C. IONIZED GAS IN A MAGNETIC FIELD

PAPER 8

IONIZED GAS IN A MAGNETIC FIELD

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ABSTRACT

The ionized gas is described as a mixture of several fluids; each obeying a quasi-hydrodynamic equation of motion with additional terms describing the mechanical interaction. Particularly, two- and three-fluid models are considered. The nature of the approximations ('quasi-neutrality', 'creeping diffusion') is discussed. Conservation-laws are formulated for the case of negligible effect of mutual encounters and of pressure diffusion. These models lead to a generalization of Ohm's law; it is shown that the additional terms are of practical importance if one has three components, of which one may be neutral.

1. INTRODUCTION AND GENERAL FORMULATION OF THE MODEL

The dynamics of an ionized gas in an electromagnetic field can be treated by different methods. An exact theory would consist of the solution of the Boltzmann equation of the kinetic theory of gases together with Maxwell's equations, taking all interactions into account. Since this is impracticable one has to rely upon approximate methods. One such approximation consists in using a hydrodynamic equation of motion for the ionized gas supplemented by a term representing the Lorentz force, together with Maxwell's equations (usually neglecting the displacement current) and a form of Ohm's law appropriate to moving conductors. So one arrives at a set of simultaneous equations, known as the 'hydromagnetic' equations

$$\rho \frac{d\mathbf{v}}{dt} = \mathbf{k} + \frac{1}{c}\,\mathbf{j}\times\mathbf{H}; \quad \operatorname{div}(\rho\mathbf{v}) = -\frac{\partial\rho}{\partial t}, \tag{1a}$$

$$\mathbf{j} = \sigma\mathbf{E}^c; \quad \mathbf{E}^c = \mathbf{E} + \frac{1}{c}\,\mathbf{v}\times\mathbf{H}, \tag{1b}$$

$$c \operatorname{curl} \mathbf{H} = 4\pi\mathbf{j}; \quad \operatorname{div} \mathbf{H} = 0, \tag{1c}$$

and

$$c \operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{E}}{\partial t} \qquad (1\,d)$$

($\mathbf{k}$ = density of all non-electromagnetic forces, for instance $\mathbf{k} = -\operatorname{grad} p + \rho \mathbf{g}$; ρ = mass density; $\mathbf{g}$ = gravitational acceleration; σ = ohmic conductivity; all other symbols have their usual meanings).

A different method can be applied, particularly when the electromagnetic field can be considered known and when the effect of pressure is small. It consists of solving the equation of motion of some representative kind of the differently charged particles—either strictly or by some approximate method, as the one first used by Alfvén [1]—and thence one infers the behaviour of the ionized gas more or less intuitively. The difficulties lie here in knowing the electromagnetic field and in selecting the representative particles; because of this some of the results gained by this method are demonstratively spurious.

I shall discuss a third method, which is in some respects intermediate between the two approaches I have mentioned. Here all particles of one kind (characterized by their charge per mass ratio) are taken to constitute a fluid, and the ionized gas is described as a mixture of (at least two) such fluids penetrating each other. The equation of motion of each fluid component is the conventional hydrodynamic one apart from additional terms describing the interaction with the Maxwell field due to the electric charge and the electric current carried by the component considered and further terms representing the frictional interaction between all components. We assume that the frictional force between the jth and the kth component is proportional to the relative velocity $\mathbf{v}_j - \mathbf{v}_k$ and proportional to the density of either component. We therefore write for it

$$\beta_{jk} \rho_j \rho_k (\mathbf{v}_j - \mathbf{v}_k); \qquad (2)$$

ρ_j = mass density of jth component.

In a free path theory the parameter β_{jk} depends on the collisional cross-section q_{ik}, on the root mean square relative speed v_{jk} and on the masses approximately

$$\beta_{jk} \approx q_{jk} v_{jk} / (m_j + m_k). \qquad (3)$$

One would therefore expect it to be essentially independent of a possible magnetic field. This is borne out by an exact kinetic treatment, the total change of β_{jk} between zero magnetic field and infinite field strength being by about a factor 2.

The equation of motion of the jth component is then

$$\rho_j \frac{d_j \mathbf{v}_j}{dt} = \rho_j \sum_k \rho_k \beta_{kj}(\mathbf{v}_k - \mathbf{v}_j) + \rho_j z_j \left\{ \mathbf{E} + \frac{1}{c}\, \mathbf{v}_j \times \mathbf{H} \right\} + \mathbf{k}_j \tag{4}$$

(d_j/dt = time derivative following the motion of the jth component, $\beta_{kj} = \beta_{jk} > 0$ is the friction parameter, z_j = charge to mass ratio, k_j = density of all non-electromagnetic forces including the gradient of the partial pressure p_j).

