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# A TOPOLOGICAL CHARACTERISATION OF FINITE DIMENSIONALITY OF $C^*$-ALGEBRAS

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A $C^*$-algebra $\mathcal{A}$ is finite-dimensional if and only if the cone $M_2^+ \otimes \mathcal{A}^+$ is closed in $M_2 \otimes \mathcal{A}$.

## 1. INTRODUCTION

In the study of the matricial order structure of a $C^*$-algebra $\mathcal{A}$, two cones in $M_n \otimes \mathcal{A}$ arise naturally: $(M_n \otimes \mathcal{A})^+$ and $M_n^+ \otimes \mathcal{A}^+$. (Notation will be explained in the next paragraph.) It is a basic fact that the cone $(M_n \otimes \mathcal{A})^+$ of all positive elements in $M_n \otimes \mathcal{A}$ is closed in $M_n \otimes \mathcal{A}$ for every positive integer $n$. We may ask the same question for $M_n^+ \otimes \mathcal{A}^+$:

**Question.** Let $n \geqslant 2$ be a positive integer. Is $M_n^+ \otimes \mathcal{A}^+$ closed in $M_n \otimes \mathcal{A}$?
We will show that this is true if and only if $\mathcal{A}$ is finite dimensional (see Remark 2 and Remark 5).

Throughout this paper, $\mathcal{A}$ denotes a $C^*$-algebra. The set of all hermitian elements (respectively positive elements) in $\mathcal{A}$ is denoted by $\mathcal{A}^h$ (respectively $\mathcal{A}^+$). We write $A \geqslant 0$ if $A \in \mathcal{A}^+$. $M_n$ denotes the algebra of all $n \times n$ complex matrices with the Hilbert space operator norm. We often identify the $C^*$-tensor product $M_n \otimes \mathcal{A}$ of $M_n$ and $\mathcal{A}$ with the $C^*$-algebra $M_n(\mathcal{A})$ of all $n \times n$ matrices over $\mathcal{A}$, by the identification

$$[\lambda_{ij}] \otimes A = [\lambda_{ij} A]$$

for $[\lambda_{ij}]$ in $M_n$ and $A$ in $\mathcal{A}$. We let

$$M_n^+ \otimes \mathcal{A}^+ = \text{ convex hull of } \{S \otimes A : S \in M_n^+,\, A \in \mathcal{A}^+\}$$

(*without closure*). All topologies considered here are those induced by $C^*$-norms. Not very much is known about $M_n^+ \otimes \mathcal{A}^+$, even for $\mathcal{A} = M_m$. It is known that $M_n^+ \otimes M_m^+$ is a proper subcone of $(M_n \otimes M_m)^+$ except when $n = 1$ or $m = 1$ (see [**1**] for pertinent results). For a linear map $\Phi\colon \mathcal{A} \to \mathcal{B}$ between $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$, let $id_n \otimes \Phi\colon M_n \otimes \mathcal{A} \to M_n \otimes \mathcal{B}$ be the map defined by $id_n \otimes \Phi(\Sigma S_i \otimes A_i) = \Sigma S_i \otimes \Phi(A_i)$. $\Phi$ is said to be *positive* if $\Phi(\mathcal{A}^+) \subseteq \mathcal{B}^+$.

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## 2. Results

We first prove the following

**Lemma 1.** *$M_n^+ \otimes M_k^+$ is closed in $M_n \otimes M_k$.*

**Proof:** By definition,

$$\begin{aligned} M_n^+ \otimes M_k^+ &= \text{ convex hull of } \{A \otimes B : A \in M_n^+, B \in M_k^+\} \\ &\subset (M_n \otimes M_k)^h. \end{aligned}$$

Since $(M_n \otimes M_k)^h$ is a real vector space of dimension (over $\mathbb{R}$) $n^2k^2$, each element in $M_n^+ \otimes M_k^+$ is a convex combination of at most $p = n^2k^2 + 1$ elements in

$$\{A \otimes B : A \in M_n^+, B \in M_k^+\}$$

(see for example, [**3**, p. 73]). Hence

$$M_n^+ \otimes M_k^+ = \{\sum_{i=1}^{p} A_i \otimes B_i : A_i \in M_n^+, B_i \in M_k^+\}.$$

