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THE CAMERON-STORVICK FUNCTION SPACE INTEGRAL: AN $L(L_p, L_{p'})$ THEORY

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0. Introduction.

In [3] Cameron and Storvick introduced a very general operator-valued function space "integral". In [3–5, 8, 9, 11, 13–20] the existence of this integral as an operator from L_2 to L_2 was established for certain functions. Recently the existence of the integral as an operator from L_1 to L_∞ has been studied [6, 7, 21]. In this paper we study the integral as an operator from L_p to $L_{p'}$ where $1 < p \le 2$. The resulting theorems extend the theory substantially and indicate relationships between the L_2-L_2 and L_1-L_∞ theories that were not apparent earlier. Even in the most studied case, $p = p' = 2$, the results below strengthen the theory.

Before giving the basic definitions, we fix some notation. $\boldsymbol{R}^\nu$ will denote ν-dimensional Euclidean space. $\boldsymbol{C}$ will denote the complex numbers and $\boldsymbol{C}^+$ the complex numbers with positive real part. If Y and Z are Banach spaces, $L(Y, Z)$ will denote the space of continuous linear operators from Y to Z. For ν a positive integer, let $C^\nu[a, b]$ denote the space of $\boldsymbol{R}^\nu$-valued continuous functions on $[a, b]$. $C_0^\nu[a, b]$ will denote those X in $C^\nu[a, b]$ such that $X(a) = 0$. $C_0^\nu[a, b]$ will be referred to as "Wiener space" and integration over $C_0^\nu[a, b]$ will always be with respect to Wiener measure.

Let $1 < p \le 2$ be given. Let F be a function from $C^\nu[a, b]$ to $\boldsymbol{C}$. Given $\lambda > 0$, ψ in $L_p(\boldsymbol{R}^\nu)$ and ξ in $\boldsymbol{R}^\nu$, let

$$(0.1)\qquad (I_\lambda(F)\psi)(\xi) \equiv \int_{C_0^\nu[a,b]} F(\lambda^{-1/2}X + \xi)\psi(\lambda^{-1/2}X(b) + \xi)dm(X) .$$

If $I_\lambda(F)\psi$ is in $L_{p'}(\boldsymbol{R}^\nu)$ as a function of ξ and if the correspondence $\psi \to I_\lambda(F)\psi$ gives an element of $L \equiv L(L_p(\boldsymbol{R}^\nu), L_{p'}(\boldsymbol{R}^\nu))$, we say that the operator-valued function space integral $I_\lambda(F)$ exists.

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Next suppose that there exists λ_0 $(0 < \lambda_0 \leq \infty)$ such that $I_\lambda(F)$ exists for all λ in $(0, \lambda_0)$ and further suppose that there exists an L-valued function which is analytic in $C^+ \cap \{\lambda : |\lambda| < \lambda_0\} \equiv C^+_{\lambda_0}$ and agrees with $I_\lambda(F)$ on $(0, \lambda_0)$; then this L-valued function is denoted $I^{\mathrm{an}}_\lambda(F)$ and is called the operator-valued function space integral of F associated with λ. We will most often be dealing with the case $\lambda_0 = \infty$.

Finally, let q be in R with $|q| < \lambda_0$. Suppose that there exists an operator $J^{\mathrm{an}}_q(F)$ in L such that for every ψ in $L_p(R^\nu)$ $\|J^{\mathrm{an}}_q(F)\psi - I^{\mathrm{an}}_\lambda(F)\psi\|_{p'} \to 0$ as $\lambda \to -iq$ through $C^+_{\lambda_0}$; then $J^{\mathrm{an}}_q(F)$ is called the operator-valued function space integral of F associated with $-iq$.

The above definitions actually vary with p and ν and so, for example, the left side of (0.1) could be written as $(I_{\lambda,p,\nu}(F)\psi)(\xi)$ to reflect this dependence. However, for a given F, it is natural to think of p and ν as fixed, and so, for this reason, and for notational convenience, we will surpress reference to p and ν.

The above definitions are very general but, from the point of view of quantum mechanics, which provided the initial motivation for the theory, it is $J^{\mathrm{an}}_{-1}(F)$ and the following special type of function which are of most interest:

$$F(X) = \exp\left[\int_a^b \theta(s, X(s))ds\right] \tag{0.2}$$

where θ is a C-valued function on $[a, b] \times R^\nu$ and X is in $C^\nu[a, b]$. As we will see the existence theorems below deal with functions related to but quite a bit more general than (0.2). Indeed the details of the theory and examples in [14 and 17] make it seem unlikely that the theory goes through for a large general class of functions F.

Next we describe briefly the results of this paper; in the process we introduce some necessary notation. Given a number d such that $1 \leq d \leq \infty$, d and d' will always be related by $1/d + 1/d' = 1$. If $1 \leq p \leq 2$ is given let γ in $[1, \infty]$ be such that

$$\gamma = \frac{1}{\frac{1}{p} - \frac{1}{p'}} = \frac{p'}{p' - 2} = \frac{p}{2 - p}\,. \tag{0.3}$$

Note that $\gamma = 1$ when $p = 1$ and $\gamma = \infty$ when $p = 2$. In our theorems ν will be a positive integer restricted so that

$$\nu < 2\gamma\,. \qquad \text{(Note that } 2\gamma > 2 \text{ for } p > 1.\text{)} \tag{0.4}$$

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For $1 \leq p < 2$ we will let r be a real number such that

$$\frac{2\gamma}{2\gamma - \nu} < r \leq \infty \,. \tag{0.5}$$

When $p = 2$ the left member of (0.5) is 1 and we can allow

$$1 \leq r \leq \infty \qquad (p = 2)\,. \tag{0.6}$$

The number $\nu/2\gamma$ will occur often and so it is worthwhile introducing a symbol for it:

$$\delta \equiv \frac{\nu}{2\gamma}\,. \qquad \text{(Note that } \delta = 0 \text{ when } p = 2.) \tag{0.7}$$

Let $L_{\gamma r} \equiv L_{\gamma r}([a, b] \times \boldsymbol{R}^\nu)$ be the space of all measurable $\boldsymbol{C}$-valued functions θ on $[a, b] \times \boldsymbol{R}^\nu$ such that $\theta(s, \cdot)$ is in $L_\gamma(\boldsymbol{R}^\nu)$ for almost all s in $[a, b]$ and $\|\theta(s, \cdot)\|_\gamma$ is in $L_r([a, b)]$.

The main existence theorem will appear in section 3 with applications in sections 4 and 5. However much of the work comes in sections 1 and 2 in dealing with functions of the form

$$F(X) = \prod_{j=1}^{m} \int_a^b \theta_j(s, X(s))ds \tag{0.8}$$

where each θ_j is in $L_{\gamma r}([a, b] \times \boldsymbol{R}^\nu)$. For such functions we will see that the operators $I_\lambda^{\text{an}}(F)$ and $J_q^{\text{an}}(F)$ exist for all appropriate values of the parameters λ and q and we will obtain explicit formulas for them.

Let $A \equiv A(p, \nu, r)$ be the class of functions of the form (0.8) as well as the function identically 1. The main existence theorem for $I_\lambda^{\text{an}}(F)$ and $J_q^{\text{an}}(F)$ deals with functions of the form

$$F(X) = \sum_{k=0}^{\infty} a_k F_k(X) \tag{0.9}$$

where each F_k is in A and where certain restrictions to guarantee summability are placed on the a_k's.

In section 4 we show that the appropriate summability conditions are satisfied for functions of the form

$$F(X) = f\left[\int_a^b \theta_1(s, X(s))ds, \cdots, \int_a^b \theta_k(s, X(s))ds\right] \tag{0.10}$$

where f is an entire function of k complex variables of order not exceeding $1/\delta$. The results of section 4 hold in particular for functions of the form (0.2) since $f(z) = \exp(z)$ has order 1 which is always less than $1/\delta$.

In section 5 we apply our results to such functions and obtain a solution to an integral equation which, in the case $\lambda = -i$, is formally equivalent to Schroedinger's equation. We also obtain an approximation formula in terms of sums of integrals over finite-dimensional spaces.

Section 6 contains some counterexamples related to the dimension restriction (0.4).

We finish this introduction by indicating how our results relate to previous theorems.

$p = 2$: The case $p = 2$ has been the most studied; in this case the most relevant earlier papers are [3, 4, 11, 15, 16]. A substantial part of [11] is not readily comparable to the present paper; when we make comparisons to [11] we have in mind section 7 and some of its preliminaries. [15] refines some of the results of [3], and [11] extends to higher dimensions the results of [3 and 4], and so we will make our comparisons with [11 and 16]. When $p = 2$, the present paper combines the advantages of [11 and 16] and goes somewhat further. As in [11] we allow an arbitrary number of dimensions ν rather than require $\nu = 1$; also as in [11], θ is required to be measurable rather than continuous almost everywhere. We treat functions of the form (0.9) as in [16] rather than strictly functions of the form (0.2). The present paper requires, as in both [11 and 16], that for almost every s in $[a, b], \theta(s, \cdot)$ is in $L_\infty(\boldsymbol{R}^\nu)$. In [11 and 16] it is required that $\|\theta(s, \cdot)\|_\infty$ also be in $L_\infty([a, b])$; here we make the weaker requirement that $\|\theta(s, \cdot)\|_\infty$ be in $L_1([a, b])$. Perhaps the major advantage this paper has over both [11 and 16] is that our treatment of the integral equation (section 5 below) is simpler both conceptually and in technical details. The treatment in [11] follows the earlier clever but very involved treatment in [3 and 4]. In [16] we simplified a major part of [4], and here we carry this simplification further while allowing more general θ's as discussed above.

$1 < p < 2$, $p = 1$: For $1 < p < 2$ the results of the present paper are new. They allow us to treat various unbounded θ's that could not be treated before, and, in particular, to solve the integral equation formally equivalent to Schroedinger's equation for such θ's.

In the $p = 1$ case the relevant earlier papers are [6, 7, and 21]. In case $p = 1$ the definitions of $I_\lambda^{\text{an}}(F)$ and $J_\lambda^{\text{an}}(F)$ are actually somewhat different than in the present paper. Let ψ be in L_1. Rather than taking the analyticity and limits connected with the definition of $I_\lambda^{\text{an}}(F)\psi$ and $J_q^{\text{an}}(F)\psi$ respectively in L_∞-norm, one uses the weak* topology on L_∞ induced by its pre-dual L_1. (Actually in [21], we have $I_\lambda^{\text{an}}(F)\psi$ and $J_q^{\text{an}}(F)\psi$ in the space $C_0(R)$ of continuous functions that vanish at ∞ and the analyticity and limits are with respect to the weak topology on $C_0(\boldsymbol{R})$ induced by its dual, the space of regular Borel measure on $\boldsymbol{R}$.) Because of this difference of definition, and because the present techniques do not yield any more in the $p=1$ case than is already contained in [21], and, finally, because the $p=1$ case is done in detail in [21], we do not formally include the $p=1$ case in the present paper. We could alternately have made the definitions of $I_\lambda^{\text{an}}(F)\psi$ and $J_q^{\text{an}}(F)\psi$ in terms of the weak* (=weak for $1<p\le 2$) topology on $L_{p'}$ induced by L_p and included all the cases $1\le p\le 2$ in this paper.

The $p=1$ and $p=2$ cases, although formally related, have appeared very different in many respects. In light of the present paper, one recognizes these cases as the extremes in a continuum of possibilities $1\le p\le 2$ and sees that they are much more intimately related than was previously apparent. For example, Haugsby [11] showed that for $p=2$ any dimension ν could be handled. In contrast for $p=1$ attempts to extend the theory beyond $\nu=1$ have failed. The restriction (0.4) along with later details of proofs give more insight into the dimension restrictions. When $p=1$, (0.4) is a severe restriction; ν must equal 1. As p varies from 1 to 2, the restriction becomes less and less severe and is no restriction at all when $p=2$. An example given in section 6 shows that the severe dimension restriction found here and in earlier papers when $p=1$ is really needed.