For later convenience we give here a list of abbreviations:

$\rho = \sum_k \rho_k$ (total mass density),

$\mathbf{k} = \sum_k \mathbf{k}_k$ (total non-electromagnetic force),

$\rho \mathbf{v} = \sum_k \rho_k \mathbf{v}_k$ (mean mass velocity),

$\mathbf{j} = \sum_k \rho_k z_k \mathbf{v}_k$ (electric current density),

$\mathbf{E}^c = \mathbf{E} + (1/c)\, \mathbf{v} \times \mathbf{H}$ (electric field in a system of reference moving with the mean mass velocity).

In addition to Eq. (4) we have Maxwell's equations including the relation between the electric field and the total charge density

$$\operatorname{div} \mathbf{E} = 4\pi \sum_k \rho_k z_k. \tag{5}$$

Together with the approximate equations determining the non-electromagnetic force (which we shall however not consider in detail) the system of equations is complete. To simplify it, we use two approximations.

1. We assume quasi-neutrality. That is, we drop Eq. (5) and replace it by

$$\sum_k \rho_k z_k = 0. \tag{6}$$

With this relation the system is again complete and suffices to determine the electric field, including its curl-free part. The approximation is consistent when it turns out that

$$|\operatorname{div} \mathbf{E}| \ll 4\pi \sum_k \rho_k |z_k|, \quad \text{say}, \tag{7}$$

so that the relative magnitude of the two sides of this inequality is of such an order that the error in the densities ρ_k induced by the approximation does not appreciably influence the solution of the equation of motion.

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2. We assume the diffusion to be 'creeping' by neglecting the difference of acceleration of the components:

$$\frac{d_j \mathbf{v}_j}{dt} = \frac{d\mathbf{v}}{dt}. \tag{8}$$

Since usually the diffusion velocities are small compared with the mean velocity, Eq. (8) holds in most practical cases too. The physical meaning of this assumption is that the diffusion equilibrium sets in instantaneously whether or not the gas as a whole is being accelerated.

The two assumptions are of quite a different nature. Assumption 1 will be good in almost all cases of astrophysical interest, except for problems like a non-linear theory of plasma oscillations of great amplitude. The simplification gained is relatively minor, however, since we have replaced one instantaneous differential equation (namely Eq. (5)) by an algebraic relation. There are however many cases where diffusion is certainly not at all 'creeping', but the inertia due to the relative velocities is of decisive importance. Particularly, plasma oscillations are excluded. The approximation 2 will be good in all cases where a 'magnetohydrodynamic' approach is reasonable. While the restriction imposed is severe, the simplification is considerable since we are now left with just as many time derivations as in ordinary hydromagnetics.

We shall first discuss the Eqs. (4) together with our assumptions in the special case of a two-component plasma and only briefly deal with the more general case of a three-component plasma.*

2. THE TWO-FLUIDS MODEL

If only two constituents are present our equations are readily solved in terms of those variables which describe the behaviour of the plasma as a whole:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\operatorname{grad} \mathbf{v}\right) = \frac{1}{c}\,\mathbf{j} \times \mathbf{H} - \operatorname{grad} p, \tag{9a}$$

$$c\mathbf{E} + \mathbf{v} \times \mathbf{H} = \frac{c}{\sigma}\,\mathbf{j} + \frac{\alpha}{\rho}\,\{\mathbf{j} \times \mathbf{H} - c \operatorname{grad} \gamma p\}, \tag{9b}$$

* For the case of a two-component field, equations corresponding to those given here were first considered by A. Schlüter[2] and independently by M. H. Johnson and E. O. Hulburt [3] (the latter without the inertial terms). The three-component case was considered by A. Schlüter[4] and applied to interstellar magnetic fields by A. Schlüter and L. Biermann[5] and to the ionosphere by I. Lucas and A. Schlüter[6]. The relation to the kinetic theory of gases was established in the case of creeping diffusion by M. H. Johnson[7].

where $\sigma = \beta_{12}/z_1 . z_2$ (ohmic conductivity),

$$\alpha = 2(\rho_1 - \rho_2)/(\rho_1 z_1 - \rho_2 z_2)$$

and $\gamma = (\rho_1 p_2 - \rho_2 p_1)/(\rho_1 - \rho_2)\, p.$

We have here taken pressure as the only non-electromagnetic force acting.