If $X_j = \sum_{i=1}^{p} A_{ji} \otimes B_{ji} \in M_n^+ \otimes M_k^+$ and $X_j \to X$ in $M_n \otimes M_k$, then there exists a positive constant $r$ such that $\|X_j\| \leqslant r$ for every $j$. Since $0 \leqslant A_{ji} \otimes B_{ji} \leqslant X_j$, $\|A_{ji} \otimes B_{ji}\| \leqslant \|X_j\| \leqslant r$ for every $1 \leqslant i \leqslant p$ and every $j$. We may assume without loss of generality that $A_{ji} \neq 0$ for every $i$ and $j$. Then

$$\frac{1}{\|A_{ji}\|} A_{ji} \otimes \|A_{ji}\| B_{ji} = A_{ji} \otimes B_{ji}$$

and

$$\| \|A_{ji}\| B_{ji}\| = \|A_{ji}\| \|B_{ji}\| = \|A_{ji} \otimes B_{ji}\| \leqslant r.$$

Since the sets $\{A \in M_n^+ : \|A\| \leqslant 1\}$ and $\{B \in M_k^+ : \|B\| \leqslant r\}$ are compact, there exist subsequence $\{j_\ell\}$ and $A_i \in M_n^+$, $B_i \in M_k^+$, $1 \leqslant i \leqslant p$, such that for every $i$,

$$\frac{1}{\|A_{j_\ell i}\|} A_{j_\ell i} \to A_i$$

and

$$\|A_{j_\ell i}\| B_{j_\ell i} \to B_i.$$

Hence for every $i$,

$$A_{j_\ell i} \otimes B_{j_\ell i} \to A_i \otimes B_i,$$

and so

$$X_{j_\ell} = \sum_{i=1}^{p} A_{j_\ell i} \otimes B_{j_\ell i} \to \sum_{i=1}^{p} A_i \otimes B_i.$$

Since $X_{i_\ell} \to X$ also, $X = \sum_{i=1}^{p} A_i \otimes B_i \in M_n^+ \otimes M_k^+$. Hence $M_n^+ \otimes M_k^+$ is closed in $M_n \otimes M_k$. ∎

**Remark 2.** Since each finite-dimensional $C^*$-algebra is $^*$-isomorphic to a direct sum of full matrix algebras (for example, [**5**, Chapter I, Theorem 11.2], we have the following result: if $\mathcal{A}$ is a finite-dimensional $C^*$-algebra, then $M_n^+ \otimes \mathcal{A}^+$ is closed in $M_n \otimes \mathcal{A}$ for every positive integer $n$.

Let $\ell_\infty$ be the algebra of all bounded sequences of complex numbers with sup norm.

Lemma 3. *$M_2^+ \otimes \ell_\infty^+$ is not closed in $M_2 \otimes \ell_\infty$.*

Proof: Let $e_k = (\delta_{ik})_{i=1}^\infty \in \ell_\infty$. Let

$$A = \begin{bmatrix} \left[\frac{1}{n}\right]_{n=1}^\infty & \left[\frac{1}{n^2}\right]_{n=1}^\infty \\ \left[\frac{1}{n^2}\right]_{n=1}^\infty & \left[\frac{1}{n^3}\right]_{n=1}^\infty \end{bmatrix} = \sum_{n=1}^\infty A(n) \otimes e_n$$

$$\text{where } A(n) = \begin{bmatrix} \frac{1}{n} & \frac{1}{n^2} \\ \frac{1}{n^2} & \frac{1}{n^3} \end{bmatrix}, \text{ and}$$

$$\begin{aligned} A_n &= \begin{bmatrix} (1, \frac{1}{2}, \dots, \frac{1}{n}, 0, \dots) & (1, \frac{1}{2^2}, \dots, \frac{1}{n^2}, 0, \dots) \\ (1, \frac{1}{2^2}, \dots, \frac{1}{n^2}, 0, \dots) & (1, \frac{1}{2^3}, \dots, \frac{1}{n^3}, 0, \dots) \end{bmatrix} \\ &= \sum_{k=1}^n A(k) \otimes e_k. \end{aligned}$$

Then $A \in (M_2 \otimes \ell_\infty)^+$, $A_n \in M_2^+ \otimes \ell_\infty^+$ and $A_n \to A$ in norm, since $\|A_n - A\| \leqslant \frac{1}{n+1} + \frac{2}{(n+1)^2} + \frac{1}{(n+1)^3}$.