The norm estimate $\|C_\lambda\|\le\left(\frac{|\lambda|}{2\pi s}\right)^{\nu/2\gamma}$ given in Lemma 1.1 below for the operator C_λ is the ultimate source for the difference in techniques of proof as p varies from 1 to 2. Note that when $p=1$, $\|C_\lambda\|$ involves $\frac{|\lambda|}{2\pi s}$ to the $\nu/2$ power. For $1<p<2$, any dimension is allowed which keeps the power on $\frac{|\lambda|}{2\pi s}$ less than 1. For $p=2$, $\gamma=\infty$ and so $\frac{|\lambda|}{2\pi s}$ does not appear at all in the estimate for $\|C_\lambda\|$ no matter what ν may be. The fact that formulas throughout the paper are much simpler when $p = 2$ traces back to the simple norm estimate on C_λ in this case.

The hypothesis put on $\theta(s, \cdot)$ (for almost every s) is such that $\theta(s, \cdot)f(\cdot)$ is in L_p if f is in $L_{p'}$ (see Lemma 1.3). For $p = 1$, this says $\theta(s, \cdot)$ should be in L_1 whereas for $p = 2$, this puts $\theta(s, \cdot)$ in L_∞; this of course corresponds to the hypotheses of the previous papers.

As one proceeds through the paper one can see in more detail how the techniques vary as p goes from 1 to 2 but the above should suffice for now.

When $p = 2$ the class $\hat{A}$ of functions F which we are able to deal with form a Banach algebra as was the case in [16]. Since the proof of this fact follows quite closely the development in [16], we will simply outline it here at the end of section 4. Also at the end of section 4 we outline some facts about the "sequential" operator valued function space integral. The sequential integral has played a major role in some earlier papers [3, 4, 5, 11, 15]. As in [16], the sequential integral exists and equals the analytic operator valued function space integral for functions in the Banach algebra $\hat{A}$. In contrast, for $p = 1$, Cameron and Storvick [6] have shown that the sequential integral may fail to exist in cases where the analytic integral does exist. We will see that for all other p $(1 < p < 2)$ the situation is as in the $p = 1$ case. Further rather extreme pathologies of the sequential integral have been exhibited in [9 and 20].

In sections 2–4 below the techniques of proof largely involve combining and extending ideas from [16 and 21]; many of these ideas were in turn based in part on earlier work of Cameron and Storvick. On the other hand several of the techniques in sections 1, 5 and 6 are quite different from techniques in earlier papers.

1. Some preliminary lemmas.

In this section we shall develop four preliminary lemmas that play a critical role in this paper.

LEMMA 1.1. *Let s be a positive number. Let $1 \le p \le 2$, let γ be given by* (0.3) *and let ν satisfy* (0.4). *Given a nonzero complex number λ with nonnegative real part, ψ in $L_p(\boldsymbol{R}^\nu)$ and ξ in $\boldsymbol{R}^\nu$, let*

$$(1.1) \qquad (C_\lambda\psi)(\xi) \equiv \left(\frac{\lambda}{2\pi s}\right)^{\nu/2} \int_{\boldsymbol{R}^\nu} \psi(V) \exp \frac{-\lambda \| V - \xi \|^2}{2s} dV .$$

Then C_λ is in $L(L_p(\boldsymbol{R}^\nu), L_{p'}(\boldsymbol{R}^\nu))$ and

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$$\|C_\lambda\| \leq \left(\frac{|\lambda|}{2\pi s}\right)^{\nu(1/2-1/p')} = \left(\frac{|\lambda|}{2\pi s}\right)^{\nu/2r} . \tag{1.2}$$

Remark. (i) When ν is odd we always choose $\lambda^{1/2}$ with nonnegative real part. (ii) When $1 < p \leq 2$ and $\operatorname{Re}\lambda = 0$ the integral in (1.1) should be interpreted in the mean just as in the theory of the L_p Fourier transform [2; Definition 5.2.8, p. 211].

Proof. We first treat the extreme cases $p = 1$ and $p = 2$. The intermediate cases $(1 < p < 2)$ will be handled by interpolation via the M. Riesz convexity theorem [26; Theorem 1.3, p. 179].

$p = 1$: Here the result is clear since for all ξ

$$|(C_\lambda\psi)(\xi)| \leq \left(\frac{|\lambda|}{2\pi s}\right)^{\nu/2} \|\psi\|_1 .$$

$p = 2$, $\lambda = -iq$ $(q \neq 0)$: In this case we may write

$$(C_{-iq}\psi)(\xi) = \exp \frac{iq\|\xi\|^2}{2s} \left\{\left(\frac{-iq}{2\pi s}\right)^{\nu/2} \int_{\boldsymbol{R}^\nu} \exp\left[\frac{iq\|V\|^2}{2s}\right]\psi(V) \times \exp\left[\frac{-iqV\cdot\xi}{s}\right]dV\right\} . \tag{1.3}$$

Hence in this case, C_{-iq} is the composition of 3 unitary operators on $L_2(\boldsymbol{R}^\nu)$: multiplication by $\exp\left[\frac{iq\|V\|^2}{2s}\right]$ followed by a scaled version of the Fourier-Plancherel transform followed by multiplication by $\exp\left[\frac{iq\|\xi\|^2}{2s}\right]$. In particular $\|C_{-iq}\| \leq 1$ as desired.

$p = 2$, $\operatorname{Re}\lambda > 0$: In this case C_λ is the operator of convolution by the L_1 function

$$e_\lambda(U) \equiv \left(\frac{\lambda}{2\pi s}\right)^{\nu/2} \exp\left[\frac{-\lambda\|U\|^2}{2s}\right] \tag{1.4}$$

and it is known [26; Theorem 3.18, p. 28] that $\|C_\lambda\| = \|\mathscr{F}(e_\lambda)\|_\infty$ where $\mathscr{F}$ denotes the Fourier transform. By a fairly routine calculation one sees that $\mathscr{F}(e_\lambda)(V) = \exp\left[\frac{-2\pi^2 s}{\lambda}\|V\|^2\right]$ and so $\|C_\lambda\| = \|\mathscr{F}(e_\lambda)\|_\infty = 1$.

$1 < p < 2$: Fix λ such that $\operatorname{Re}\lambda \geq 0$ $(\lambda \neq 0)$. In the terminology of the M. Riesz convexity theorem as given in [26; Theorem 1.3, p. 179] we have shown above that C_λ is of type $(1, \infty)$ with $(1, \infty)$ norm dominated by $\left(\frac{|\lambda|}{2\pi s}\right)^{\nu/2}$ and of type (2,2) with (2,2) norm dominated by 1. Applying the convexity theorem we have that C_λ is in $L(L_p, L_{p'})$ with

$$\|C_\lambda\| \leq \left[\left(\frac{|\lambda|}{2\pi s}\right)^{\nu/2}\right]^{1-2/p'} [1]^{2/p'} = \left(\frac{|\lambda|}{2\pi s}\right)^{\nu(1/2-1/p')} .$$

This finishes the proof of Lemma 1.1.

Remark. In the case $p = 2$ $(\nu > 1)$ Haugsby [11; pp. 8–16] gives a longer but more elementary proof; in particular his proof does not depend on a use of [26; Theorem 3.18, p. 28].

LEMMA 1.2. *Let $1 < p \leq 2$, let ψ be in $L_p(\boldsymbol{R}^\nu)$ where ν satisfies* (0.4), *and let q be a nonzero real number. Then*

$$\|C_\lambda\psi - C_{-iq}\psi\|_{p'} \to 0 \qquad \text{as } \lambda \to -iq \text{ through } \boldsymbol{C}^+ . \tag{1.5}$$

Proof. Fix p such that $1 < p \leq 2$. Since $\{\|C_\lambda\|\}$ is uniformly bounded for λ in a neighborhood of $-iq$, it suffices to establish (1.5) for ψ in a fundamental subset of L_p; i.e. a subset of L_p whose span is dense. It will be convenient to take the ν-dimensional Hermite functions as the fundamental subset since $C_\lambda\psi$ can be specifically calculated for such functions. The n^{th} (1-dimensional) Hermite polynomial is defined by

$$h_n(u) \equiv (-1)^n e^{u^2/2} \frac{d^n}{du^n} e^{-u^2/2} . \tag{1.6}$$

Then n^{th} (1-dimensional) Hermite function is defined by

$$H_n(u) = h_n(u) e^{-u^2/2} . \tag{1.7}$$

The fact that the Hermite functions are a fundamental subset of $L_p(\boldsymbol{R})$ is given in [2; Corollary 3.1.9, p. 122]. The ν-dimensional Hermite functions have the form

$$H_{n_1,\cdots,n_\nu}(u_1, \cdots, u_\nu) \equiv \prod_{i=1}^{\nu} H_{n_i}(u_i) \tag{1.8}$$

where $n_1, \cdots, n_\nu$ are nonnegative integers. The fact that the ν-dimensional Hermite functions are fundamental in $L_p(\boldsymbol{R}^\nu)$ follows from the 1-dimensional case and the fact that the following collection of functions is fundamental in $L_p(\boldsymbol{R}^\nu)$;

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$$\left\{\prod_{i=1}^{\nu} \phi_i(u_i) : \phi_i \text{ is in } L_p(\boldsymbol{R}) \text{ for } i = 1, \cdots, n\right\}.$$

We first need to calculate $C_\lambda H_{n_1,\cdots,n_\nu}$. A calculation by Cameron and Storvick [5; pp. 361–362] shows that for $\operatorname{Re}\lambda > 0$,

$$(1.9)\quad \begin{aligned}(C_\lambda H_n)(u) &\equiv \left(\frac{\lambda}{2\pi s}\right)^{1/2} \int_{\boldsymbol{R}} H_n(w) \exp\left[\frac{-\lambda(u-w)^2}{2s}\right] dw \\ &= \left(\frac{\lambda}{\lambda+s}\right)^{(n+1)/2} h_n\left[\left(\frac{\lambda}{\lambda+s}\right)^{1/2} u\right] \exp\left[-\left(\frac{\lambda}{\lambda+s}\right)\frac{u^2}{2}\right].\end{aligned}$$

Arguing by analytic continuation, we see that (1.9) is also true for $\operatorname{Re}\lambda = 0$ $(\lambda \neq 0)$. Hence we have

$$(1.10)\quad \begin{aligned}(C_\lambda H_{n_1,\cdots,n_\nu})(\xi) &\equiv \left(\frac{\lambda}{2\pi s}\right)^{\nu/2} \int_{\boldsymbol{R}^\nu} H_{n_1,\cdots,n_\nu}(W) \exp\left[\frac{-\lambda\|W-\xi\|^2}{2s}\right] dW \\ &= \left(\frac{\lambda}{2\pi s}\right)^{\nu/2} \int_{\boldsymbol{R}^\nu} \prod_1^\nu \left[H_{n_i}(w_i) \exp\left(\frac{-\lambda(w_i-\xi_i)^2}{2s}\right)\right] dw_1 \cdots dw_n \\ &= \prod_1^\nu \left(\frac{\lambda}{2\pi s}\right)^{1/2} \int_{\boldsymbol{R}} H_{n_i}(w_i) \exp\left(\frac{-\lambda(w_i-\xi_i)^2}{2s}\right) dw_i \\ &= \prod_1^\nu \left(\frac{\lambda}{\lambda+s}\right)^{(n_i+1)/2} h_{n_i}\left[\left(\frac{\lambda}{\lambda+s}\right)^{1/2} \xi_i\right] \exp\left[-\left(\frac{\lambda}{\lambda+s}\right)\frac{\xi_i^2}{2}\right].\end{aligned}$$

