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C^*-ALGEBRAS OF SELF-SIMILAR ACTION OF GROUPOIDS ON ROW-FINITE DIRECTED GRAPHS

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Abstract

For amenable discrete groupoids $\mathcal{G}$ and row-finite directed graphs E, let $(\mathcal{G}, E)$ be a self-similar groupoid and let $C^*(\mathcal{G}, E)$ be the associated C^*-algebra. We introduce a weaker faithfulness condition than those in the existing literature that still guarantees that $C^*(\mathcal{G})$ embeds in $C^*(\mathcal{G}, E)$. Under this faithfulness condition, we prove a gauge-invariant uniqueness theorem.

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1. Introduction

Roughly speaking, if parts of an object are similar to the whole, repeating the structure of the object at all scales, then we call the object self-similar. If a group or a groupoid acts self-similarly on a space, then we simply call it a self-similar group or a self-similar groupoid. Self-similar groups were introduced by Grigorchuk in [2] and Gupta and Sidki in [3] to answer the question of existence of groups with intermediate growth. Recently, operator algebraists have made use of self-similar groups to study C^*-algebras (for example, [1, 4]). Since a groupoid is a generalisation of a group, it is then natural to think of this notion of self-similarity on a groupoid, as introduced in [5]. Self-similar groups act on the path-spaces of graphs with a single vertex. To study self-similar actions on more general directed graphs and the associated Cuntz–Krieger algebras, Laca *et al.* in [5] introduced the notion of a self-similar groupoid. In [5], the authors are primarily interested in computing KMS states, so, informed by results about graph C^*-algebras, they restricted their attention to finite graphs. They also built their self-similar groupoids by generalising the process whereby automata are used to build self-similar groups, so by definition their self-similar actions satisfy a

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faithfulness condition that simplifies their analysis and, in particular, guarantees that $C^*(\mathcal{G})$ embeds in $C^*(\mathcal{G}, E)$.

Another approach to self-similar actions on graphs with multiple vertices was developed by Exel and Pardo [1] and does not require a faithfulness condition. We combine and generalise the constructions in [1, 5]. We consider self-similar actions of groupoids $\mathcal{G}$ on the path spaces E^* of row-finite directed graphs E that are not necessarily faithful in the sense of [5]. We develop a new faithfulness condition that is weaker than both faithfulness as in [5] and pseudo-faithfulness as in [1], but still guarantees that $C^*(\mathcal{G})$ embeds in $C^*(\mathcal{G}, E)$, and we prove a gauge-invariant uniqueness theorem. In particular, our theorems apply to conventional actions of groups on graphs (see Example 3.7). We also depart from [5] in that we work solely with generators and relations, without employing the machinery of Hilbert modules and Cuntz–Pimsner algebras.

The paper is organised as follows. We define our notion of a self-similar groupoid $(\mathcal{G}, E)$ in Definition 2.1 and construct the associated C^*-algebras $C^*(\mathcal{G}, E)$ in Section 3 following the approach of [6]. We introduce our injectivity condition in our key technical result Proposition 3.6. We analyse the fixed-point algebra $C^*(\mathcal{G}, E)^\gamma$ for the gauge action γ in Section 4. By applying all the results in the previous sections, we prove the gauge-invariant uniqueness theorem in Theorem 5.1.

2. Self-similar groupoids

Recall that a groupoid $\mathcal{G}$ is a small category with inverses. We write $\mathcal{G}^{(0)}$ for the set of identity morphisms and $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ for the maps induced by the codomain and domain range maps. Throughout this paper, $\mathcal{G}$ will denote a countable discrete groupoid. We will assume that $\mathcal{G}$ is amenable in the sense of [7]. Since $\mathcal{G}$ is discrete, this is equivalent to requiring that its full and reduced C^*-algebras coincide, and is also equivalent to requiring that each of its isotropy groups is amenable.

As in [6], a (directed) graph is a quadruple $E = (E^0, E^1, r, s)$ consisting of countable sets E^0, E^1 and maps $r, s : E^1 \to E^0$. Elements of E^1 are called *edges* and elements of E^0 are called *vertices*. We will assume that all our graphs are *row-finite* and have *no sources* in the sense that $0 < |r^{-1}(v)| < \infty$ for all $v \in E^0$. Let $e, f \in E^1$ with $s(e) = r(f)$. Then, ef is a path of length 2 and we write $|ef| = 2$. In general, a path μ of length n in E is a sequence $\mu_1\mu_2 \cdots \mu_n$ such that $s(\mu_i) = r(\mu_{i+1})$ for $1 \le i \le n-1$. The vertices are viewed as paths of length 0. The paths of length n are collected in a set denoted by E^n. We let $E^* := \bigcup_{k\ge 0} E^k$. It is natural to extend the maps r, s to E^* by putting $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_{|\mu|})$ where $|\mu| > 1$, and $r(v) = v = s(v)$ for $v \in E^0$.

DEFINITION 2.1. Let E be a row-finite graph with no sources and let $\mathcal{G}$ be a groupoid with $\mathcal{G}^{(0)} = E^0$. Write

$$\mathcal{G} * E^* := \{(g, \mu) \in \mathcal{G} \times E^* \mid s(g) = r(\mu)\}$$

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and

$$E^* * \mathcal{G} := \{(\mu, g) \mid s(\mu) = r(g)\}.$$

We will often denote the element $(\mu, g) \in E^* * \mathcal{G}$ by the shorthand μg. A *self-similar action* of $\mathcal{G}$ on E^* consists of two maps: (1) an action $(g, \mu) \mapsto g \cdot \mu$ of $\mathcal{G}$ on the set E^* and (2) a map $\varphi : \mathcal{G} * E^* \to \mathcal{G}$ such that:

(i) $g \cdot (\mu\beta) = (g \cdot \mu)(\varphi(g, \mu) \cdot \beta)$;
(ii) $r(g \cdot \mu) = g \cdot r(\mu)$ and $s(g \cdot \mu) = \varphi(g, \mu) \cdot s(\mu)$;
(iii) $|g \cdot \mu| = |\mu|$;
(iv) $\varphi(g, v) = g$;
(v) $\varphi(gh, \mu) = \varphi(g, h \cdot \mu)\varphi(h, \mu)$;
(vi) $\varphi(g, \mu\beta) = \varphi(\varphi(g, \mu), \beta)$; and
(vii) $\varphi(g^{-1}, \mu) = (\varphi(g, g^{-1} \cdot \mu))^{-1}$.