Claim: $A \notin M_2^+ \otimes \ell_\infty^+$. For suppose that $A \in M_2^+ \otimes \ell_\infty^+$; then there exist $\lambda_i$, $\mu_i$ in C, $|\lambda_i|^2 + |\mu_i|^2 \neq 0$ and $f_i \neq 0$ in $\ell_\infty^+ (1 \leqslant i \leqslant r)$ such that

$$\begin{aligned} A &= \sum_{i=1}^r \begin{bmatrix} \lambda_i \\ \mu_i \end{bmatrix} [\overline{\lambda_i} \quad \overline{\mu_i}] \otimes f_i \\ &= \sum_{i=1}^r \begin{bmatrix} |\lambda_i|^2 f_i & \lambda_i \overline{\mu_i} f_i \\ \mu_i \overline{\lambda_i} f_i & |\mu_i|^2 f_i \end{bmatrix}. \end{aligned}$$

Fix i. If $f_i(n) \neq 0$, then

$$\begin{bmatrix} \frac{1}{n} & \frac{1}{n^2} \\ \frac{1}{n^2} & \frac{1}{n^3} \end{bmatrix} = A(n) = \sum_{j=1}^r \begin{bmatrix} |\lambda_j|^2 & \lambda_j \overline{\mu_j} \\ \mu_j \overline{\lambda_j} & |\mu_j|^2 \end{bmatrix} f_j(n)$$

is of rank 1 and so

$$\begin{bmatrix} |\lambda_i|^2 & \lambda_i \overline{\mu_i} \\ \mu_i \overline{\lambda_i} & |\mu_i|^2 \end{bmatrix} = t_{in} \begin{bmatrix} \frac{1}{n} & \frac{1}{n^2} \\ \frac{1}{n^2} & \frac{1}{n^3} \end{bmatrix} \text{ for some } t_{in} > 0.$$

Hence if $f_i(n) \neq 0$ and $f_i(m) \neq 0$, then

$$t_{in}\begin{bmatrix} \frac{1}{n} & \frac{1}{n^2} \\ \frac{1}{n^2} & \frac{1}{n^3} \end{bmatrix} = t_{im}\begin{bmatrix} \frac{1}{m} & \frac{1}{m^2} \\ \frac{1}{m^2} & \frac{1}{m^3} \end{bmatrix} \text{ where } t_{in},\, t_{im} > 0.$$

So $n = m$. Therefore for each $i$, there exists a unique $n_i$ such that $f_i(n_i) \neq 0$, that is, $f_i = f_i(n_i)e_{n_i}$. Since $r$ is finite, if $n > \max(n_1, \dots, n_r)$, then $A(n)$ would be 0 in $M_2$, which is a contradiction. Hence $A \notin M_2^+ \otimes \ell_\infty^+$. Thus $M_2^+ \otimes \ell_\infty^+$ is not closed in $M_2 \otimes \ell_\infty$. ∎

Combining Remark 2 and Lemma 3, we obtain a characterisation of the finite-dimensionality of a $C^*$-algebra $\mathcal{A}$ in terms of a topological property of the cone $M_2^+ \otimes \mathcal{A}^+$ in $M_2 \otimes \mathcal{A}$.

THEOREM 4. *Let $\mathcal{A}$ be a $C^*$-algebra. Then $\mathcal{A}$ is finite-dimensional if and only if $M_2^+ \otimes \mathcal{A}^+$ is closed in $M_2 \otimes \mathcal{A}$.*