α is a constant which only depends on the nature of the plasma, not on its actual state. For a mixture of ions (of mass m_i and charge $+e$) and electrons it is practically $\alpha = m_i/e$. γ depends on temperature only as far as the ratio of the partial pressures does, so in a simple plasma where the electron temperature equals that of the ions, $\gamma = \frac{1}{2}$.

The difference between these equations and the hydromagnetic equations (1) lies only in the two terms multiplied by α. These are the Hall term $(\alpha/\rho c)$ $(\mathbf{j} \times \mathbf{H})$ which produces an electric field when a current flows across the lines of force, and the pressure diffusion term (α/ρ) grad γp. Both terms are the more important the smaller the density becomes. The Hall term is more important than the Ohm term $\mathbf{j}/\sigma$ if $\alpha H/c > \rho/\sigma$, that is if the mean of the gyro-frequencies is larger than the collision frequency, and this is the case for practically all cosmical magnetic fields, except for the interior of stars, planets, and the like. We are particularly interested in the deviations from ordinary hydromagnetics and shall therefore discuss the extreme case of vanishing ohmic resistivity.

If $\sigma \to \infty$, the Eqs. (9) allow a number of transformations. One of these is gained by eliminating in Eq. (9*a*) the Lorentz term by means of the Hall term of Eq. (9*b*). We then obtain:

$$\rho \frac{d\mathbf{v}}{dt} = \frac{\rho}{\alpha}\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{H}\right) - \text{grad}\,(1-\gamma)\,p. \tag{10}$$

This is the equation of motion of a charged fluid with a mass-to-charge ratio α and a pressure $(1-\gamma)\,p \approx p/2$. So, if the electromagnetic field is known, it is essentially the same problem to solve the equation of motion for a quasi-neutral plasma as for a gas consisting of charged particles of one kind only. This is one of the many possible transformations in this field which are correct in a formal sense but completely useless in practice. The point is, in this case, that we never know the electromagnetic field beforehand. There are situations where we may neglect the influence on the magnetic field by the currents flowing in the plasma, or where we may treat it as a perturbation. But the electric field is always determined by the space charges in the fluid, whenever the motion of a quasi-neutral plasma is applicable. There is no other way but to solve both the equation of motion

(9*a*) (or equivalently (10)) *and* the diffusion equation (9*b*) simultaneously. Because of the relative unimportance of **E** it is advisable to eliminate **E** in Eq. (9*b*) by taking its curl:

$$\frac{\partial \mathbf{H}}{\partial t} = -\operatorname{curl}\left(\frac{c^2}{4\pi}\operatorname{curl}\mathbf{H}\right) + \operatorname{curl}\left\{\left(\mathbf{v}-\frac{\alpha}{\rho}\mathbf{j}\right)\times\mathbf{H}\right\} - \frac{\alpha c}{\rho^2}(\operatorname{grad}\rho\times\operatorname{grad}\beta p). \tag{11}$$

We consider again the case of vanishing resistivity $1/\sigma$. If furthermore $\alpha/\rho \to 0$, we have the well-known hydromagnetic relation

$$\frac{\partial \mathbf{H}}{\partial t} = \operatorname{curl}(\mathbf{v}\times\mathbf{H}). \tag{12}$$

This equation has a simple meaning: the magnetic flux through every closed line which is moving with the fluid is constant—the magnetic lines of force are frozen in. Returning to the case $\alpha \neq 0$, it is tempting to introduce instead of the mean mass velocity a slightly different velocity $\mathbf{v}'$ by

$$\mathbf{v}' = \mathbf{v} - \frac{\alpha}{\rho}\mathbf{j}$$

or

$$\rho\mathbf{v}' = \rho_2\mathbf{v}_1 + \rho_1\mathbf{v}_2. \tag{13}$$

If we use this mean velocity Eq. (11) reads (with $\sigma \to \infty$)

$$\frac{\partial \mathbf{H}}{\partial t} = \operatorname{curl}(\mathbf{v}'\times\mathbf{H}) - \frac{\alpha c}{\rho^2}[\operatorname{grad}\rho\times\operatorname{grad} p]. \tag{14}$$

The only effect of the Hall term is therefore that the lines of force are not moving with the mean mass velocity ($\mathbf{v}$) but with a velocity which differs from this by a quantity of the order of the relative diffusion velocity. This result is not surprising, since the concept of the mean mass velocity has been introduced for its obvious importance for the equation of motion, but it is certainly not appropriate to the diffusion problem when the forces are not proportional to the masses. Besides this motional-induction term we have the pressure term; it describes real creation and annihilation of the magnetic lines of force and is of particular importance if one wants to treat the first origin of magnetic fields in fluid conductors. It disappears if p and ρ are uniquely related to each other as they are in many simple cases of interest.