We write this self-similar action of the groupoid $\mathcal{G}$ on E^* as a pair $(\mathcal{G}, E)$ and call it a *self-similar groupoid* $(\mathcal{G}, E)$.

3. The universal C^*-algebra $C^*(\mathcal{G}, E)$

Recall that a Toeplitz–Cuntz–Krieger family for a row-finite directed graph E with no sources consists of partial isometries $\{T_e \mid e \in E^1\}$ and mutually orthogonal projections $\{W_v \mid v \in E^0\}$ satisfying $T_e^* T_e = W_{s(e)}$ and $W_v \geq \sum_{e \in vE^1} T_e T_e^*$ for all $v \in E^0$. It is a Cuntz–Krieger E-family if $W_v = \sum_{e \in vE^1} T_e T_e^*$ for all $v \in E^0$. A unitary representation in a unital C^*-algebra A of a discrete groupoid $\mathcal{G}$ is a family $\{U_g \mid g \in \mathcal{G}\}$ of partial isometries such that $U_g U_h = \delta_{s(g), r(h)} U_{gh}$ and $U_{g^{-1}} = U_g^*$ for all $g, h \in \mathcal{G}$, and such that $\sum_{v \in \mathcal{G}^{(0)}} U_v = 1_A$.

DEFINITION 3.1. Let $(\mathcal{G}, E)$ be a self-similar groupoid. A *Toeplitz $(\mathcal{G}, E)$-family* consists of partial isometries $\{T_e \mid e \in E^1\}$ and a unitary representation $\{W_g \mid g \in \mathcal{G}\}$ of $\mathcal{G}$ such that $\{T_e \mid e \in E^1\} \cup \{W_v \mid v \in E^0\}$ is a Toeplitz–Cuntz–Krieger E-family. It is a *Cuntz–Krieger $(\mathcal{G}, E)$-family* if $\{T_e, W_v\}$ is a Cuntz–Krieger E-family.

EXAMPLE 3.2. Suppose that $\mathcal{G}$ acts self-similarly on E. Let $\mathcal{H} := l^2(E^* * \mathcal{G})$ with orthonormal basis $\{e_{\mu g} \mid \mu \in E^*, g \in \mathcal{G}\}$. For $e \in E^1$ and $h \in \mathcal{G}$, let $T_e, W_h \in B(\mathcal{H})$ be the operators such that

$$T_e e_{\mu g} = \begin{cases} e_{e\mu g} & \text{if } s(e) = r(\mu), \\ 0 & \text{otherwise,} \end{cases}$$

$$W_h e_{\mu g} = \begin{cases} e_{(h\cdot\mu)(\varphi(h,\mu)\cdot g)} & \text{if } s(h) = r(\mu), \\ 0 & \text{otherwise.} \end{cases}$$

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For $v \in E^0$, W_v is the projection onto $l^2(\{\mu g \mid r(\mu) = v\}) \subset \mathcal{H}$, and a routine calculation shows that for $e \in E^1$,

$$T_e^* e_{\mu g} = \begin{cases} e_{\mu' g} & \text{if } \mu = e\mu', \\ 0 & \text{otherwise.} \end{cases}$$

It is routine to check that the family $\{T_e \mid e \in E^1\} \cup \{W_h \mid h \in \mathcal{G}\}$ is a Toeplitz $(\mathcal{G}, E)$-family in $B(\mathcal{H})$.

The proofs of the following two lemmas are more or less identical to those of the cited results in [4–6].

LEMMA 3.3 (See [4, Lemma 3.4] and [5, Lemma 4.6]). *Let $(\mathcal{G}, E)$ be a self-similar groupoid. Suppose that $\{T_e, W_g\}$ is a Toeplitz $(\mathcal{G}, E)$-family in a C^*-algebra B. Then for all $\mu, \beta, \alpha, \rho \in E^*$, $g, h \in \mathcal{G}$,*

$$(T_\mu W_g T_\beta^*)(T_\alpha W_h T_\rho^*) = \begin{cases} T_{\mu(g\cdot\alpha')} W_{\varphi(g,\alpha')h} T_\rho^* & \text{if } \alpha = \beta\alpha', \\ T_\mu W_{g\varphi(h,h^{-1}\cdot\beta')} T^*_{\rho(h^{-1}\cdot\beta')} & \text{if } \beta = \alpha\beta', \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 3.4 (See [6, Corollary 1.16]). *Let $(\mathcal{G}, E)$ be a self-similar groupoid. Suppose that $\{T_e, W_g\}$ is a Toeplitz $(\mathcal{G}, E)$-family. Then*

$$C^*(T, W) = \overline{\operatorname{span}}\{T_\mu W_g T_\beta^* \mid \mu, \beta \in E^*, g \in \mathcal{G}_{s(\beta)}^{s(\mu)}, s(\mu) = g \cdot s(\beta)\}.$$

A standard argument along the lines of Propositions 1.20 and 1.21 of [6] shows that there exists a C^*-algebra $\mathcal{T}C^*(\mathcal{G}, E)$ generated by a Toeplitz $(\mathcal{G}, E)$-family $\{t_e, w_g\}$ that is universal in the sense that for any Toeplitz $(\mathcal{G}, E)$-family $\{T_e, W_g\}$, there is a homomorphism $\pi_{T,W} : \mathcal{T}C^*(\mathcal{G}, E) \to C^*(T, W)$ such that $\pi_{T,W}(t_e) = T_e$ for all $e \in E^1$ and $\pi_{T,W}(w_g) = W_g$ for all $g \in \mathcal{G}$.

Let I be the ideal of $\mathcal{T}C^*(\mathcal{G}, E)$ generated by $\{w_v - \sum_{r(e)=v} t_e t_e^* \mid v \in E^0\}$. Then $s_e := t_e + I$ for all $e \in E^1$ and $u_g := w_g + I$ for all $g \in \mathcal{G}$ defines a Cuntz–Krieger $(\mathcal{G}, E)$-family and $C^*(\mathcal{G}, E) := \mathcal{T}C^*(\mathcal{G}, E)/I$ is universal for Cuntz–Krieger $(\mathcal{G}, E)$-families. We will need to know that the generators of $C^*(\mathcal{G}, E)$ are nonzero. For this, we construct a concrete Cuntz–Krieger $(\mathcal{G}, E)$-family (see Proposition 3.6).