To show that $\|C_\lambda H_{n_1,\cdots,n_\nu} - C_{-iq} H_{n_1,\cdots n_\nu}\|_{p'} \to 0$ as $\lambda \to -iq$ through $\boldsymbol{C}^+$ it suffices to consider a sequence $\{\lambda_j\}$ from $\boldsymbol{C}^+$ such that $\lambda_j \to -iq$ and show that $\|C_{\lambda_j} H_{n_1,\cdots,n_\nu} - C_{-iq} H_{n_1,\cdots,n_\nu}\|_{p'} \to 0$ as $j \to \infty$. But to show that a sequence of functions in $L_{p'}(\boldsymbol{R}^\nu)$ converges in the p'-norm to another function, it suffices, by the Dominated Convergence Theorem, to show that pointwise convergence holds and that there is a dominating $L_{p'}(\boldsymbol{R}^\nu)$ function for the sequence. In our setting, the pointwise convergence is clear and we need only find the dominating $L_{p'}(\boldsymbol{R}^\nu)$ function. Note that $\operatorname{Re}\left(\frac{-iq}{-iq+s}\right) = \frac{q^2}{s^2+q^2} \equiv D > 0$. Now $\operatorname{Re}\left(\frac{\lambda_j}{\lambda_j+s}\right) \to D$ as $j \to \infty$ and so, for sufficiently large j, $\operatorname{Re}\left(\frac{\lambda_j}{\lambda_j+s}\right) \geq \frac{D}{2}$. Next observe that since h_{n_i} is a polynomial there exists constants A_i and B_i such that $|h_{n_i}(u_i)| \leq A_i \exp[B_i |u_i|]$ for all u_i in C. Also there exists E_0 such that $\left|\frac{\lambda_j}{\lambda_j+s}\right|^{1/2} \leq E_0$ for all j. Thus

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$$\left|h_{n_i}\left[\left(\frac{\lambda_j}{\lambda_j+s}\right)^{1/2}\xi_i\right]\right| \leq A_i \exp\left[B_i\left|\left(\frac{\lambda_j}{\lambda_j+s}\right)^{1/2}\xi_i\right|\right]$$
$$= A_i \exp\left[B_i\left|\frac{\lambda_j}{\lambda_j+s}\right|^{1/2}|\xi_i|\right] \leq A_i \exp\left[E_0 B_i\,|\xi_i|\right]$$

for sufficiently large j. Clearly there also exists a constant E_i such that $\left|\frac{\lambda_j}{\lambda_j+s}\right|^{(n_i+1)/2} \leq E_i$ for all j. We can now see that for sufficiently large j, $|(C_\lambda H_{n_1,\cdots,n_\nu})(\xi)|$ is dominated by the $L_{p'}(\boldsymbol{R}^\nu)$ function

$$\prod_{i=1}^{\nu} E_i[A_i e^{E_0 B_i|\xi_i|}]e^{-D\xi_i^2/4}\,.$$

The proof of Lemma 1.2 is now complete.

The following lemma follows quite easily from Hölder's inequality and is well known [22; p. 129]. We state it formally for convenience.

LEMMA 1.3. *Let* $1 \leq p \leq 2$, *let* γ *be given by* (0.3) *and let* ν *satisfy* (0.4). *If* f *is in* $L_{p'}(\boldsymbol{R}^\nu)$ *and* θ *is in* $L_\gamma(\boldsymbol{R}^\nu)$, *then* $f\theta$ *is in* $L_p(\boldsymbol{R}^\nu)$ *and*

$$\|f\theta\|_p \leq \|f\|_{p'}\,\|\theta\|_\gamma\,. \tag{1.11}$$

Hence the operator M_θ *of multiplication by* θ *is in* $L(L_{p'}, L_p)$ *and* $\|M_\theta\| \leq \|\theta\|_\gamma$.

LEMMA 1.4. *Let* $1 \leq p \leq 2$, *let* γ *be given by* (0.3) *and let* ν *satisfy* (0.4). *Suppose that* $\theta_1, \cdots, \theta_m$ *are functions in* $L_\gamma(\boldsymbol{R}^\nu)$. *Let* $a = s_0 < s_1 < \cdots < s_m < s_{m+1} = b$. *Let* λ *be a nonzero complex number with* $\operatorname{Re}\lambda \geq 0$. *Given* ψ *in* $L_p(\boldsymbol{R}^\nu)$ *and* ξ *in* $\boldsymbol{R}^\nu$, *let* $(G_\lambda(s_1, \cdots, s_m)\psi)(\xi)$ *be defined by*

$$\begin{aligned}
&(G_\lambda(s_1,\cdots,s_m)\psi)(\xi)\\
&= \lambda^{\nu/2}[2\pi(s_1-a)]^{-\nu/2}\int_{\boldsymbol{R}^\nu}\theta_1(V_1)\exp\frac{-\lambda\,\|V_1-\xi\|^2}{2(s_1-a)}\\
&\quad\times \lambda^{\nu/2}[2\pi(s_2-s_1)]^{-\nu/2}\int_{\boldsymbol{R}^\nu}\theta_2(V_2)\exp\frac{-\lambda\,\|V_2-V_1\|^2}{2(s_2-s_1)}\\
&\quad\cdots\\
&\quad\times \lambda^{\nu/2}[2\pi(s_m-s_{m-1})]^{-\nu/2}\int_{\boldsymbol{R}^\nu}\theta_m(V_m)\exp\frac{-\lambda\,\|V_m-V_{m-1}\|^2}{2(s_m-s_{m-1})}\\
&\quad\times \lambda^{\nu/2}[2\pi(b-s_m)]^{-\nu/2}\int_{\boldsymbol{R}^\nu}\psi(V_{m+1})\exp\frac{-\lambda\,\|V_{m+1}-V_m\|^2}{2(b-s_m)}\\
&\quad\times dV_{m+1}\cdots dV_1\,.
\end{aligned} \tag{1.12}$$

Then $G_\lambda(s_1, \cdots, s_m)$ *is in* $L(L_p(\boldsymbol{R}^\nu), L_{p'}(\boldsymbol{R}^\nu))$ *and*

$$\|G_\lambda(s_1, \cdots, s_m)\| \leq \left[\prod_{j=1}^{m} \|\theta_j\|_r\right]\left[\prod_{j=1}^{m+1}\left(\frac{|\lambda|}{2\pi(s_j - s_{j-1})}\right)^{\nu/2\gamma}\right]. \tag{1.13}$$

Further for $1 < p \leq 2$, nonzero real q, and ψ in $L_p(\mathbf{R}^\nu)$ we have

$$\|G_\lambda(s_1, \cdots, s_m)\psi - G_{-iq}(s_1, \cdots, s_m)\psi\|_{p'} \to 0 \tag{1.14}$$

as $\lambda \to -iq$ through $\mathbf{C}^+$.

Proof. The fact that $G_\lambda(s_1, \cdots, s_m)$ is in $L(L_p(\mathbf{R}^\nu), L_{p'}(\mathbf{R}^\nu))$ and satisfies (1.13) follows immediately from the fact that $G_\lambda(s_1, \cdots, s_m)$ is the composition of convolution operators as dealt with in Lemma 1.1 and multiplication operators as dealt with in Lemma 1.3.

Now suppose $1 < p \leq 2$. The individual convolution operators involved in $G_\lambda(s_1, \cdots, s_m)$ converge (by Lemma 1.2) in the strong operator topology to the convolution operators involved in $G_{-iq}(s_1, \cdots, s_m)$ as $\lambda \to -iq$. The multiplication operators involved in $G_\lambda(s_1, \cdots, s_m)$ are independent of λ and so certainly converge in the strong operator topology as $\lambda \to -iq$. Now in general the composition of operators which converge in the strong operator topology does not necessarily converge in the strong operator topology. However, continuity of operator composition in the strong operator topology does hold if the operators lie in a norm bounded set; that is the case in our situation because of inequalities (1.2) and (1.11) and the fact that we only need consider λ's in a bounded neighborhood of $-iq$.

2. The existence of $J_q^{\mathrm{an}}(F)$; $F \in A$.

In this section, for F in A we shall establish the existence of $I_\lambda^{\mathrm{an}}(F)$ and $J_q^{\mathrm{an}}(F)$ for all λ in $\mathbf{C}^+$ and all real $q \neq 0$ respectively.

Now that we have Lemmas 1.1–1.4, the proof of the following theorem depends on combining and modifying the techniques from [16; Theorem 1.1, p. 133] and [21; Theorem 1.1].

THEOREM 2.1. *Let $1 < p < 2$, let γ be given by* (0.3) *and let ν and r satisfy* (0.4) *and* (0.5) *respectively. Let F be given by* (0.8) *with each θ_j in $L_{\gamma r}$. Then $I_\lambda^{\mathrm{an}}(F)$ and $J_q^{\mathrm{an}}(F)$ exist for all λ in $\mathbf{C}^+$ and all real $q \neq 0$ respectively. Further for λ in $\mathbf{C}^+$, ψ in $L_p(\mathbf{R}^\nu)$ and ξ in $\mathbf{R}^\nu$*

$$(I_\lambda^{\mathrm{an}}(F)\psi)(\xi) = \left(\frac{\lambda}{2\pi}\right)^{(m+1)\nu/2} \sum_\tau$$

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$$\times (B)\int_{S(\tau)} [(s_{\tau(1)} - a)(s_{\tau(2)} - s_{\tau(1)})\cdots(b - s_{\tau(m)})]^{-\nu/2}$$

$$\times \int_{R^\nu} (m+1)\int_{R_\nu}\left[\prod_{j=1}^{m}\theta_{\tau(j)}(s_{\tau(j)}, V_j)\right]\psi(V_{m+1})$$

$$\times \exp\left[-\frac{\lambda}{2}\sum_{1}^{m+1}\frac{\|V_j - V_{j-1}\|^2}{(s_{\tau(j)} - s_{\tau(j-1)})}\right]dV_{m+1}\cdots dV_1 dS$$

$$(2.1)\quad \begin{aligned} &= \sum_{\tau}(B)\int_{S(\tau)}\lambda^{\nu/2}[2\pi(s_{\tau(1)} - a)]^{-\nu/2}\int_{R^\nu}\theta_{\tau(1)}(s_{\tau(1)}, V_1) \\ &\times \exp\left[\frac{-\lambda\|V_1 - \xi\|^2}{2(s_{\tau(1)} - a)}\right]\lambda^{\nu/2}[2\pi(s_{\tau(2)} - s_{\tau(1)})]^{-\nu/2} \\ &\times \int_{R^\nu}\theta_{\tau(2)}(s_{\tau(2)}, V_2)\exp\left[\frac{-\lambda\|V_2 - V_1\|^2}{2(s_{\tau(2)} - s_{\tau(1)})}\right] \\ &\cdots \\ &\times \lambda^{\nu/2}[2\pi(s_{\tau(m)} - s_{\tau(m-1)})]^{-\nu/2}\int_{R^\nu}\theta_{\tau(m)}(s_{\tau(m)}, V_m) \\ &\times \exp\left[\frac{-\lambda\|V_m - V_{m-1}\|^2}{2(s_{\tau(m)} - s_{\tau(m-1)})}\right]\lambda^{\nu/2}[2\pi(b - s_{\tau(m)})]^{-\nu/2} \\ &\times \int_{R^\nu}\psi(V_{m+1})\exp\left[\frac{-\lambda\|V_{m+1} - V_m\|^2}{2(b - s_{\tau(m)})}\right]dV_{m+1}\cdots dV_1 dS \end{aligned}$$

where $V_0 \equiv \xi, s_{\tau(0)} \equiv a, s_{\tau(m+1)} \equiv b$, *the sum is taken over all* $m!$ *permutations* τ *of* $\{1, \cdots, m\}$,

$$S(\tau) \equiv \{(s_1, \cdots, s_m) \in (a, b)^m : a < s_{\tau(1)} < \cdots < s_{\tau(m)} < b\}$$

and where $(B)\int_{S(\tau)} f(s_1, \cdots, s_m)dS$ *refers to the Bochner integral* [12, pp. 78–89] *with respect to Lebesgue measure on* $S(\tau)$. (*In Corollary* 2.2 *we will see that this integral can also be interpreted as an ordinary Lebesgue integral of a scalar-valued function over* $S(\tau)$.)