Another useful relation is obtained by taking the curl of Eq. (10). Introducing the vorticity $\boldsymbol{\omega} = \text{curl } \mathbf{v}$, we have:

$$\frac{\partial(\mathbf{H}+\alpha c\boldsymbol{\omega})}{\partial t} = \text{curl } \{\mathbf{v} \times (\mathbf{H}+\alpha c\boldsymbol{\omega})\} + \frac{\alpha c}{\rho^2} \{\text{grad } \rho \times \text{grad } (1-\gamma)\, p\}. \tag{15}$$

We have seen that the magnetic field moves with a velocity which is different from the mean mass velocity, now we see which quantity it is that is transported by $\mathbf{v}$; in the case of a mixture of ions and electrons the conserved quantity is

$$\frac{e\mathbf{H}}{(m_i - m_e)\, c} + \text{curl } \mathbf{v}. \tag{16}$$

For the influence of the pressure term the same remarks apply as above. Eq. (15) has two interesting limiting cases. If $\alpha \to 0$, we return to the hydromagnetic case previously discussed. For $\mathbf{H} \to 0$ we obtain, however, the well-known vorticity theorem of hydrodynamics, but with a small deviation:

$$\frac{\partial \text{ curl } \mathbf{v}}{\partial t} = \text{curl } \{\mathbf{v} \times \text{curl } \mathbf{v}\} + \frac{1}{\rho^2} \{\text{grad } \rho \times \text{grad } (1-\gamma)\, p\}, \tag{17}$$

the deviation consisting in the term with grad γp, which we have already found to be responsible for the creation of magnetic flux. The two quantities, the sum of which is conserved in the considered sense, are comparable to one another when curl $\mathbf{v}$ has the order of the smaller gyrofrequency of either component. So again, the modification to hydromagnetics due to the Hall terms is very small indeed for all cases where the application of hydromagnetics is at all reasonable.

The fact that the real importance of the Hall effect is very small is contrary to what one would expect, if one describes its effect as a reduction of conductivity across the lines of force. In our case, where we have considered the case of vanishing ohmic resistivity ($\sigma \to \infty$), the cross-conductivity would indeed be zero.

3. THE THREE-FLUIDS MODEL

From the treatment of the two-fluids model we have learned how to handle the diffusion equations: we have to solve them with respect to the electric field $\mathbf{E}$ or $\mathbf{E}^c$ in terms of the magnetic field (and thereby the electric current) and the partial pressures only. So we arrive at a modification of Ohm's law, which is then used to determine curl $\mathbf{E}$ and thereby $\partial\mathbf{H}/\partial t$.

If we carry through this programme for the case where we have three

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different constituents, and solve the Eqs. (4), remembering our assumptions (6) and (8), we arrive at the following somewhat lengthy formula:

$$
\begin{aligned}
&(\beta_{23}\rho_1 z_1^2 + \beta_{13}\rho_2 z_2^2 + \beta_{12}\rho_3 z_3^2)\ \mathbf{E}^c + z_1 z_2 z_3(\mathbf{E}^c \times \mathbf{H}/c) \\
&\quad = (\beta_{12}\beta_{13}\rho_1 + \beta_{12}\beta_{23}\rho_2 + \beta_{13}\beta_{32}\rho_3)\ \mathbf{j} \\
&\qquad - (1/\rho)\ (\beta_{23} z_1(\rho - 2\rho_1) + \beta_{13} z_2(\rho - 2\rho_2) + \beta_{12} z_3(\rho - 3\rho_3))\ (\mathbf{j} \times \mathbf{H}/c) \\
&\qquad + (1/\rho^2)\ (\rho_1 z_2 z_3 + \rho_2 z_3 z_1 + \rho_3 z_1 z_2)\ (\mathbf{j} \times \mathbf{H}/c) \times \mathbf{H}/c \\
&\qquad + (1/\rho)\ \{\beta_{23}\rho_1 z_1 + \beta_{13}\rho_2 z_2 + \beta_{12}\rho_3^3 z_3\}\ \mathbf{k} \\
&\qquad - \{\beta_{23}\rho_1 z_1 \mathbf{k}_1 + \beta_{13}\rho_2 z_2 \mathbf{k}_2 + \beta_{12}\rho_3 z_3 \mathbf{k}_3\} \\
&\qquad + (1/\rho^2)\ \{\rho_1 z_2 z_3 + \rho_2 z_1 z_3 + \rho_3 z_1 z_2\}\ \mathbf{k} \times \mathbf{H}/c \\
&\qquad - (1/\rho)\ \{\rho_1 z_2 z_3 \mathbf{k}_1 + \rho_2 z_1 z_3 \mathbf{k}_2 + \rho_3 z_1 z_2 \mathbf{k}_3\}\ \times \mathbf{H}/c.
\end{aligned}
$$