LEMMA 3.5. *Let $(\mathcal{G}, E)$ be a self-similar groupoid. Let $\pi : C^*(T, W) \to B(l^2(E^* * \mathcal{G}))$ be the representation induced by the Toeplitz $(\mathcal{G}, E)$-family $\{T_e, W_g\}$ of Example 3.2. For every $a \in I$ and every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,*

$$\|\pi(a)|_{\overline{\operatorname{span}}\{e_{\lambda g} \mid \lambda \in E^n, g \in \mathcal{G}^{s(\lambda)}\}}\| < \varepsilon.$$

PROOF. First, note that for $v \in E^0$, $\lambda \in E^*$ and $k \in \mathcal{G}^{s(\lambda)}$,

$$\Big(W_v - \sum_{e \in vE^1} T_e T_e^*\Big) e_{\lambda k} = \begin{cases} 0 & \text{if } \lambda \neq v, \\ e_{\lambda k} & \text{otherwise.} \end{cases} \tag{3.1}$$

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Now fix $v \in E^0, \mu, \beta \in E^*$ and $g \in \mathcal{G}_v^{s(\mu)}, h \in \mathcal{G}_v^{s(\beta)}$. Then,

$$T_\mu W_g\Big(W_v - \sum_{e \in vE^1} T_e T_e^*\Big) W_h^* T_\beta^* e_{\lambda k}$$
$$= \begin{cases} T_\mu W_g(W_v - \sum_{e \in vE^1} T_e T_e^*) e_{(h^{-1}\cdot\lambda')\varphi(h^{-1},\lambda')k} & \text{if } \lambda = \beta\lambda', \\ 0 & \text{otherwise.} \end{cases}$$

By (3.1), this equals 0 if $|\lambda'| > 0$. Hence,

$$\Big\| T_\mu W_g\Big(W_v - \sum_{e \in vE^1} T_e T_e^*\Big) W_h^* T_\beta^* e_{\lambda k} \Big\| = 0 \quad \text{whenever } |\lambda| > |\beta|. \tag{3.2}$$

Fix a finite linear combination $a_0 = \sum_{a_{\mu,g,h,\beta}} T_\mu W_g(W_v - \sum_{e \in vE^1} T_e T_e^*) W_h^* T_\beta^*$. Let $N = \max\{|\beta| \mid a_{\mu,g,h,\beta} \neq 0\}$. Then (3.2) implies that $\|a_0 e_{\lambda k}\| = 0$ whenever $|\lambda| > N$.

Finally, fix $a \in I$, and $\varepsilon > 0$. A routine argument gives

$$I = \overline{\operatorname{span}}\Big\{ t_\mu w_g\Big(w_v - \sum_{e \in vE^1} t_e t_e^*\Big) w_h^* t_\beta^* \mid \mu, \beta \in E^*, g, h \in \mathcal{G}, v \in E^0 \Big\}.$$

So there exists

$$a_0 \in \operatorname{span}\Big\{ T_\mu W_g\Big(W_v - \sum_{e \in vE^1} T_e T_e^*\Big) W_h^* T_\beta^* \mid \mu, \beta \in E^*, g, h \in \mathcal{G}, v \in E^0 \Big\},$$

such that $\|\pi(a) - a_0\| < \varepsilon$.

Take N as above and fix $n \geq N$. Then,

$$\|\pi(a)|_{\overline{\operatorname{span}}\{e_{\lambda k} | \lambda \in E^n, k \in \mathcal{G}^{s(\lambda)}\}}\| \leq \|\pi(a) - a_0\| + \|a_0|_{\overline{\operatorname{span}}\{e_{\lambda k} | \lambda \in E^n, k \in \mathcal{G}^{s(\lambda)}\}}\| < \varepsilon. \qquad \square$$

The following proposition will be used in describing our fixed-point algebra in the next section. Let G be a discrete group and let $\mathcal{H} = l^2(G) = \overline{\operatorname{span}}\{\delta_g \mid g \in G\}$. For $g \in G$, define $\lambda_g \in \mathcal{U}(l^2(G))$ by $\lambda_g(\delta_h) = \delta_{gh}$ for all $h \in G$. We get a representation $\lambda : C^*(G) \to B(\mathcal{H})$ such that $\lambda(u_g) = \lambda_g$ for all $g \in G$; we call this the *regular representation*. If G is amenable, then the representation λ is faithful. Since our groupoid is an amenable (discrete) groupoid, its (discrete) isotropy groups are also amenable.

PROPOSITION 3.6. *Let $(\mathcal{G}, E)$ be a self-similar groupoid. Let $\{s_e, u_g\}$ be the universal Cuntz–Krieger $(\mathcal{G}, E)$-family in $C^*(\mathcal{G}, E)$. Then each s_e and each u_g is nonzero. Fix $v \in E^0$. The universal property of $C^*(\mathcal{G}_v^v)$ gives a homomorphism $\pi_u : C^*(\mathcal{G}_v^v) \to C^*(\mathcal{G}, E)$ such that $\pi_u(\delta_h) = u_h$ for all $h \in \mathcal{G}_v^v$. Suppose that for each $k \in \mathbb{N}$, there exists $\lambda \in vE^k$ such that the map $g \mapsto (g \cdot \lambda)\varphi(g, \lambda)$ is injective. Then π_u is injective.*

PROOF. By the universal property, it suffices to construct a Cuntz–Krieger $(\mathcal{G}, E)$-family $\{S_e, U_g\}$ consisting of nonzero partial isometries. If $\{S_e, U_g\}$ is a Cuntz–Krieger $(\mathcal{G}, E)$-family and each $U_v \neq 0$, then $S_e \neq 0$ for all $e \in E^1$ and $U_g \neq 0$

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for all $g \in \mathcal{G}$, because $U_{s(e)} = S_e^* S_e$ and $U_{s(g)} = U_g^* U_g$. So, it suffices to construct a $(\mathcal{G}, E)$-family with $U_v \neq 0$ for all $v \in E^0$.