For real $q \neq 0$, $J_q^{an}(F)$ *is given by the third expression in* (2.1) *with* $\lambda = -iq$ *and where the integrals with respect to* $V_1, \cdots, V_{m+1}$ *are interpreted in the mean.*

Finally we have

$$(2.2)\qquad \|I_\lambda^{an}(F)\| \leq \frac{\left(\frac{|\lambda|}{2\pi}\right)^{(m+1)\delta}\|g\|_r^m (m!)^{1/r'}(b-a)^{m(1-r'\delta)/r'}\{\Gamma(1 - r'\delta)\}^{(m+1)/r'}}{(b-a)^\delta\{\Gamma[(m+1)(1 - r'\delta)]\}^{1/r'}}$$

where Γ *denotes the Gamma function and where* $g : [a, b] \to \boldsymbol{R}$ *is given by*

$$(2.3) \qquad g(s) \equiv \max\{\|\theta_1(s,\cdot)\|_r, \cdots, \|\theta_m(s,\cdot)\|_r\} .$$

The bound (2.3) *also holds for* $\|J_q^{\text{an}}(F)\|$ *with* $|\lambda|$ *replaced by* $|q|$.

Proof. Fix p such that $1 < p < 2$. For real $\lambda > 0$ let $(K_\lambda(F)\psi)(\xi)$ denote the third expression in (2.1). We begin by showing that $K_\lambda(F)$ is in $L(L_p, L_{p'})$ with $\|K_\lambda(F)\|$ bounded by the right side of (2.2). Because there are $m!$ terms in the sum (2.1) and because $1/r' = 1 - 1/r$, it suffices to examine one term, say $(K_{\lambda,\tau}(F)\psi)(\xi)$, in the sum and show that $K_{\lambda,\tau}(F)\psi$ is in $L_{p'}$ with

$$(2.4) \qquad \|K_{\lambda,\tau}(F)\psi\|_{p'} \le \frac{\|\psi\|_p\left(\dfrac{|\lambda|}{2\pi}\right)^{(m+1)\delta}\|g\|_r^m(m!)^{-1/r}(b-a)^{m(1-r'\delta)/r'}\{\Gamma(1-r'\delta)\}^{(m+1)/r'}}{(b-a)^\delta\{\Gamma[(m+1)(1-r'\delta)]\}^{1/r'}} .$$

We will go through the estimates needed to establish (2.4) in detail. Closely related arguments will be needed several times further on but will not be carried out in detail.

Let $G_{\lambda,\tau}: S(\tau) \to L_{p'}$ be defined by

$$(2.5) \qquad \begin{aligned} &G_{\lambda,\tau}(s_1, \cdots, s_m)(\xi) \\ &= \lambda^{\nu/2}[2\pi(s_{\tau(1)} - a)]^{-\nu/2}\int_{R^\nu} \theta_{\tau(1)}(s_{\tau(1)}, V_1)\exp\left[\frac{-\lambda\|V_1 - \xi\|^2}{2(s_{\tau(1)} - a)}\right] \\ &\quad \cdots \\ &\quad \times \lambda^{\nu/2}[2\pi(s_{\tau(m)} - s_{\tau(m-1)})]^{-\nu/2}\int_{R^\nu} \theta_{\tau(m)}(s_{\tau(m)}, V_m) \\ &\quad \times \exp\left[\frac{-\lambda\|V_m - V_{m-1}\|^2}{2(s_{\tau(m)} - s_{\tau(m-1)})}\right]\lambda^{\nu/2}[2\pi(b - s_{\tau(m)})]^{-\nu/2} \\ &\quad \times \int_{R^\nu} \psi(V_{m+1})\exp\left[\frac{-\lambda\|V_{m+1} - V_m\|^2}{2(b - s_{\tau(m)})}\right]dV_{m+1}\cdots dV_1 . \end{aligned}$$

By Lemma 1.4 for almost all $(s_1, \cdots, s_m)$ in $S(\tau)$, $G_{\lambda,\tau}(s_1, \cdots, s_m)(\xi)$ is an $L_{p'}$ function of ξ with

$$(2.6) \qquad \begin{aligned} &\|G_{\lambda,\tau}(s_1, \cdots, s_m)(\cdot)\|_{p'} \\ &\le \|\psi\|_p\left[\prod_1^m \|\theta_{\tau(j)}(s_{\tau(j)}, \cdot)\|_r\right]\left(\frac{|\lambda|}{2\pi}\right)^{(m+1)\delta} \\ &\quad \times[(s_{\tau(1)} - a)(s_{\tau(2)} - s_{\tau(1)})\cdots(b - s_{\tau(m)})]^{-\delta} . \end{aligned}$$

We will show further on that $G_{\lambda,\tau}$ is strongly measurable [12; Definition

3.5.4]. Once this is done the following argument will show that $G_{\lambda,\tau}$ is Bochner integrable over $S(\tau)$ and that (2.4) holds.

$$\begin{aligned}\|K_{\lambda,\tau}(F)\psi\|_{p'} &\leq \left\|(B)\int_{S(\tau)} G_{\lambda,\tau}(s_1, \cdots, s_m)(\,\cdot\,)dS\right\|_{p'} \\ &\leq \int_{S(\tau)} \left\|G_{\lambda,\tau}(s_1, \cdots, s_m)(\,\cdot\,)\right\|_{p'} dS \\ &\leq \|\psi\|_p \left(\frac{|\lambda|}{2\pi}\right)^{(m+1)\delta} .\end{aligned}$$

$$\begin{aligned}&\int_{S(\tau)} \left[\prod_1^m \|\theta_{\tau(j)}(s_{\tau(j)}, \cdot)\|_r\right][(s_{\tau(1)} - a)\cdots(b - s_{\tau(m)})]^{-\delta} dS \\ &\leq \|\psi\|_p\left(\frac{|\lambda|}{2\pi}\right)^{(m+1)\delta} \int_{S(\tau)} \left[\prod_1^m g(s_j)\right][(s_{\tau(1)} - a)\cdots(b - s_{\tau(m)})]^{-\delta} dS \\ &\leq \|\psi\|_p\left(\frac{|\lambda|}{2\pi}\right)^{(m+1)\delta} \left\{\int_{S(\tau)} \left(\prod_1^m g^r(s_j)\right)ds\right\}^{1/r} \\ &\quad\times \left\{\int_{S(\tau)} [(s_{\tau(1)} - a)\cdots(b - s_{\tau(m)})]^{-r'\delta} dS\right\}^{1/r'} .\end{aligned} \tag{2.7}$$

But

$$\begin{aligned}\left\{\int_{S(\tau)} \left[\prod_1^m g^r(s_j)\right]dS\right\}^{1/r} &= \left\{\frac{1}{m!}\int_{[a,b]^m} \left[\prod_1^m g^r(s_j)\right]ds_1\cdots ds_m\right\}^{1/r} \\ &= (m!)^{-1/r}\|g\|_r^m .\end{aligned} \tag{2.8}$$

Next we note that the restrictions on ν and r were chosen so that $r'\delta < 1$. Thus the quantity $[(s_{\tau(1)} - a)\cdots(b - s_{\tau(m)})]^{-r'\delta}$ is integrable over $S(\tau)$. We now proceed to compute $\int_{S(\tau)} [(s_{\tau(1)} - a)\cdots(b - s_{\tau(m)})]^{-r'\delta} dS$ or, equivalently,

$$\int_a^b \int_a^{s_m} \cdots \int_a^{s_2} [(s_1 - a)(s_2 - s_1)\cdots(b - s_m)]^{-r'\delta} ds_1\cdots ds_m . \tag{2.9}$$

Making the substitution $x = \dfrac{s_1 - a}{s_2 - a}$, we see that

$$\begin{aligned}&\int_a^{s_2} [(s_1 - a)(s_2 - s_1)]^{-r'\delta} ds_1 \\ &= (s_2 - a)^{1-2r'\delta}\int_0^1 x^{(1-r'\delta)-1}(1 - x)^{(1-r'\delta)-1}dx \\ &= (s_2 - a)^{1-2r'\delta}\beta(1 - r'\delta, 1 - r'\delta)\end{aligned} \tag{2.10}$$

where β denotes the Beta function. At the next stage we make the substitution $x = \dfrac{s_2 - a}{s_3 - a}$ and obtain

$$\int_a^{s_3} (s_2 - a)^{1-2r'\delta}(s_3 - s_2)^{-r'\delta} ds_2 = (s_3 - a)^{2-3r'\delta}\beta(2 - 2r'\delta, 1 - r'\delta) . \tag{2.11}$$

We continue in this fashion until at the mth and last stage we make the substitution $x = \dfrac{s_m - a}{b - a}$ and obtain

$$\begin{aligned} &\int_a^{b} (s_m - a)^{(m-1)-mr'\delta}(b - s_m)^{-r'\delta} ds_m \\ &\quad = (b - a)^{m-(m+1)r'\delta}\beta(m - mr'\delta, 1 - r'\delta) . \end{aligned} \tag{2.12}$$

Combining the result of these m integrations we see that (2.9) equals

$$\begin{aligned} &(b - a)^{m-(m+1)r'\delta} \prod_1^m \beta(j - jr'\delta, 1 - r'\delta) \\ &\quad = \frac{(b - a)^{m-(m+1)r'\delta}[\Gamma(1 - r'\delta)]^{m+1}}{\Gamma[(m + 1)(1 - r'\delta)]} . \end{aligned} \tag{2.13}$$

Combining this last computation with (2.7) and (2.8) we obtain (2.4) as desired.

