_0 = 1$, we have that $\|d\|^2 + |f(d)|^2 = 2$, therefore $\|d\| + |f(d)| \leq 2$. Since $\|d\| > 1$ (because d is not in $C \cup -C$), we deduce that $|f(d)| < 1$. Therefore, $\|d\| + |f(d)| < 2$. Let c be in C. Let us see that $\|(c+d)/2\|_0 < 1$, which is a contradiction. We have that

$$\begin{aligned}
\left\|\frac{c+d}{2}\right\|_0 &= \frac{1}{\sqrt{2}}\sqrt{\left\|\frac{c+d}{2}\right\|^2 + \left|f\left(\frac{c+d}{2}\right)\right|^2} \\
&= \frac{1}{2\sqrt{2}}\sqrt{\|c+d\|^2 + |1+f(d)|^2} \\
&\leq \frac{1}{2\sqrt{2}}\sqrt{\|c\|^2 + \|d\|^2 + 2\|c\|\|d\| + 1 + |f(d)|^2 + 2|f(d)|} \\
&= \frac{1}{2\sqrt{2}}\sqrt{4 + 2(\|d\| + |f(d)|)} = \frac{1}{2}\sqrt{2 + \|d\| + |f(d)|} < 1.
\end{aligned}$$

So the proof is concluded. □

COROLLARY 3.3. *Let X be a Banach space and let $x \in \mathsf{S}_X$. The following statements are equivalent*:

(i) *x is an exposed point of B_X.*
(ii) *There exists an equivalent norm $\|\cdot\|_0$ on X such that $\mathsf{B}_X \subseteq \mathsf{B}_{X_0} \subseteq \sqrt{2}\mathsf{B}_X$, $\mathsf{S}_{X_0} \cap \mathsf{S}_X = \{x, -x\}$, and x is a rotund point of B_{X_0}, where X_0 denotes the space X with the norm $\|\cdot\|_0$.*

THEOREM 3.4. *Let X be a Banach space and let $x \in \mathsf{S}_X$. The following statements are equivalent*:

(i) *x is a strongly exposed point of* B_X.

(ii) *There exists an equivalent norm* $\|\cdot\|_0$ *on* X *such that* $\mathsf{B}_X \subseteq \mathsf{B}_{X_0} \subseteq \sqrt{2}\mathsf{B}_X$, $\mathsf{S}_{X_0} \cap \mathsf{S}_X = \{x, -x\}$, *and* x *is a locally uniformly rotund point of* B_{X_0}, *where* X_0 *denotes the space* X *with the norm* $\|\cdot\|_0$.

PROOF. It is an easy exercise to see that (ii) implies (i). Assume that (i) holds. Let $f \in \mathsf{S}_{X^*}$ be the functional that characterizes x as a strongly exposed point of B_X. Consider the equivalent norm on X given by (1). We know that $\mathsf{B}_X \subseteq \mathsf{B}_{X_0} \subseteq \sqrt{2}\mathsf{B}_X$ and $\mathsf{S}_{X_0} \cap \mathsf{S}_X = \{x, -x\}$. We show that x is a locally uniformly rotund point of B_{X_0}. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in S_{X_0} such that the sequence

$$\left(\left\|\frac{x_n + x}{2}\right\|_0\right)_{n\in\mathbb{N}}$$

converges to 1. We assume, without loss of generality, that there exist α and β in $\mathbb{R}$ such that $(\|x_n\|)_{n\in\mathbb{N}}$ converges to α and $(f(x_n))_{n\in\mathbb{N}}$ converges to β. For every $n \in \mathbb{N}$, since $\|x_n\|_0 = 1$, we have that $\|x_n\|^2 + |f(x_n)|^2 = 2$ and $\|x_n\| + |f(x_n)| \leq 2$. On the other hand, for every $n \in \mathbb{N}$, we have

$$\begin{aligned}\left\|\frac{x_n + x}{2}\right\|_0 &= \frac{1}{\sqrt{2}}\sqrt{\left\|\frac{x_n + x}{2}\right\|^2 + \left|f\left(\frac{x_n + x}{2}\right)\right|^2} \\ &= \frac{1}{2\sqrt{2}}\sqrt{\|x_n + x\|^2 + |f(x_n) + 1|^2} \\ &\leq \frac{1}{2\sqrt{2}}\sqrt{\|x_n\|^2 + \|x\|^2 + 2\|x_n\|\|x\| + 1 + |f(x_n)|^2 + 2|f(x_n)|} \\ &= \frac{1}{2\sqrt{2}}\sqrt{4 + 2(\|x_n\| + |f(x_n)|)} = \frac{1}{2}\sqrt{2 + \|x_n\| + |f(x_n)|} \leq 1.\end{aligned}$$

Therefore,

$$1 \leq \frac{1}{2}\sqrt{2 + \alpha + |\beta|} \leq 1.$$

We deduce that $\alpha = |\beta| = 1$. Suppose that $\beta = -1$. Then $((-f)(x_n))_{n\in\mathbb{N}}$ converges to 1, therefore $(x_n)_{n\in\mathbb{N}}$ $\|\cdot\|$-converges to $-x$. Then, for every $n \in \mathbb{N}$, we have

$$\|x_n + x\|_0 = \sqrt{\|x_n + x\|^2 + |f(x_n + x)|^2}.$$

Therefore, $(\|x_n + x\|_0)_{n\in\mathbb{N}}$ converges to 0. We have a contradiction, because $(\|(x_n + x)/2\|_0)_{n\in\mathbb{N}}$ converges to 1. We deduce that $\alpha = \beta = 1$. Then, $(f(x_n))_{n\in\mathbb{N}}$ converges to 1, therefore $(x_n)_{n\in\mathbb{N}}$ $\|\cdot\|$-converges to x. Then, for every $n \in \mathbb{N}$, we have

$$\|x_n - x\|_0 = \sqrt{\|x_n - x\|^2 + |f(x_n - x)|^2}.$$

Therefore, $(x_n)_{n\in\mathbb{N}}$ $\|\cdot\|_0$-converges to x. $\square$

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