Let $\{T_e, W_g\}$ be the Toeplitz $(\mathcal{G}, E)$-family of Example 3.2. For $v \in E^0$, we have $W_v \cdot e_{\mu s(\mu)} = e_{\mu s(\mu)}$ for all $\mu \in vE^*$. So,

$$\|W_v|_{\overline{\operatorname{span}}\{e_{\lambda g} | \lambda \in vE^n, g \in \mathcal{G}^{s(\lambda)}\}}\| = 1.$$

Thus, Lemma 3.5 gives $W_v \notin I$. Therefore, $S_e := T_e + I$ and $U_g := W_g + I$ is a $(\mathcal{G}, E)$-family with each $U_v \neq 0$.

Now fix $v \in E^0$. Let $\pi_W : C^*(\mathcal{G}_v^v) \to B(l^2(E^* * \mathcal{G}))$ be the homomorphism such that $\pi_W(\delta_h) = W_h$. Fix $k \in \mathbb{N}$. Choose $\lambda \in vE^*$ such that the map $h \mapsto (h \cdot \lambda)\varphi(h, \lambda)$ is injective. Let $\mathcal{H}_\lambda := \overline{\operatorname{span}}\{e_{(h\cdot\lambda)\varphi(h,\lambda)} \mid h \in \mathcal{G}_v^v\} \subseteq l^2(E^* * \mathcal{G})$. By construction, $\mathcal{H}_\lambda$ is invariant for π_W.

Since the map $g \mapsto (g \cdot \lambda)\varphi(g, \lambda)$ is injective, there is an inner-product preserving map $\phi_\lambda : l^2(\mathcal{G}_v^v) \to \mathcal{H}_\lambda$ that maps the element e_g of the orthonormal basis of $l^2(\mathcal{G}_v^v)$ to the element $e_{(g\cdot\lambda)\varphi(g,\lambda)}$ of the orthonormal basis of $\mathcal{H}_\lambda$. For $h \in \mathcal{G}_v^v$, define $V_h^\lambda \in \mathcal{U}(l^2(\mathcal{G}_v^v))$ by $V_h^\lambda = \phi_\lambda^* W_h \phi_\lambda$. We get

$$\begin{aligned} V_h^\lambda e_g = \phi_\lambda^* W_h \phi_\lambda e_g = \phi_\lambda^* W_h e_{(g\cdot\lambda)(\varphi(g,\lambda))} &= \phi_\lambda^* e_{(h\cdot(g\cdot\lambda))(\varphi(h,g\cdot\lambda)\varphi(g,\lambda))} \\ &= \phi_\lambda^* e_{((hg)\cdot\lambda)(\varphi(hg,\lambda))} = e_{hg}. \end{aligned}$$

Hence, $\{V_h^\lambda \mid h \in \mathcal{G}_v^v\} \subseteq B(l^2(\mathcal{G}_v^v))$ is the regular representation of $\mathcal{G}_v^v$ and induces a faithful representation of $C^*(\mathcal{G}_v^v)$. Hence, the reduction of π_W to $\mathcal{H}_\lambda$ is injective, so its reduction to $l^2(E^k * \mathcal{G})$ is injective. Since k was arbitrary, the reduction of π_W to $l^2(E^k * \mathcal{G})$ is injective, and hence isometric for all k.

Now, fix $a \in C^*(\mathcal{G}_v^v) \setminus \{0\}$. Then for all k,

$$\|\pi_W(a)|_{\overline{\operatorname{span}}\{e_{\mu g} | \mu \in E^k, g \in \mathcal{G}^{s(\mu)}\}}\| = \|a\| \neq 0.$$

Thus, Lemma 3.5 implies $a \notin I$. We have $\pi_u(a) = a + I \neq 0$. Therefore, the homomorphism π_u is injective. □

To see that our faithfulness condition is strictly weaker than that of [5], we provide the following example.

Example 3.7. Let E be the graph with one vertex and n edges $e_0, \ldots, e_{n-1}$ and let $\mathcal{G} = \mathbb{Z}$. Define an action of $\mathcal{G}$ on E by $m \cdot e_i = e_{i+m}$ where addition is mod n, and define $\varphi(m, e_i) = m$ for all m. Then $\mathcal{G}$ does not act faithfully in the sense of [5], because $n \cdot e_i = e_i$ for all i. However, the map $(m, \lambda) \mapsto (m \cdot \lambda, \varphi(m, \lambda))$ is injective for each λ because $\varphi(m, \lambda) = m$ and then $\lambda = \varphi(m, \lambda)^{-1} \cdot (m \cdot \lambda)$. It is routine to see using universal properties that $C^*(\mathcal{G}, E) = \mathcal{O}_n \rtimes \mathbb{Z}$.

4. The gauge action and the core

Let $\{s_e, u_g\}$ be the universal Cuntz–Krieger $(\mathcal{G}, E)$-family in $C^*(\mathcal{G}, E)$. Then for $z \in \mathbb{T}$, the family $\{zs_e, u_g\}$ is also a Cuntz–Krieger $(\mathcal{G}, E)$-family. So, the universal property gives a homomorphism $\gamma_z : C^*(\mathcal{G}, E) \to C^*(\mathcal{G}, E)$ such that $\gamma_z(s_e) = zs_e$ and

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$\gamma_z(u_g) = u_g$ for all e, g. Since γ_1 agrees with the identity and $\gamma_z \circ \gamma_w$ agrees with γ_{zw} on generators, $z \mapsto \gamma_z$ is an action. A standard $\varepsilon/3$ argument shows that it is a strongly continuous action, which we call the *gauge action* on $C^*(\mathcal{G}, E)$. The *fixed-point algebra* of γ is the $*$-subalgebra

$$C^*(\mathcal{G}, E)^\gamma := \{a \in C^*(\mathcal{G}, E) \mid \gamma_z(a) = a \text{ for all } z \in \mathbb{T}\}$$

of $C^*(\mathcal{G}, E)$. The following corollary describes $C^*(\mathcal{G}, E)^\gamma$ concretely.

COROLLARY 4.1. *Let $(\mathcal{G}, E)$ be a self-similar groupoid and let $\Phi : C^*(\mathcal{G}, E) \to C^*(\mathcal{G}, E)^\gamma$ be the conditional expectation, $\Phi(a) = \int_{\mathbb{T}} \gamma_z(a)\, dz$. Then,*

$$\Phi(s_\mu u_g s_\beta^*) = \delta_{|\mu|,|\beta|} s_\mu u_g s_\beta^* \quad \textit{for } \mu, \beta \in E^* \textit{ and } g \in \mathcal{G}_{s(\beta)}^{s(\mu)}.$$

Further, $C^*(\mathcal{G}, E)^\gamma = \overline{\operatorname{span}}\{s_\mu u_g s_\beta^* \mid s(\mu) = g \cdot s(\beta) \text{ and } |\mu| = |\beta|\}$.