To finish the proof that $K_\lambda(F)$ is in $L(L_p, L_{p'})$ with $\|K_\lambda(F)\|$ bounded by the right side of (2.2) it remains only to show that $G_{\lambda,\tau}(s_1, \cdots, s_m)(\cdot)$ as given by (2.5) is a strongly measurable function of $(s_1, \cdots, s_m)$. To show that $G_{\lambda,\tau}$ is strongly measurable, it suffices, since $L_{p'}$ is separable, to show that it is weakly measurable [12; Theorem 3.5.3]. So given ϕ in L_p, we must show that

$$\int_{R^\nu} \phi(\xi) G_{\lambda,\tau}(s_1, \cdots, s_m)(\xi) d\xi \tag{2.14}$$

is a measurable function of $s_1, \cdots, s_m$. However

$$\begin{aligned} &\phi(\xi)\left(\frac{\lambda}{2\pi}\right)^{(m+1)\nu/2}[(s_{\tau(1)} - a)\cdots(b - s_{\tau(m)})]^{-\nu/2}\psi(V_{m+1}) \\ &\quad \times \left[\prod_1^m \theta_{\tau(j)}(s_{\tau(j)}, V_j)\right] \exp\left[-\frac{\lambda}{2}\sum_1^{m+1}\frac{\|V_j - V_{j-1}\|^2}{(s_{\tau(j)} - s_{\tau(j-1)})}\right] \end{aligned} \tag{2.15}$$

is a measurable function of $\xi, s_1, \cdots, s_m, V_1, \cdots, V_{m+1}$ which is also integrable with respect to all these variables as can be seen via the Fubini-Tonelli Theorem as follows: Integrate the absolute value of

(2.15) with respect to $V_{m+1}, \cdots, V_1$ and ξ. One obtains a function of $(s_1, \cdots, s_m)$ which by Lemma 1.4 is bounded by

$$\|\psi\|_p \|\phi\|_p \left[\prod_1^m \|\theta_{\tau(j)}(s_{\tau(j)}, \cdot)\|_r\right]\left(\frac{|\lambda|}{\operatorname{Re}\lambda}\right)^{(m+1)\nu/2}\left(\frac{\operatorname{Re}\lambda}{2\pi}\right)^{(m+1)\nu/2r} \times [(s_{\tau(1)} - a)\cdots(b - s_{\tau(m)})]^{-\nu/2r} . \tag{2.16}$$

But the function in (2.16) is an integrable function of $s_1, \cdots, s_m$ as we have already shown as part of the argument in (2.7)–(2.13). Thus, by the Fubini Theorem, the function obtained after integrating (2.15) with respect to $V_{m+1}, \cdots, V_1$ and ξ, namely (2.14), is measurable in $(s_1, \cdots, s_m)$ as desired.

So now we know that for λ in $C^+, K_\lambda(F)$ is in $L(L_p, L_{p'})$ and $\|K_\lambda(F)\|$ is bounded by the right side of (2.2). Furthermore, another application of the Fubini Theorem shows that $K_\lambda(F)\psi$ is also given by the second expression in (2.1).

Next we wish to show that $K_\lambda(F)$ is an $L(L_p, L_{p'})$-valued analytic function of λ in C^+. It suffices to fix ψ and ϕ in L_p and show that $(K_\lambda(F)\psi, \phi)$ is a scalar-valued analytic function of λ in C^+. In fact, as before, we may concentrate on one term $K_{\lambda,\tau}(F)\psi$ from (2.1) and show that

$$h(\lambda) \equiv \int_{R^\nu} \phi(\xi)\left[(B)\int_{S(\tau)} G_{\lambda,\tau}(s_1, \cdots, s_m)(\xi)dS\right]d\xi \tag{2.17}$$

is analytic in C^+. Now by [12; Theorem 3.7.12 and following remark, pp. 83–84], we have

$$h(\lambda) = \int_{S(\tau)}\int_{R^\nu} \phi(\xi)G_{\lambda,\tau}(s_1, \cdots, s_m)(\xi)d\xi dS \tag{2.18}$$

where the integral over $S(\tau)$ may now be interpreted as an ordinary Lebesgue integral. Since the function in (2.15) is integrable, as we observed earlier, we can use the Fubini Theorem to write

$$\begin{aligned} h(\lambda) = \int_{S(\tau)}\int_{R^\nu}(m+2)\int_{R^\nu}\left(\frac{\lambda}{2\pi}\right)^{(m+1)\nu/2}[(s_{\tau(1)} - a)\cdots(b - s_{\tau(m)})]^{-\nu/2}\phi(\xi) \\ \times \left[\prod_1^m \theta_{\tau(j)}(s_{\tau(j)}, V_j)\right]\exp\left[-\frac{\lambda}{2}\sum_1^{m+1}\frac{\|V_j - V_{j-1}\|^2}{(s_{\tau(j)} - s_{\tau(j-1)})}\right]\psi(V_{m+1}) \\ \times dV_{m+1}\cdots dV_1 d\xi dS . \end{aligned} \tag{2.19}$$

We will use Morera's Theorem to show that $h(\lambda)$ is analytic in C^+.

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First an application of the Dominated Convergence Theorem shows that $h(\lambda)$ is continuous in C^+; an appropriate dominating function is obtained almost exactly as in the following argument and so the argument will be omitted here. Now let Δ be a triangular path in C^+. We need only show that $\int_\Delta h(\lambda)d\lambda = 0$. But this will clearly follow from the Cauchy Integral Theorem if we can justify moving the integral with respect to λ inside all the other integrals. Let $D \equiv \sup\{|\lambda|: \lambda \in \Delta\}$ and $E \equiv \inf\{\operatorname{Re}\lambda: \lambda \in \Delta\}$. Then

$$\begin{aligned}(2.20)\quad &\left(\frac{D}{E}\right)^{(m+1)\nu/2}\left(\frac{E}{2\pi}\right)^{(m+1)\nu/2}[(s_{\tau(1)} - a)\cdots(b - s_{\tau(m)})]^{-\nu/2}|\phi(\xi)\psi(V_{m+1})| \\ &\times\left[\prod_1^m |\theta_{\tau(j)}(s_{\tau(j)}, V_j)|\right]\exp\left[-\frac{E}{2}\sum_1^{m+1}\frac{\|V_j - V_{j-1}\|^2}{(s_{\tau(j)} - s_{\tau(j-1)})}\right]\end{aligned}$$

is a dominating function for the integral in (2.19) which is integrable with respect to $V_{m+1}, \cdots, V_1, \xi, S$ and λ; the integrability of the function in (2.20) is established by minor modifications of the arguments in (2.7) –(2.13).

Now let $(K_{-iq}(F)\psi)(\xi)$ denote the third expression in (2.1) when $\lambda = -iq$ $(q \neq 0)$. We will show that $K_\lambda(F) \to K_{-iq}(F)$ in the strong operator topology as $\lambda \to -iq$ through C^+. In the process we will also see that $K_{-iq}(F)$ makes sense. Let ψ in L_p be fixed. By Lemma 1.4, for almost every $(s_1, \cdots, s_m)$

$$(2.21)\quad \|G_{\lambda,\tau}(s_1, \cdots, s_m)(\cdot) - G_{-iq,\tau}(s_1, \cdots, s_m)(\cdot)\|_{p'} \to 0 \quad \text{as} \quad \lambda \to -iq\ .$$

Also for λ in a neighborhood of $-iq$, (2.6) yields

$$\begin{aligned}(2.22)\quad &\|G_{\lambda,\tau}(s_1, \cdots, s_m)(\cdot)\|_{p'} \\ &\le \|\psi\|_p\left[\prod_1^m \|\theta_{\tau(j)}(s_{\tau(j)}, \cdot)\|_r\right]\left(\frac{2|q|}{2\pi}\right)^{(m+1)\delta} \\ &\times [(s_{\tau(1)} - a)\cdots(b - s_{\tau(m)})]^{-\delta}\ .\end{aligned}$$

But the right hand side of (2.22) is independent of λ and is an integrable function of $s_1, \cdots, s_m$ over $S(\tau)$ as is essentially argued in (2.7)–(2.13). We may now apply the Dominated Convergence Theorem for Bochner integrals [12; Theorem 3.7.9] and conclude that $G_{-iq,\tau}(s_1, \cdots, s_m)$ is Bochner integrable so that $K_{-iq}(F)\psi$ makes sense and, furthermore,

$$(2.23)\quad \|K_\lambda(F)\psi - K_{-iq}(F)\psi\|_{p'} \to 0 \quad \text{as} \quad \lambda \to -iq\ .$$

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We remark that the use of the Dominated Convergence Theorem for Bochner integrals is a key step in the present paper as it was in [16; p. 137 and p. 146]. The ordinary Dominated Convergence Theorem is not applicable in establishing (2.23). In the case that $\lambda = -iq$, if one puts absolute values inside all the integrals in the second or third expression in (2.1), one gets an integrand that is not in general integrable. The Bochner version of the Dominated Convergence Theorem allows us to go inside the integral with respect to S with absolute values without going all the way inside; this allows one to take advantage of the exponential kernels through Lemmas 1.4 and 1.1 even when $\lambda = -iq$.

Now that we know that $G_{-iq,\tau}(s_1, \cdots, s_m)$ is Bochner integrable, the argument used to establish (2.4), and hence to show that $\|K_\lambda(F)\|$ is dominated by the right side of (2.2) for $\operatorname{Re}\lambda > 0$, also works in the case $\lambda = -iq$.

The proof will be finished if we can show that for $\lambda > 0$,

$$(K_\lambda(F)\psi)(\xi) = \int_{C_0^\nu[a,b]} F(\lambda^{-1/2}X + \xi)\psi(\lambda^{-1/2}X(b) + \xi)dm(X) . \tag{2.24}$$

The measurability of the functions involved in the following arguments is a consequence of Lemmas 4A, 5A and 5B of Haugsby's thesis [11].

We begin our proof of the equality in (2.24) by showing that the right hand side of (2.24) is an $L_{p'}$ function of ξ. The third equality below comes from a fundamental Wiener integration formula. (The fact that we can use Bochner integrals below instead of ordinary Lebesgue integrals can be justified as in Corollary 2.2 to follow.)

$$\begin{aligned}(2.25)\quad &\left|\int_{C_0^\nu[a,b]} F(\lambda^{-1/2}X + \xi)\psi(\lambda^{-1/2}X(b) + \xi)dmX)\right| \\ &\le \int_{C_0^\nu[a,b]} \left[\prod_1^m \int_a^b |\theta_j(s, \lambda^{-1/2}X(s) + \xi)|\, ds\right] |\psi(\lambda^{-1/2}X(b) + \xi)|\, dm(X) \\ &= \int_a^b (m) \int_a^b \int_{C_0^\nu[a,b]} \left[\prod_1^m |\theta_j(s_j, \lambda^{-1/2}X(s_j) + \xi)|\right] |\psi(\lambda^{-1/2}X(b) + \xi)| \\ &\quad \times dm(X)ds_1 \cdots ds_m \\ &= \sum_\tau (B) \int_{S(\tau)} \int_{C_0^\nu[a,b]} \left[\prod_1^m |\theta_{\tau(j)}(s_{\tau(j)}, \lambda^{-1/2}X(s_{\tau(j)}) + \xi)|\right] \\ &\quad \times |\psi(\lambda^{-1/2}X(b) + \xi|\, dm(X)dS \\ &= \sum_\tau (B) \int_{S(\tau)} \left(\frac{1}{2\pi}\right)^{(m+1)\nu/2} [(s_{\tau(1)} - a) \cdots (b - s_{\tau(m)})]^{-\nu/2}\end{aligned}$$

$$\times \int_{R^\nu} (m+1) \int_{R^\nu} \left[\prod_1^m |\theta_{\tau(j)}(s_{\tau(j)}, \lambda^{-1/2}U_j + \xi)| \right] |\psi(\lambda^{-1/2}U_{m+1} + \xi)|$$
$$\times \exp\left[-\frac{1}{2} \left\{ \frac{\|U_1\|^2}{(s_{\tau(1)} - a)} + \frac{\|U_2 - U_1\|^2}{(s_{\tau(2)} - s_{\tau(1)})} + \cdots \right.\right.$$
$$\left.\left. + \frac{\|U_{m+1} - U_m\|^2}{(b - s_{\tau(m)})} \right\} \right] dU_{m+1} \cdots dU_1 dS$$
$$= \sum_\tau (B) \int_{S(\tau)} \left(\frac{\lambda}{2\pi} \right)^{(m+1)\nu/2} [(s_{\tau(1)} - a) \cdots (b - s_{\tau(m)})]^{-\nu/2}$$
$$\times \int_{R^\nu} (m+1) \int_{R_\nu} \left[\prod_1^m |\theta_{\tau(j)}(s_{\tau(j)}, V_j)| \right] |\psi(V_{m+1})|$$
$$\times \exp\left[-\frac{\lambda}{2} \left\{ \frac{\|V_1 - \xi\|^2}{(s_{\tau(1)} - a)} + \frac{\|V_2 - V_1\|^2}{(s_{\tau(2)} - s_{\tau(1)})} + \cdots \right.\right.$$
$$\left.\left. + \frac{\|V_{m+1} - V_m\|^2}{(b - s_{\tau(m)})} \right\} \right] dV_{m+1} \cdots dV_1 dS .$$

Now the last expression above, except for absolute values on the θ's and ψ, is the same as the second expression in (2.1). Since we showed earlier that the second expression in (2.1) is an $L_{p'}$ function of ξ and since that proof is unaffected by absolute values on the θ's and ψ, we see that the right hand side of (2.24) is an $L_{p'}$ function of ξ.