We have by this formulation not completely fulfilled our aim, the term $\mathbf{E}^c \times \mathbf{H}$ not being removed. This could easily be done, but in the case of greatest interest—namely if one component is not charged—its coefficient disappears. The essential novel features appearing here are the terms which contain the square of H/ρ. They are larger than the Hall term in the ratio given by a certain average value of the gyro-frequencies relative to the collision frequencies. Their occurrence is most easily explained in the case of one neutral component. Then, the Lorentz force $\mathbf{j} \times \mathbf{H}/c$ acts on the charged components only, hence these move relative to the neutral component ('ambipolar diffusion'). It is then the mean velocity of the charged components which determines the motional induction $\mathbf{v} \times \mathbf{H}$ and this velocity differs from the mean mass velocity by a term proportional to the Lorentz force. By this effect the dissipation of energy is really increased if an electric current flows perpendicular to the lines of force and—as found by A. Schlüter and L. Biermann [5]—it might well be that this sink of energy is of importance in the case of magnetic fields in interstellar H II-regions. It also seems that by this mechanism the tidal currents in the ionosphere are effectively limited to the lower layers (I. Lucas [8]). A further effect of this term is a modification of shock conditions compared to the hydromagnetic case, while the Hall term does not contribute.

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Discussion

Cowling: I have during the last year derived results essentially equivalent to those given by Schlüter. It appears that the question normally posed as to how the conductivity is affected by a magnetic field is too imprecise for the answer to have any value. The more important question is how collision processes affect the dissipation of a magnetic field; an answer can be given to this but only if the physical circumstances are clearly defined.

Buneman: Conservation of vortices (in the electrodynamic sense, i.e. vortices of momentum plus vector potential) is a very fundamental property. It applies to each species separately when there are no collisions, even under extreme relativistic conditions. When there are collisions the vortices of the total momentum are conserved—hence Dr Schlüter's result. Conservation of vortices is an extremely useful fact for resolution of problems and has been employed successfully by myself in calculations for conditions where no collisions take place, such as in interplanetary space.

Piddington: Dr Schlüter has considered each component of the gas as having a separate motion. This is undoubtedly necessary to obtain a complete solution but it may lead to great complexity in some astrophysical problems. These problems are usually so complicated in any case that some simplifying assumptions are necessary. One simplification which is often permissible is to consider two or perhaps more of the different gas components as a single gas with a single mass motion. An example is a hydromagnetic disturbance in a gas containing heavy ions and electrons and perhaps neutral particles. There is no doubt that, because of their greater mobility, the electrons move to some degree separate from the heavy ions and so cause space-charge electric fields within the ion plasma. This results in 'an electron plasma wave' or space-charge wave as an integral part of the whole hydromagnetic wave. However, the electric current which flows to cause this wave is small; in fact it is equal to the displacement current which, as Dr Schlüter has pointed out, is negligible, except when relativistic effects are significant.

Perhaps it is desirable to examine each particular astrophysical problem with a view to reducing as far as possible the total number of gas components considered. This may avoid the development of equations which cannot be solved.

von Engel: What is the relative importance of the production of charges (e.g. in the ionosphere) which has not been considered in the theory?

Schlüter: The equations which I have given describe only the balance of momentum, so they hold irrespective of the presence of ionization and recombination processes. These may, however, influence the coefficients of friction. In the case of three co-existing fluids one has to introduce a condition on the transmutations between the constituents. In the work on the tidal currents in the ionosphere the approximation was made that the degree of ionization is controlled by the instantaneous equilibrium between radiative ionization and recombination, independent of the state of motion.

Terletzsky: Do you agree with me that for extremely rarefied gases it is better to solve your first equations—the equations of mutually penetrating ideal gases?

Schlüter: As far as the assumptions (quasi-neutrality, creeping diffusion) hold, both approaches are mathematically equivalent. Otherwise, one has either to solve the original equations directly or to use transformations which do not imply the correctness of these approximations.