PROOF. We have $\gamma_z(s_\mu u_g s_\beta^*) = z^{|\mu|-|\beta|} s_\mu u_g s_\beta^*$, so $\Phi(s_\mu u_g s_\beta^*) = \delta_{|\mu|,|\beta|} s_\mu u_g s_\beta^*$. Moreover, $\Phi(C^*(\mathcal{G}, E)) = \overline{\operatorname{span}}\{s_\mu u_g s_\beta^* \mid s(\mu) = g \cdot s(\beta)$ and $|\mu| = |\beta|\}$. Proposition 3.2 of [6] shows that $\Phi(C^*(\mathcal{G}, E)) = C^*(\mathcal{G}, E)^\gamma$. □

Let $(\mathcal{G}, E)$ be a self-similar groupoid and let $\{S_e, U_g\}$ be a Cuntz–Krieger $(\mathcal{G}, E)$-family. For $k \in \mathbb{N}$, we define

$$\mathcal{F}_k(S, U) := \overline{\operatorname{span}}\{S_\mu U_g S_\beta^* \mid \mu, \beta \in E^k, g \in \mathcal{G}_{s(\beta)}^{s(\mu)}, s(\mu) = g \cdot s(\beta)\}.$$

We define a relation $\sim$ on E^0 by $v \sim w$ if and only if $\mathcal{G}_w^v \neq \emptyset$. Then $\sim$ is an equivalence relation. For $\xi \in E^0/\!\sim$, define

$$\mathcal{F}_k(S, U, \xi) := \overline{\operatorname{span}}\{S_\mu U_g S_\beta^* \mid \mu, \beta \in E^k, g \in \mathcal{G}_{s(\beta)}^{s(\mu)}, s(\mu) = g \cdot s(\beta) \in \xi\}.$$

When $\{S_e, U_g\}$ is the universal family $\{s_e, u_g\}$ in $C^*(\mathcal{G}, E)$, we write $\mathcal{F}_k := \mathcal{F}_k(s, u)$ and $\mathcal{F}_k(\xi) := \mathcal{F}_k(s, u, \xi)$.

NOTATION 4.2. For the next few results, fix a self-similar groupoid $(\mathcal{G}, E)$, an element $\xi \in E^0/\sim$, a vertex $v \in \xi$ and for each $u \in \xi$, an element $g_u \in \mathcal{G}_v^u$ (take $g_v = v$). We call $\{g_u \mid u \in \xi\}$ *a spanning tree for* $\mathcal{G}|_\xi$. We denote $E^k\xi := \{\mu \in E^k \mid s(\mu) \in \xi\}$.

PROPOSITION 4.3. *With Notation 4.2, let $\{S_e, U_g\}$ be a Cuntz–Krieger $(\mathcal{G}, E)$-family. For $h \in \mathcal{G}_v^v$ and $\mu \in E^*$, define $V_{h,\mu} := S_\mu U_{g_{s(\mu)}} U_h (S_\mu U_{g_{s(\mu)}})^*$. For each $k \in \mathbb{N}$, the series $\sum_{\mu \in E^k\xi} V_{h,\mu}$ converges strictly to a partial unitary $\overline{V}_h$ in $\mathcal{M}C^*(S, U)$ and $\overline{V}_h \mathcal{F}_k(S, U, \xi) \subseteq \mathcal{F}_k(S, U, \xi)$.*

PROOF. Fix $h \in \mathcal{G}_v^v$. For $\mu \in E^k\xi$,

$$V_{h,\mu} V_{h,\mu}^* = S_\mu S_\mu^* = V_{h,\mu}^* V_{h,\mu}. \tag{4.1}$$

For $\mu \neq \beta \in E^k\xi$, $S_\mu S_\mu^* S_\beta S_\beta^* = 0$. So, $S_\mu^* S_\beta = 0$. Therefore, for $F \subseteq E^k\xi$ finite,

$$\Big(\sum_{\mu \in F} V_{h,\mu}\Big)\Big(\sum_{\beta \in F} V_{h,\beta}\Big)^* = \sum_{\mu \in F} V_{h,\mu} V_{h,\mu}^* = \sum_{\mu \in F} S_\mu S_\mu^*.$$

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Now, fix $a \in C^*(S,U)$. Then $a = \lim_K \sum_{v\in K} P_v a$, where K ranges over all finite subsets of E^0. Let $P_K = \sum_{v\in K} P_v$. Fix $\varepsilon > 0$. There exists a finite set $K' \subseteq E^0$ such that $\|P_K a - a\| < \varepsilon/2$ for all finite $K \supseteq K'$.

Let $F \subseteq E^k\xi$ be the finite set $F = KE^k\xi$. For $F', F'' \supseteq F$,

$$\Big\|\sum_{\mu\in F'} S_\mu S_\mu^* a - \sum_{\beta\in F''} S_\beta S_\beta^* a\Big\| \le \Big\|\sum_{\mu\in F'\setminus F''} S_\mu S_\mu^* a\Big\| + \Big\|\sum_{\beta\in F''\setminus F'} S_\beta S_\beta^* a\Big\|$$
$$\le \|(1-P_K)a\| + \|(1-P_K)a\| < \varepsilon.$$

So, $(\sum_{\mu\in F} S_\mu S_\mu^* a)_{F\subseteq E^k\xi}$ is Cauchy and hence converges. Thus, $\sum_{\mu\in E^k\xi} S_\mu S_\mu^*$ converges strictly to a projection $P_\xi \in \mathcal{M}C^*(S,U)$. Equation (4.1) shows that $\sum_{\mu\in F} V_{h,\mu}^* V_{h,\mu}$ also converges strictly to P_ξ. Therefore, $\sum_{\mu\in E^k\xi} V_{h,\mu}$ converges strictly to a unitary $\overline{V}_h \in P_\xi \mathcal{M}C^*(S,U)P_\xi$.