Since we now know that both sides of (2.24) are $L_{p'}$ functions, their equality will be established if we show that for any ϕ in L_p,

$$\int_{R^\nu} \phi(\xi) \int_{C_0^\nu[a,b]} F(\lambda^{-1/2}X + \xi)\psi(\lambda^{-1/2}X(b) + \xi) dm(X) d\xi$$
$$= \int_{R^\nu} \phi(\xi)(K_\lambda(F)\psi)(\xi) d\xi . \tag{2.26}$$

The formal part of the argument needed to establish (2.26) proceeds just as in (2.25) except for the absence of absolute values and the presence of the extra integral with respect to ξ. The fact that the appropriate integrals exist so that the Fubini Theorem and the fundamental Wiener integration formula may be applied follows from the fact that the last expression in (2.25) is an $L_{p'}$ function of ξ so that the product of it with $|\phi(\xi)|$ is integrable.

This finally completes the proof.

Next we wish to establish a corresponding result for the case $p = 2$. When p is equal to 2 we can handle the endpoint $r = 1$ and, since $L_r([a, b]) \subseteq L_1([a, b])$ for $1 \leq r \leq \infty$, we will always take $r = 1$. It then turns out to be appropriate to think of $r'\delta$ as equal to zero ($\delta = 0$ when $\gamma = \infty$). In this case the bound in (2.2) simplifies to $\|I_\lambda^{\mathrm{an}}(F)\| \leq \|g\|_1^m$; however, as we will see, this bound can be improved.

COROLLARY 2.1. *Let $p = 2$ and let ν be an arbitrary positive integer. Let F be given by* (0.8) *with each θ_j in $L_{\infty 1}$. Then $I_\lambda^{\mathrm{an}}(F)$ and $J_q^{\mathrm{an}}(F)$ exist for all λ in C^+ and all real $q \neq 0$ respectively. Once again, for ψ in L_p and ξ in R^ν, $I_\lambda^{\mathrm{an}}(F)\psi$ and $J_q^{\mathrm{an}}(F)\psi$ are given by* (2.1). *In addition $I_\lambda^{\mathrm{an}}(F)$ and $J_q^{\mathrm{an}}(F)$ satisfy the bounds*

$$\|I_\lambda^{\mathrm{an}}(F)\| \leq \prod_1^m \|\theta_j\|_{\infty 1} \quad \text{and} \quad \|J_q^{\mathrm{an}}(F)\| \leq \prod_1^m \|\theta_j\|_{\infty 1}\,, \tag{2.27}$$

for all values of the parameters λ and q.

Proof. The proof given above for Theorem 2.1 also works when $p = 2$ (In fact the proof can be simplified considerably.) and establishes everything but the bounds (2.27). To establish (2.27) fix λ in C^+ and apply Lemma 1.4 to the second expression in (2.1) to obtain

$$\begin{aligned}
\|I_\lambda^{\mathrm{an}}(F)\psi\|_2 &\leq \sum_\tau \int_{S(\tau)} \prod_1^m \|\theta_{\tau(j)}(s_{\tau(j)}, \cdot)\|_\infty \, dS \\
&= \sum_\tau \int_{S(\tau)} \prod_1^m \|\theta_j(s_j, \cdot)\|_\infty \, dS \\
&= \int_a^b (m) \int_a^b \prod_1^m \|\theta_j(s_j, \cdot)\|_\infty \, ds_1 \cdots ds_m \\
&= \prod_1^m \|\theta_j\|_{\infty 1}\,.
\end{aligned}$$

COROLLARY 2.2. *In* (2.1), *each integral over $S(\tau)$ may be interpreted either as a Bochner integral or as a Lebesgue integral.*

Proof. Let $\operatorname{Re}\lambda \geq 0$, $\lambda \neq 0$. We consider a typical term $(K_{\lambda,\tau}(F)\psi)(\xi)$ of the sum (2.1). (One should keep in mind that when $\operatorname{Re}\lambda = 0$, $(K_\lambda(F)\psi)(\xi)$ is given just by the third expression in (2.1).) Let ϕ be in L_p. By (2.6) we can write

$$\begin{aligned}
&\int_{S(\tau)} \int_{R^\nu} |\phi(\xi) G_{\lambda,\tau}(s_1, \cdots, s_m)(\xi)| \, d\xi dS \\
&\quad \leq \|\phi\|_p \|\psi\|_p \left(\frac{|\lambda|}{2\pi}\right)^{(m+1)\delta} \int_{S(\tau)} \left[\prod_1^m \|\theta_{\tau(j)}(s_{\tau(j)}, \cdot)\|_r\right] \\
&\qquad \times [(s_{\tau(1)} - a) \cdots (b - s_{\tau(m)})]^{-\delta} dS\,.
\end{aligned} \tag{2.28}$$

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But this last integral is finite as the arguments in (2.7)–(2.13) show. Hence we may apply Fubini's Theorem and [12; Theorem 3.7.12 and following remark, pp. 83–84] and write

$$\begin{aligned}(2.29)\qquad &\int_{R^\nu}\phi(\xi)\int_{S(\tau)}G_{\lambda,\tau}(s_1,\cdots,s_m)(\xi)dSd\xi\\ &=\int_{S(\tau)}\int_{R^\nu}\phi(\xi)G_{\lambda,\tau}(s_1,\cdots,s_m)(\xi)d\xi dS\\ &=\int_{R^\nu}\phi(\xi)(B)\int_{S(\tau)}G_{\lambda,\tau}(s_1,\cdots,s_m)(\xi)dSd\xi\end{aligned}$$

where we note in particular that the inner integral in the first expression in (2.29) must exist for almost all ξ. Since ϕ in L_p was arbitrary the desired equality

$$\int_{S(\tau)}G_{\lambda,\tau}(s_1,\cdots,s_m)(\xi)dS=(B)\int_{S(\tau)}G_{\lambda,\tau}(s_1,\cdots,s_m)(\xi)dS$$

follows for almost all ξ.

Remark. The finiteness of the left hand side of (2.28) was used to justify a use of the Fubini Theorem. When $\lambda=-iq$, it is again important to note that we can make the argument without taking absolute values all the way inside the integrals which define $G_{\lambda,\tau}(s_1,\cdots,s_m)(\xi)$.

We finish this section by dealing with the function $F\equiv 1$. The necessary arguments here are easy because we already have Lemmas 1.1 and 1.2; hence we will merely state the proposition and indicate formally where the formula comes from. We do not need any dimension restriction here even when $1<p<2$.

PROPOSITION 2.1. *Let p in $(1,2]$ and a positive integer ν be given. Let $F\equiv 1$. Then $I_\lambda^{\mathrm{an}}(F)$ and $J_q^{\mathrm{an}}(F)$ exist for all λ in C^+ and all real $q\neq 0$ respectively. Further for λ in C^+, ψ in L_p and ξ in R^ν,*

$$(2.30)\quad (I_\lambda^{\mathrm{an}}(F)\psi)(\xi)=\left(\frac{\lambda}{2\pi(b-a)}\right)^{\nu/2}\int_{R^\nu}\psi(V)\exp\left[\frac{-\lambda\|V-\xi\|^2}{2(b-a)}\right]dV\,.$$

$J_q^{\mathrm{an}}(F)$ is also given by the right hand side of (2.30) with $\lambda=-iq$. In addition we have

$$(2.31)\qquad \|I_\lambda^{\mathrm{an}}(F)\|\le\left(\frac{\lambda}{2\pi(b-a)}\right)^{\nu/2r}\text{ and }\|J_q^{\mathrm{an}}(F)\|\le\left(\frac{|q|}{2\pi(b-a)}\right)^{\nu/2r}.$$

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Remark. Formula (2.30) comes from a fundamental Wiener integration formula as follows:

$$\begin{aligned}\int_{C_0^\nu[a,b]} & F(\lambda^{-1/2}X+\xi)\psi(\lambda^{-1/2}X(b)+\xi)dm(X)\\ &= \int_{C_0^\nu[a,b]} \psi(\lambda^{-1/2}X(b)+\xi)dm(X)\\ &= \left(\frac{1}{2\pi(b-a)}\right)^{\nu/2}\int_{R^\nu}\psi(\lambda^{-1/2}U+\xi)\exp\left[\frac{-\|U\|^2}{2(b-a)}\right]dU\\ &= \left(\frac{\lambda}{2\pi(b-a)}\right)^{\nu/2}\int_{R^\nu}\psi(V)\exp\left[\frac{-\lambda\|V-\xi\|^2}{2(b-a)}\right]dV\,.\end{aligned}$$

3. The main existence theorem.

In this section we shall establish the existence of $I_\lambda^{\text{an}}(F)$ and $J_q^{\text{an}}(F)$ for series of the form (0.9) where each F_k is in $A\equiv A(p,\nu,r)$. For each k, F_k is either the function identically one or is given by

$$F_k(X)=\prod_1^{m_k}\int_a^b \theta_{k,j}(s,X(s))ds \tag{3.1}$$

where each $\theta_{k,j}$ is in $L_{\gamma r}$. Given F_k as in (3.1), we let $b_k(|\lambda|)$ be the right side of (2.2) with m and g replaced by m_k and g_k respectively, where

$$g_k(s)\equiv\max\{\|\theta_{k,1}(s,\cdot)\|_r,\cdots,\|\theta_{k,m_k}(s,\cdot)\|_r\}\,. \tag{3.2}$$

THEOREM 3.1. *Let* $1<p<2$, *let* γ *be given by* (0.3) *and let* ν *and* r *satisfy* (0.4) *and* (0.5) *respectively. Let* F *be given by* (0.9) *with the* F_k*'s as above. Suppose that* $\{a_k\}$ *is a sequence of complex numbers such that for every* λ *in* $C_{\lambda_0}^+$

$$\sum_{k=0}^\infty |a_k|\,b_k(|\lambda|)<\infty\,. \tag{3.3}$$

Then for every λ *in* $(0,\lambda_0)$, *the series* $\sum_0^\infty a_kF_k(\lambda^{-1/2}X+\xi)$ *converges absolutely for almost all* (X,ξ) *in* $C_0^\nu[a,b]\times R^\nu$. *Also* $I_\lambda^{\text{an}}(F)$ *and* $J_q^{\text{an}}(F)$ *exist respectively for all* λ *in* $C_{\lambda_0}^+$ *and all real* $q\neq 0$ *such that* $|q|<\lambda_0$. *Furthermore*

$$I_\lambda^{\text{an}}(F)=\sum_0^\infty a_kI_\lambda^{\text{an}}(F_k) \tag{3.4}$$

and

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(3.5) $$J_q^{\text{an}}(F) = \sum_0^\infty a_k J_q^{\text{an}}(F_k)$$

with $I_\lambda^{\text{an}}(F_k)$ *and* $J_q^{\text{an}}(F_k)$ *given by* (2.1) *with appropriate replacements to account for the fact that* F_k *is given by* (3.1) *rather than by* (0.8). *Also the series in* (3.4) *and* (3.5) *converge in operator norm.*

Proof. Let p be fixed in (1,2). First using (2.2) and (3.3), we see that for each λ in $C_{\lambda_0}^+$

$$\sum_0^\infty \|a_k I_\lambda^{\text{an}}(F_k)\| \leq \sum_0^\infty |a_k| b_k(|\lambda|) < \infty .$$

Hence the right hand side of (3.4) defines an element of $L(L_p, L_{p'})$ for all λ in $C_{\lambda_0}^+$. Similarly the right hand side of (3.5) defines an element of $L(L_p, L_{p'})$ for all q in $(-\lambda_0, 0) \cup (0, \lambda_0)$. In fact since $b_k(|\lambda|)$ is an increasing function of $|\lambda|$, the series in (3.4) and (3.5) actually converge uniformly on C_λ^+ and $(-\lambda, 0) \cup (0, \lambda)$ respectively for each fixed λ in $(0, \lambda_0)$.