PROOF: The "only if" part is in Remark 2. Conversely, suppose that $\mathcal{A}$ is infinite dimensional. We will show that $M_2^+ \otimes \mathcal{A}^+$ is not closed in $M_2 \otimes \mathcal{A}$. Since $\mathcal{A}$ is infinite dimensional, $\mathcal{A}$ has a hermitian element $A$ with infinite spectrum [**2**]. By spectral theory, there exist a sequence $\{E_i\}_{i=1}^\infty$ of positive elements of norm one in $\mathcal{A}$ and states $\{\phi_i\}_{i=1}^\infty$ on $\mathcal{A}$ such that $E_iE_j = 0$ if $i \neq j$ and $\phi_i(E_j) = \delta_{ij}$. (See also [**4**, proof of Lemma 2.3]). Let

$$X = \begin{bmatrix} \sum_{n=1}^\infty \frac{1}{n}E_n & \sum_{n=1}^\infty \frac{1}{n^2}E_n \\ \sum_{n=1}^\infty \frac{1}{n^2}E_n & \sum_{n=1}^\infty \frac{1}{n^3}E_n \end{bmatrix} = \sum_{n=1}^\infty \begin{bmatrix} \frac{1}{n} & \frac{1}{n^2} \\ \frac{1}{n^2} & \frac{1}{n^3} \end{bmatrix} \otimes E_n$$

and

$$X_n = \begin{bmatrix} \sum_{k=1}^n \frac{1}{k}E_k & \sum_{k=1}^n \frac{1}{k^2}E_k \\ \sum_{k=1}^n \frac{1}{k^2}E_k & \sum_{k=1}^n \frac{1}{k^3}E_k \end{bmatrix} = \sum_{k=1}^n \begin{bmatrix} \frac{1}{k} & \frac{1}{k^2} \\ \frac{1}{k^2} & \frac{1}{k^3} \end{bmatrix} \otimes E_k.$$

Then $X \in (M_2 \otimes \mathcal{A})^+$, $X_n \in M_2^+ \otimes \mathcal{A}^+$ and $X_n \to X$ in norm, since $\|X_n - X\| \leqslant \frac{1}{n+1} + \frac{2}{(n+1)^2} + \frac{1}{(n+1)^3}$ (since the $\{E_j\}_{j=1}^\infty$ are mutually orthogonal).
Define

$$\Phi\colon \mathcal{A} \to \ell_\infty$$

by

$$\Phi(T) = (\phi_i(T))_{i=1}^\infty,\ T \in \mathcal{A}.$$

Then $\Phi$ is positive and contractive, $\Phi(E_n) = (\delta_{in})_{i=1}^\infty = e_n$ and

$$id_2 \otimes \Phi(X) = \sum_{n=1}^\infty \begin{bmatrix} \frac{1}{n} & \frac{1}{n^2} \\ \frac{1}{n^2} & \frac{1}{n^3} \end{bmatrix} \otimes e_n \notin M_2^+ \otimes \ell_\infty^+$$

by Lemma 3. Hence $X \notin M_2^+ \otimes \mathcal{A}^+$. For if $X = \Sigma B_i \otimes T_i \in M_2^+ \otimes \mathcal{A}^+$ where $B_i \in M_2^+$ and $T_i \in \mathcal{A}^+$, then $id_2 \otimes \Phi(X) = \Sigma B_i \otimes \Phi(T_i) \in M_2^+ \otimes \ell_\infty^+$, a contradiction. ∎

**Remark 5.** A similar argument as in the proof of Lemma 3 shows that $M_n^+ \otimes \ell_\infty^+$ is not closed in $M_n \otimes \ell_\infty$ for each positive integer $n \geqslant 2$. Using this and the same idea as in the proof of Theorem 4, we can show that if $\mathcal{A}$ is infinite dimensional, then $M_n^+ \otimes \mathcal{A}^+$ is not closed in $M_n \otimes \mathcal{A}$ for each $n \geqslant 2$.

## References

[1] M. -D. Choi, 'Positive linear maps', *Proc. Sympos. Pure Math.* **38** (1982), 583–590.

[2] T. Ogasawara, 'Finite dimensionality of certain Banach algebras', *J. Sci. Hiroshima Univ. Ser. A.* **17** (1954), 359–364.

[3] W. Rudin, *Functional Analysis* (McGraw-Hill, New York, 1973).

[4] R.R. Smith, 'Completely bounded maps between $C^*$-algebras', *J. London Math. Soc. (2)* **27** (1983), 157–166.

[5] M. Takesaki, *Theory of Operator Algebras I* (Springer Verlag, Berlin, 1979).

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