Now fix a spanning element $S_\alpha U_l S_\beta^*$ of $\mathcal{F}_k(S,U,\xi)$. For each $\mu \in E^k\xi$, we obtain

$$V_{h,\mu} S_\alpha U_l S_\beta^* = \delta_{\mu,\alpha} S_\mu U_{g'l} S_\beta^* \quad \text{for some } g' \in \mathcal{G}_{s(\mu)}^{s(\mu)},$$

which implies that

$$\overline{V}_h S_\alpha U_l S_\beta^* = \sum_{\mu\in E^k\xi} \delta_{\mu,\alpha} S_\mu U_{g'l} S_\beta^* = S_\alpha U_{g'l} S_\beta^* \quad \text{for some } g' \in \mathcal{G}_{s(\alpha)}^{s(\alpha)},$$
$$\in \mathcal{F}_k(S,U,\xi).$$

Hence, $\overline{V}_h \mathcal{F}_k(S,U,\xi) \subseteq \mathcal{F}_k(S,U,\xi)$. □

PROPOSITION 4.4. *Fix $\xi \in E^0/\sim$ and $v \in \xi$. Let $\{S_e, U_g\}$ be a Cuntz–Krieger $(\mathcal{G},E)$-family. For $h \in \mathcal{G}_v^v$, let $\overline{V}_h$ be as in Proposition 4.3. Then there is a homomorphism $\pi_{\overline{V}} : C^*(\mathcal{G}_v^v) \to \mathcal{M}C^*(S,U)$ that maps δ_h to $\overline{V}_h$.*

PROOF. Let $h,k \in \mathcal{G}_v^v$. Routine calculations show that for $k \ge 1$ and $\mu \in E^k\xi$, we have $V_{h,\mu}V_{k,\mu} = V_{hk,\mu}$ and $V_{h,\mu}^* = S_\mu U_{g_{s(\mu)}} U_{h^{-1}} (S_\mu U_{g_{s(\mu)}})^* = V_{h^{-1},\mu}$. This implies that for any finite $F \subseteq E^k\xi$,

$$\sum_{\mu\in F} V_{h,\mu} \sum_{\mu\in F} V_{k,\mu} = \sum_{\mu\in F} V_{hk,\mu}.$$

Thus, $\overline{V}_h\overline{V}_k = \sum_{\mu\in E^k\xi} V_{h,\mu} \sum_{\mu\in E^k\xi} V_{k,\mu} = \overline{V}_{hk}$ and

$$\overline{V}_h^* = \sum_{\mu\in E^k\xi} V_{h,\mu}^* = \sum_{\mu\in E^k\xi} V_{h^{-1},\mu} = \overline{V}_{h^{-1}}.$$

So, the universal property of $C^*(\mathcal{G}_v^v)$ gives a homomorphism

$$\pi_{\overline{V}} : C^*(\mathcal{G}_v^v) \to \mathcal{M}(C^*(S,U) \quad \text{such that } \pi_{\overline{V}}(\delta_h) = \overline{V}_h.$$ □

PROPOSITION 4.5. *Fix $\xi \in E^0/\sim$. Let $\{S_e, U_g\}$ be a Cuntz–Krieger $(\mathcal{G},E)$-family. For $\mu,\beta \in E^k\xi$, let $e_\mu \otimes e_\beta^*$ denote the rank-one operator on the Hilbert space $l^2(\{E^k\xi\})$, and*

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let $\Theta_{\mu,\beta} := S_\mu U_{g_{s(\mu)}} U^*_{g_{s(\beta)}} S^*_\beta \in C^*(S,U)$. *Then there is an injective homomorphism*

$$\theta : \mathcal{K}(l^2(\{E^k\xi\})) \to \overline{\operatorname{span}}\{\Theta_{\mu,\beta} \mid \mu,\beta \in E^k\xi\}$$

such that $\theta(e_\mu \otimes e^*_\beta) = \Theta_{\mu,\beta}$.

Proof. We claim that the elements $\Theta_{\mu,\beta}$ are matrix units. Let $\mu,\beta,\alpha,\rho \in E^k\xi$. Then,

$$\begin{aligned}\Theta_{\mu,\beta}\Theta_{\alpha,\rho} &= (S_\mu U_{g_{s(\mu)}} U^*_{g_{s(\beta)}} S^*_\beta)(S_\alpha U_{g_{s(\alpha)}} U^*_{g_{s(\rho)}} S^*_\rho)\\ &= \begin{cases} S_\mu U_{g_{s(\mu)}} U^*_{g_{s(\rho)}} S^*_\rho & \text{if } \beta = \alpha,\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

and $(\Theta_{\mu,\beta})^* = S_\beta U_{g_{s(\beta)}} U^*_{g_{s(\mu)}} S^*_\mu = \Theta_{\beta,\mu}$. Hence, $\{\Theta_{\mu,\beta} \mid \mu,\beta \in E^k\xi\}$ is a family of matrix units. Since

$$\|S_\mu U_{g_{s(\mu)}} U^*_{g_{s(\beta)}} S^*_\beta\|^2 = \|S_\beta U_{g_{s(\beta)}} U^*_{g_{s(\mu)}} S^*_\mu S_\mu U_{g_{s(\mu)}} U^*_{g_{s(\beta)}} S^*_\beta\| = \|P_v\| = 1,$$

by Lemma 3.3, these are nonzero matrix units. Hence, by Corollary A.9 of [6], we get the injective homomorphism θ as claimed. □

Proposition 4.6. *Fix* $\xi \in E^0/\sim$ *and* $v \in \xi$. *Let* $\{S_e, U_g\}$ *be a Cuntz–Krieger* $(\mathcal{G}, E)$*-family. Let* $\pi_{\overline{V}}$ *and* θ *be as in Propositions 4.4 and 4.5, respectively. Then, there exists a homomorphism*

$$\theta \otimes \pi_{\overline{V}} : \mathcal{K}(l^2(\{E^k\xi\})) \otimes C^*(\mathcal{G}^v_v) \to \mathcal{F}_k(S,U,\xi)$$

such that

$$\theta \otimes \pi_{\overline{V}}((e_\mu \otimes e^*_\beta) \otimes \delta_h) = \theta(e_\mu \otimes e^*_\beta)\pi_{\overline{V}}(\delta_h) = \pi_{\overline{V}}(\delta_h)\theta(e_\mu \otimes e^*_\beta)$$

for all $e_\mu \otimes e^*_\beta \in \mathcal{K}(l^2(\{E^k\xi\}))$ *and for all* $\delta_h \in C^*(\mathcal{G}^v_v)$.