Let λ be in $(0, \lambda_0)$. Next we give an argument which will be used in showing

(i) $\sum_0^\infty a_k F_k(\lambda^{-1/2}X + \xi)$ converges absolutely for almost every

(X, ξ) in $C_0^\nu[a, b] \times \boldsymbol{R}^\nu$, and

(ii) $$(I_\lambda(F)\psi)(\xi) \equiv \int_{C_0^\nu[a,b]} \left[\sum_0^\infty a_k F_k(\lambda^{-1/2}X + \xi)\right]\psi(\lambda^{-1/2}X(b) + \xi)dm(X)$$

in an $L_{p'}$ function of ξ and equals the $L_{p'}$ function

$$\sum_0^\infty a_k(I_\lambda(F_k)\psi)(\xi) .$$

Let ψ and ϕ be in $L_p(\boldsymbol{R}^\nu)$. Then we can write

(3.6) $$\begin{aligned} &\int_{\boldsymbol{R}^\nu} \int_{C_0^\nu[a,b]} \left[\sum_0^\infty |a_k F_k(\lambda^{-1/2}X + \xi)|\right] |\phi(\xi)| \, |\psi(\lambda^{-1/2}X(b) + \xi)| \, dm(X)d\xi \\ &= \int_{\boldsymbol{R}^\nu} \left|\phi(\xi)\right| \sum_0^\infty \left|a_k\right| \int_{C_0^\nu[a,b]} |F_k(\lambda^{-1/2}X + \xi)| \, |\psi(\lambda^{-1/2}X(b) + \xi)| \, dm(X)d\xi \\ &= \sum_0^\infty |a_k| \int_{\boldsymbol{R}^\nu} |\phi(\xi)| \, (I_\lambda(|F_k|) \, |\psi|)(\xi)d\xi \\ &\leq \sum_0^\infty |a_k| \, \|\phi\|_p \, \|I_\lambda(|F_k|) \, |\psi|\|_{p'} \end{aligned}$$

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$$\leq \|\phi\|_p \|\psi\|_p \sum_0^\infty |a_k| b_k(|\lambda|) .$$

Now the last expression in (3.6) is finite by (3.3). Hence the integrand in the first expression in (3.6) must be an integrable function of X and ξ; in particular it must be finite for almost every (X, ξ). By considering ϕ and ψ which never vanish one sees that claim (i) is justified.

Also since (3.6) is finite for every L_p function ϕ and since

$$|(I_\lambda(F)\psi)(\xi)| \leq \int_{C_0^\nu[a,b]} \left[\sum_0^\infty |a_k F_k(\lambda^{-1/2}X + \xi)|\right] |\psi(\lambda^{-1/2}X(b) + \xi)| \, dm(X) ,$$

it follows that $(I_\lambda(F)\psi)(\xi)$ is an $L_{p'}$ function of ξ. To show that the equality claimed in (ii) holds, it suffices to show that for any L_p function ϕ

$$\int_{R^\nu} \phi(\xi)(I_\lambda(F)\psi)(\xi)d\xi = \int_{R^\nu} \phi(\xi) \sum_0^\infty a_k(I_\lambda(F_k)\psi)(\xi)d\xi . \tag{3.7}$$

But (3.7) follows from the equalities

$$\begin{aligned} &\int_{R^\nu} \phi(\xi)(I_\lambda(F)\psi)(\xi)d\xi \\ &= \int_{R^\nu} \phi(\xi) \int_{C_0^\nu[a,b]} \left[\sum_0^\infty a_k F_k(\lambda^{-1/2}X + \xi)\right] \psi(\lambda^{-1/2}X(b) + \xi)dm(X)d\xi \\ &= \int_{R^\nu} \phi(\xi) \sum_0^\infty a_k \int_{C_0^\nu[a,b]} F_k(\lambda^{-1/2}X + \xi)\psi(\lambda^{-1/2}X(b) + \xi)dm(X)d\xi \\ &= \int_{R^\nu} \phi(\xi) \sum_0^\infty a_k(I_\lambda(F_k)\psi)(\xi)d\xi \end{aligned} \tag{3.8}$$

where the interchange of integral and sum in (3.8) follows from (3.6) and the Fubini-Tonelli Theorem.

Now by Theorem 2.1, for each $k, I_\lambda(F_k)$ is an $L(L_p, L_{p'})$-valued analytic function of λ in C^+. Hence by the uniform convergence of the sum in (3.4) the right hand side of (3.4) is an $L(L_p, L_{p'})$-valued analytic function of λ in $C_{\lambda_0}^+$. By the equality in (ii) and the fact that $I_\lambda(F_k) = I_\lambda^{\text{an}}(F_k)$ for $0 < \lambda < \lambda_0$ and $k = 0, 1, 2, \cdots$ we see that $I_\lambda^{\text{an}}(F)$ exists and the equality in (3.4) holds.

Next fix q in $(-\lambda_0, 0) \cup (0, \lambda_0)$. By Theorem 2.1, for each k, $\lim_{\lambda \to -iq} I_\lambda^{\text{an}}(F_k) = J_q^{\text{an}}(F_k)$, the limit being taken in the strong operator sense. Also the right hand side of (3.4) converges uniformly for λ in $C_{(\lambda_0+q)/2}^+$. Thus for each ψ in L_p,

$$\begin{aligned}\lim_{\lambda\to -iq} I_\lambda^{\rm an}(F)\psi &= \lim_{\lambda\to -iq}\sum_0^\infty a_k I_\lambda^{\rm an}(F_k)\psi\\ &= \lim_{\lambda\to -iq}\lim_{N\to\infty}\sum_0^N a_k I_\lambda^{\rm an}(F_k)\psi\\ &= \lim_{N\to\infty}\lim_{\lambda\to -iq}\sum_0^N a_k I_\lambda^{\rm an}(F_k)\psi\\ &= \lim_{N\to\infty}\sum_0^N a_k J_q^{\rm an}(F_k)\psi\\ &= \sum_0^\infty a_k J_q^{\rm an}(F_k)\psi\end{aligned}$$

with all the limits in $L_{p'}$-norm. The fact that $J_q^{\rm an}(F)$ exists and is given by the right hand side of (3.5) follows and the proof is complete.

Next we wish to establish a corresponding result for the case $p=2$. Recall that the bounds in (2.27) are independent of λ and q; hence $I_\lambda^{\rm an}(F)$ and $J_q^{\rm an}(F)$ will exist for all λ in C^+ and all real $q\neq 0$ respectively. Thus we obtain the following theorem whose proof parallels that of Theorem 3.1 above. Note that in this case, the function $F\equiv 1$ is in $A=A(2,\nu,1)$ since $\theta(s,U)\equiv(b-a)^{-1}$ is in $L_{\infty 1}$.

THEOREM 3.2. *Let $p=2$ and let ν be an arbitrary positive integer. Let $F(X)=\sum_0^\infty a_kF_k(X)$ where each F_k is given by* (3.1) *with each $\theta_{k,j}$ in $L_{\infty 1}$. Assume that the sequence of complex numbers $\{a_k\}$ is such that*

$$\sum_{k=0}^{\infty}|a_k|\left[\sum_{j=1}^{m_k}\|\theta_{k,j}\|_{\infty 1}\right]<\infty\,. \tag{3.9}$$

Then for every $\lambda>0$, the series $\sum_0^\infty a_kF_k(\lambda^{-1/2}X+\xi)$ converges absolutely for almost all (X,ξ) in $C_0^\nu[a,b]\times \boldsymbol{R}^\nu$. Also $I_\lambda^{\rm an}(F)$ and $J_q^{\rm an}(F)$ exist for all λ in C^+ and all real $q\neq 0$ respectively. Furthermore $I_\lambda^{\rm an}(F)=\sum_0^\infty a_kI_\lambda^{\rm an}(F_k)$ and $J_q^{\rm an}(F)=\sum_0^\infty a_kJ_q^{\rm an}(F_k)$ with $I_\lambda^{\rm an}(F_k)$ and $J_q^{\rm an}(F_k)$ given by (2.1) *with appropriate replacements to account for the fact that F_k is given by* (3.1) *rather than by* (0.8).

4. Analytic functions of k single integrals.

First of all in this section we shall consider functionals $F(X)$ of the form

$$F(X)=f\left[\int_a^b\theta_1(s,X(s))ds,\cdots,\int_a^b\theta_k(s,X(s))ds\right] \tag{4.1}$$

where

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$$f(z_1, \cdots, z_k) = \sum_{n_1, \cdots, n_k = 0}^{\infty} a_{n_1, \cdots, n_k} z_1^{n_1} \cdots z_k^{n_k} \tag{4.2}$$

is an entire function of k complex variables of growth (ρ, σ). (An entire function is said to be of growth (ρ, σ) if and only if it is of order not exceeding ρ, and, if its order is ρ, its type does not exceed σ.) For the convenience of the reader we will include a brief discussion of "order" and "type" of an entire function of k complex variables. For a complete discussion see [10; pp. 338–356]. (For the case $k = 1$ see [1; pp. 8–12].)

Let f be given by (4.2). Let the domain D be the polycylinder $D = D(R_1, \cdots, R_k) = \{(z_1, \cdots, z_k) : |z_j| < R_j < \infty, \ j = 1, \cdots, k\}$. Let $D_R = D_R(R_1, \cdots, R_k) = \left\{(z_1, \cdots, z_k) : \left(\frac{z_1}{R}, \cdots, \frac{z_k}{R}\right) \in D\right\}$. Let

$$M_f(R) = \sup_{D_R} |f(z_1, \cdots, z_j)| .$$

The order ρ and the type σ of f are defined by the equations;

$$\rho \equiv \limsup_{R \to \infty} \left\{ \frac{\ln \ln M_f(R)}{\ln R} \right\} \tag{4.3}$$

$$\sigma \equiv \sigma_D \equiv \limsup_{R \to \infty} \left\{ \frac{\ln M_f(R)}{R^\rho} \right\} . \tag{4.4}$$

A theorem of Goldberg [10; p. 339] allows us to express ρ and σ in terms of the coefficients $a_{n_1, \cdots, n_k}$;

$$\rho = \limsup_{n_1 + \cdots + n_k \to \infty} \left\{ \frac{(n_1 + \cdots + n_k) \ln (n_1 + \cdots + n_k)}{-\ln |a_{n_1, \cdots, n_k}|} \right\} \tag{4.5}$$

$$(e\rho\sigma)^{1/\rho} = \limsup_{n_1 + \cdots + n_k \to \infty} \{(n_1 + \cdots + n_k)^{1/\rho} [|a_{n_1, \cdots, n_k}| R_1^{n_1} \cdots R_k^{n_k}]^{1/(n_1 \cdots + n_k)}\} . \tag{4.6}$$

THEOREM 4.1. *Let $1 < p < 2$, let γ be given by (0.3) and let ν and r satisfy (0.4) and (0.5) respectively. Let $F(X)$ be of the form (4.1) with f given by (4.2) an entire function of growth $\left(\frac{2\gamma}{\nu}, \sigma\right)$ where $\sigma = \sigma_D < \infty$ and each θ_j is in $L_{\gamma r}$.*

Case 1: growth $\left(\frac{2\gamma}{\nu}, 0\right)$. In this case $I_\lambda^{\text{an}}(F)$ and $J_q^{\text{an}}(F)$ exist for all λ in C^+ and all real $q \neq 0$ respectively.