Proof. We have $\theta(e_\mu \otimes e^*_\beta) = \Theta_{\mu,\beta}$ and $\pi_{\overline{V}}(\delta_h) = \overline{V}_h$ in $\mathcal{F}_k(\xi)$ for all $e_\mu \otimes e^*_\beta \in \mathcal{K}(l^2(\{E^k\xi\}))$ and for all $\delta_h \in C^*(\mathcal{G}^v_v)$. Then,

$$\begin{aligned}\Theta_{\mu,\beta}\overline{V}_h &= S_\mu U_{g_{s(\mu)}} U^*_{g_{s(\beta)}} S^*_\beta \overline{V}_h\\ &= S_\mu U_{g_{s(\mu)}} U^*_{g_{s(\beta)}} S^*_\beta \sum_{\gamma \in E^k\xi} S_\gamma U_{g_{s(\gamma)}} U_h U^*_{g_{s(\gamma)}} S^*_\gamma\\ &= \sum_{\gamma \in E^k\xi} S_\mu U_{g_{s(\mu)}} U^*_{g_{s(\beta)}} S^*_\beta S_\gamma U_{g_{s(\gamma)}} U_h U^*_{g_{s(\gamma)}} S^*_\gamma\\ &= S_\mu U_{g_{s(\mu)}} U_h U^*_{g_{s(\beta)}} S^*_\beta.\end{aligned}$$

A similar calculation gives $\overline{V}_h\Theta_{\mu,\beta} = S_\mu U_{g_{s(\mu)}} U_h U^*_{g_{s(\beta)}} S^*_\beta$. Hence, $\Theta_{\mu,\beta}\overline{V}_h = \overline{V}_h\Theta_{\mu,\beta}$. We claim that

$$\overline{\operatorname{span}}\{\Theta_{\mu,\beta}\overline{V}_h \mid \mu,\beta \in E^k\xi, h \in \mathcal{G}^v_v\} = \mathcal{F}_k(S,U,\xi).$$

Let $\mu, \beta, \alpha, \rho \in E^k\xi$ and $h_1, h_2 \in \mathcal{G}_v^v$. Then,

$$\Theta_{\mu,\beta}\overline{V}_{h_1}\Theta_{\alpha,\rho}\overline{V}_{h_2} = \Theta_{\mu,\beta}\Theta_{\alpha,\rho}\overline{V}_{h_1}\overline{V}_{h_2} = \delta_{\beta,\alpha}\Theta_{\mu,\rho}\overline{V}_{h_1h_2}$$

and $(\Theta_{\mu,\beta}\overline{V}_h)^* = \overline{V}_{h^{-1}}\Theta_{\beta,\mu} = \Theta_{\beta,\mu}\overline{V}_{h^{-1}}$. So, $\overline{\operatorname{span}}\{\Theta_{\mu,\beta}\overline{V}_h \mid \mu,\beta \in E^k\xi, h \in \mathcal{G}_v^v\}$ is a C^*-subalgebra of $\mathcal{F}_k(S,U,\xi)$. Moreover, it contains the generators of $\mathcal{F}_k(S,U,\xi)$, so it is all of $\mathcal{F}_k(S,U,\xi)$.

Now the universal property of the (maximal) tensor product gives the desired homomorphism $\theta \otimes \pi_{\overline{V}}$. □

We show next the homomorphism $\theta \otimes \pi_{\overline{V}}$ is faithful. To show this, we need to verify that both θ and $\pi_{\overline{V}}$ are injective. From Proposition 4.5, we already know that θ is injective, so it suffices to show that $\pi_{\overline{V}}$ is injective as well.

Lemma 4.7. *Fix $\xi \in E^0/\sim$ and $v \in \xi$. Let $\{S_e, U_g\}$ be a Cuntz–Krieger $(\mathcal{G},E)$-family. Suppose that the homomorphism $\pi_U : C^*(\mathcal{G}_v^v) \to C^*(S,U)$ that maps δ_h to U_h is injective. Fix $k \in \mathbb{N}$ and $v \in E^0$. Let $\overline{V}_h$ be as in Proposition 4.3. Then, the homomorphism $\pi_{\overline{V}}^{(v,k)} : C^*(\mathcal{G}_v^v) \to \mathcal{F}_k(S,U,\xi)$ that maps δ_h to $\overline{V}_h$ is injective.*

Proof. Fix $\lambda \in E^k\xi$ and let $Y_\lambda = S_\lambda U_{g_{s(\lambda)}}$. Then,

$$Y_\lambda^*\overline{V}_hY_\lambda = \sum_{\mu\in E^k\xi} U^*_{g_{s(\lambda)}}S^*_\lambda S_\mu U_{g_{s(\mu)}}U_hU^*_{g_{s(\mu)}}S^*_\mu S_\lambda U_{g_{s(\lambda)}} = U_h.$$

Define $\mathrm{Ad}_{Y_\lambda} : \mathcal{F}_k(S,U,\xi) \to C^*(S,U)$ by $\mathrm{Ad}_{Y_\lambda}(a) = Y_\lambda^* a Y_\lambda$. By linearity and continuity, $\mathrm{Ad}_{Y_\lambda} \circ \pi_{\overline{V}}^{(v,k)} = \pi_U$. Hence, $\mathrm{Ad}_{Y_\lambda} \circ \pi_{\overline{V}}^{(v,k)}$ is injective, so $\pi_{\overline{V}}^{(v,k)}$ is also injective. □

Since $\mathcal{K}(l^2(\{E^k\xi\}))$ is simple and nuclear, Proposition 4.5 and Lemma 4.7 show that if π_U is injective on $C^*(\mathcal{G}_v^v)$, then the homomorphism of Proposition 4.6 is an isomorphism. So,

$$\mathcal{F}_k(\xi) \cong \mathcal{K}(l^2(\{E^k\xi\})) \otimes C^*(\mathcal{G}_v^v). \tag{4.2}$$

Moreover, we obtain the following corollary. Recall that

$$\mathcal{F}_k = \overline{\operatorname{span}}\{s_\mu u_g s_\beta^* \mid s(\mu) = g \cdot s(\beta), \text{ and } |\mu| = |\beta| = k\}.$$

Corollary 4.8. *Let $(\mathcal{G},E)$ be a self-similar groupoid. Fix $\xi \in E^0/\sim$ and $v \in \xi$. Suppose that for each $k \in \mathbb{N}$, there exists $\lambda \in vE^k$ such that the map $g \mapsto (g\cdot\lambda)\varphi(g,\lambda)$ is injective. Then,*

$$\mathcal{F}_k \cong \bigoplus_{\xi\in E^0/\sim} \mathcal{F}_k(\xi) \cong \bigoplus_{\xi\in E^0/\sim} \mathcal{K}(l^2(\{E^k\xi\})) \otimes C^*(\mathcal{G}_v^v).$$

Proof. For $\mu,\beta,\alpha,\rho \in E^k$ with $s(\mu) = g\cdot s(\beta) \in \xi_1$ and $s(\alpha) = h \cdot s(\rho) \in \xi_2$, the equation of Lemma 3.3 gives

$$(s_\mu u_g s_\beta^*)(s_\alpha u_h s_\rho^*) = \begin{cases} s_\mu u_g u_h s_\rho^* & \text{if } \beta = \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

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Hence, $\mathcal{F}_k(\xi_1)\mathcal{F}_k(\xi_2) = 0$, when $\xi_1 \neq \xi_2$, so Corollary A.11 of [6] combined with (4.2) gives an isomorphism of $\bigoplus_{\xi \in E^0/\sim} \mathcal{F}_k(\xi)$ onto $\mathcal{F}_k$. Equation (4.2) gives the second isomorphism. □

COROLLARY 4.9. *Let $(\mathcal{G}, E)$ be a self-similar groupoid. Then,*

$$C^*(\mathcal{G}, E)^\gamma = \overline{\bigcup_k \mathcal{F}_k} = \overline{\bigcup_k \Big(\bigoplus_{\xi \in E^0/\sim} \mathcal{F}_k(\xi) \Big)}.$$

PROOF. For any k, we claim that $\mathcal{F}_k \subset \mathcal{F}_{k+1}$. Fix $\mu, \beta \in E^k$, $g \in \mathcal{G}$ with $s(\mu) = g \cdot s(\beta)$. We have

$$s_\mu u_g s_\beta^* = s_\mu u_g u_{s(g)} s_\beta^* = \sum_{e \in s(g)E^1} s_\mu u_g s_e s_e^* s_\beta^* = \sum_{e \in s(g)E^1} s_{\mu(g\cdot e)} u_{\varphi(g,e)} s_{\beta e}^* \in \mathcal{F}_{k+1}.$$

Hence, $\mathcal{F}_k \subset \mathcal{F}_{k+1}$ for all k. By Corollary 4.1, the claim follows. □

LEMMA 4.10. *Let $(\mathcal{G}, E)$ be a self-similar groupoid. Suppose that $\{T_e, W_g\}$ is a $(\mathcal{G}, E)$-family in a C^*-algebra B. Let*

$$\pi_{T,W} : C^*(\mathcal{G}, E) \to C^*(T, W)$$

be the homomorphism induced by the universal property. Suppose that for each $v \in E^0$, the homomorphism $\pi_{v,W} : C^*(\mathcal{G}_v^v) \to C^*(T, W)$ such that $\pi_{v,W}(\delta_g) = W_g$ for all g is injective. Then, $\pi_{T,W}$ is isometric on $C^*(\mathcal{G}, E)^\gamma$.

PROOF. Fix $\xi \in E^0/\sim$ and $v \in \xi$. Choose elements $g_w \in \mathcal{G}_v^w$ for $w \in \xi$ with $g_v = v$. For $h \in \mathcal{G}_v^v$ and $k \in \mathbb{N}$, let $\overline{W}_h = \sum_{\mu \in E^k \xi} T_\mu W_{g_{s(\mu)}} W_h W_{g_{s(\mu)}}^* T_\mu^*$ as in Proposition 4.3. Lemma 4.7 shows that the homomorphism $\pi_{\overline{W}} : C^*(\mathcal{G}_v^v) \to \mathcal{M}C^*(T, W)$ is injective.

Let $\theta \otimes \pi_{\overline{W}}$ be as in Proposition 4.6. Since $\mathcal{K}(l^2(E^k\xi))$ is simple and nuclear, and since each $T_\mu T_\beta^* \neq 0$, the map $\pi_{T,W} \circ (\theta \otimes \pi_{\overline{u}}) = \theta \otimes \pi_{\overline{W}}$ is injective on each $\mathcal{F}_k(\xi)$. Therefore, it is also injective on $\mathcal{F}_k = \bigoplus_{\xi \in E^0/\sim} \mathcal{F}_k(\xi)$. Because every injective C^*-algebra homomorphism is isometric, $\pi_{T,W}$ is isometric on $\mathcal{F}_k$. Hence, $\pi_{T,W}$ is isometric on $\bigcup_k \mathcal{F}_k$ and hence on $\overline{U_k \mathcal{F}_k} = C^*(\mathcal{G}, E)^\gamma$. □

5. The gauge-invariant uniqueness theorem

THEOREM 5.1. *Let $(\mathcal{G}, E)$ be a self-similar groupoid. Suppose that (T, W) is a $(\mathcal{G}, E)$-family in a C^*-algebra B. The universal property of $C^*(\mathcal{G}, E)$ gives a homomorphism*

$$\pi_{T,W} : C^*(\mathcal{G}, E) \to C^*(T, W).$$

If there is a continuous action $\eta : \mathbb{T} \to \mathrm{Aut}B$ such that $\eta_z(T_e) = zT_e$ and $\eta_z(W_g) = W_g$ for all $e \in E^1$ and $g \in G$, and if the homomorphism $\pi_{v,W}$ is injective for each $v \in E^0$, then $\pi_{T,W}$ is an isomorphism of $C^*(\mathcal{G}, E)$ onto $C^*(T, W)$.

PROOF. Let $\Phi : C^*(\mathcal{G}, E) \to C^*(\mathcal{G}, E)^\gamma$ be the faithful conditional expectation of Corollary 4.1. Let $\Psi : C^*(T, W) \to C^*(T, W)^\eta$ be the corresponding expectation

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obtained from η. Since $\eta_z \circ \pi_{T,W}$ and $\pi_{T,W} \circ \gamma_z$ agree on generators, they are equal. Hence, $\Psi \circ \pi_{T,W} = \pi_{T,W} \circ \Phi$. By [8, Lemma 3.14], $\pi_{T,W}$ is injective if it is injective on $C^*(\mathcal{G}, E)^\gamma$, which it is by Lemma 4.10. □

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