Case 2: order $f = \frac{2\gamma}{\nu}$, *type* $f = \sigma = \sigma_{D(R_1,\cdots,R_k)} \in (0, \infty)$, $R_1, \cdots, R_k$ *any k positive numbers. In this case* $I_\lambda^{\mathrm{an}}(F)$ *and* $J_q^{\mathrm{an}}(F)$ *exist for all* λ *in* $C_{\lambda_0}^+$ *and all real* $q \neq 0$ *such that* $|q| < \lambda_0$ *where* $\left(\text{recall } \delta = \frac{\nu}{2\gamma}\right)$

$$\lambda_0^\delta = \frac{(2\pi\delta)^\delta \min\{R_1, \cdots, R_k\}}{\sigma^\delta \|g\|_r [\Gamma(1 - r'\delta)]^{1/r'}} \left(\frac{1 - r'\delta}{b - a}\right)^{(1-r'\delta)/r'} \tag{4.7}$$

and where $g(s) = \max\{\|\theta_1(s, \cdot)\|_r, \cdots, \|\theta_k(s, \cdot)\|_r\}$.

Proof. Let

$$F_{n_1,\cdots,n_k}(X) = a_{n_1,\cdots,n_k}\left[\int_a^b \theta_1(s, X(s))ds\right]^{n_1} \cdots \left[\int_a^b \theta_k(s, X(s))ds\right]^{n_k}.$$

We note that $F_{n_1,\cdots,n_k}$ is in A for all choices of $n_1, \cdots, n_k$. By Theorem 3.1 we see that it will suffice to establish the convergence of the series

$$\sum_{n_1,\cdots,n_k=0}^{\infty} \frac{|a_{n_1,\cdots,n_k}|\,|\lambda|^{(N+1)\delta}\,\|g\|_r^N (N!)^{1/r'}(b - a)^{N(1-r'\delta)/r'}}{(2N)^{(N+1)\delta}(b - a)^\delta} \times \frac{[\Gamma(1 - r'\delta)]^{(N+1)/r'}}{\{\Gamma[(N + 1)(1 - r'\delta)]\}^{1/r'}} \tag{4.8}$$

for the appropriate λ, where for notational purposes we let $N = n_1 + \cdots + n_k$.

Now $\Gamma(z) = z^{z-1/2}e^{-z}\sqrt{2\pi}(1 + 0(1))$ and hence for positive z sufficiently large

$$\frac{1}{\Gamma(z)} < \frac{2e^z\sqrt{z}}{\sqrt{2\pi}z^z}. \tag{4.9}$$

Also by Stirling's formula

$$N! \leq \left(\frac{N}{e}\right)^N (2\pi N)^{1/2} \exp\left(\frac{1}{12N}\right). \tag{4.10}$$

Next we claim that for sufficiently large N

$$\frac{N^{N/r'}N^{1/2r'}(N + 1)^{1/2r'}}{(N + 1)^{(N+1)(1-r'\delta)/r'}} \leq N^{(N+1)\delta}, \tag{4.11}$$

or, equivalently,

$$\frac{N^{(2N+1-2r'\delta N-2r'\delta)/2r'}}{(N + 1)^{(2N+1-2r'\delta N-2r'\delta)/2r'}} \leq 1. \tag{4.12}$$

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But (4.12) follows since $r'\delta < 1$ implies that $\dfrac{2N+1-2r'\delta N-2r'\delta}{2r'} \geq 0$. Now combining (4.9), (4.10) and (4.11) we see that for sufficiently large N

$$\left\{\frac{N'}{\Gamma[(N+1)(1-r'\delta)]}\right\}^{1/r'} \leq \frac{2^{1/r'}\exp\left[\dfrac{12N+1}{12Nr'}\right]N^{(N+1)\delta}}{e^{(N+1)\delta}(1-r'\delta)^{(2(N+1)(1-r'\delta)-1)/2r'}}. \tag{4.13}$$

(i) We first consider the case where $0 < \text{order } f < \dfrac{2\gamma}{\nu}$. Using (4.5) we see that there exists an α in $\left(0, \dfrac{2\gamma}{\nu}\right)$ such that for N sufficiently large $|a_{n_1,\cdots,n_k}| < N^{-N/\alpha}$. Using this fact and (4.13) we see that for N sufficiently large the series (4.8) is dominated by the series

$$\sum_0^\infty \frac{|\lambda|^{(N+1)\delta}\,\|g\|_r^N(b-a)^{N(1-r'\delta)/r'}N^\delta 2^{1/r'}\exp\left[\dfrac{12N+1}{12Nr'}\right][\Gamma(1-r'\delta)]^{(N+1)/r'}}{(2\pi e)^{(N+1)\delta}(b-a)^\delta N^{(1/\alpha-\delta)}(1-r'\delta)^{(2(N+1)(1-r'\delta)-1)/2r'}}.$$

Since $\dfrac{1}{\alpha}-\delta = \dfrac{1}{\alpha}-\dfrac{\nu}{2\gamma} > 0$, the convergence of (4.8) throughout $\operatorname{Re}\lambda \geq 0$, uniformly on compact subsets, follows by the root test.

(ii) Next we consider the case where f has order $\dfrac{2\gamma}{\nu}$ and type $\sigma = \sigma_{D(R_1,\cdots,R_k)} \in [0,\infty)$, $R_1,\cdots,R_k$ any k fixed positive numbers. Using (4.6) and the fact that order $f = \dfrac{2\gamma}{\nu} = \dfrac{1}{\delta}$ we have

$$\limsup_{N\to\infty}\{N^{N\delta}|a_{n_1,\cdots,n_k}|R_1^{n_1}\cdots R_k^{n_k}\}^{1/N} = \left(\frac{e\sigma}{\delta}\right)^{1/\delta}. \tag{4.14}$$

In this case the series (4.8) is dominated by the series

$$\sum_{n_1,\cdots,n_k=0}^\infty \left\{\frac{|\lambda|^{(N+1)\delta}\|g\|_r^N(b-a)^{N(1-r'\delta)/r'}2^{1/r'}\exp\left[\dfrac{12N+1}{12Nr'}\right]}{(2\pi e)^{N\delta}[2\pi e(b-a)]^\delta[\min\{R_1,\cdots,R_k\}]^N} \times \frac{N^\delta[\Gamma(1-r'\delta)]^{(N+1)/r'}}{(1-r'\delta)^{(2(N+1)(1-r'\delta)-1)/2r'}}\right\}\{N^{N\delta}|a_{n_1,\cdots,n_k}|R_1^{n_1}\cdots R_k^{n_k}\}.$$

Using the root test, the above series will converge for all $\operatorname{Re}\lambda \geq 0$ such that

$$\frac{|\lambda|^\delta\|g\|_r^N\left(\dfrac{e\sigma}{\delta}\right)^\delta[\Gamma(1-r'\delta)]^{1/r'}}{(2\pi e)^\delta\min\{R_1,\cdots,R_k\}}\left(\frac{b-a}{1-r'\delta}\right)^{(1-r'\delta)/r'} < 1.$$

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If $\sigma = 0$ we obtain the convergence of (4.8) throughout $\operatorname{Re}\lambda \geq 0$, uniformly on compact subsets, while if $\sigma > 0$ we obtain (4.7). This completes the proof of Theorem 4.1.

In the case $p = 2$ we are able to allow much more general analytic functions f; as the following theorem shows, f need not be entire much less of finite growth.

THEOREM 4.2. *Let $p = 2$ and let ν be an arbitrary positive integer. Let $F(X)$ be of the form (4.1) with f given by (4.2) a function of k complex variables which is analytic in a region containing $\{(z_1, \cdots, z_k): |z_j| \leq \|\theta_j\|_{\infty 1}$ for $j = 1, \cdots, k\}$ and each θ_j is in $L_{\infty 1}$. Then $I_\lambda^{\text{an}}(F)$ and $J_q^{\text{an}}(F)$ exist for all λ in $\boldsymbol{C}^+$ and all real $q \neq 0$ respectively.*

Proof. First note that for each j in $\{1, \cdots, k\}$,

$$\left|\int_a^b \theta_j(s, \lambda^{-1/2}X(s) + \xi)ds\right| \leq \int_a^b |\theta_j(s, \lambda^{-1/2}X(s) + \xi)|\, ds$$
$$\leq \int_a^b \|\theta_j(s, \cdot)\|_\infty\, ds = \|\theta\|_{\infty 1}\ .$$

Hence $F(\lambda^{-1/2}X + \xi)$ is defined for any (X, ξ) for which each of $\int_a^b \theta_j(s, \lambda^{-1/2}X(s) + \xi)ds$ are defined; in particular, $F(\lambda^{-1/2}X + \xi)$ is defined for almost every (X, ξ) in $C_0^\nu[a, b] \times \boldsymbol{R}^\nu$. Now let

$$F_{n_1,\cdots,n_k}(X) = a_{n_1,\cdots,n_k}\left[\int_a^b \theta_1(s, X(s))ds\right]^{n_1}\cdots\left[\int_a^b \theta_k(s, X(s))ds\right]^{n_k}.$$

By Corollary 2.1 we have that $F_{n_1,\cdots,n_k}$ is in $A \equiv A(2, \nu, 1)$ for all choices of $n_1, \cdots, n_k$ and that for each ψ in $L_2(\boldsymbol{R}^\nu)$,

$$\|I_\lambda^{\text{an}}(F_{n_1,\cdots,n_k})\psi\|_2 \leq |a_{n_1,\cdots,n_k}|\, \|\psi\|_2\, \|\theta_1\|_{\infty 1}^{n_1}\cdots\|\theta_k\|_{\infty 1}^{n_k}$$

and

$$\|J_q^{\text{an}}(F_{n_1,\cdots,n_k})\psi\|_2 \leq |a_{n_1,\cdots,n_k}|\, \|\psi\|_2\, \|\theta_1\|_{\infty 1}^{n_1}\cdots\|\theta_k\|_{\infty 1}^{n_k}\ .$$

By Theorem 3.2 the present theorem will be proved if we establish the convergence of the series

$$\sum_{n_1,\cdots,n_k=0}^{\infty} |a_{n_1,\cdots,n_k}|\, \|\theta_1\|_{\infty 1}^{n_1}\, \|\theta_2\|_{\infty 1}^{n_2}\cdots\|\theta_k\|_{\infty 1}^{n_k};$$

but this follows from our assumption that f is given by (4.2) and is

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analytic in a region containing

$$\{(z_1, \cdots, z_k) : |z_j| \leq \|\theta_j\|_{\infty 1},\ j = 1, \cdots, k\} .$$

Much as in [16] the class of functions described in Theorem 3.2 form a Banach algebra. Since the development proceeds much as in [16] we simply outline the facts.

We introduce a "norm germ" N_0 on $A = A(2, \nu, 1)$ by letting

$$N_0(F) \equiv \prod_1^m \|\theta_j\|_{\infty 1} \tag{4.15}$$

where F in A is of the form $F(X) = \prod_1^m \int_a^b \theta_j(s, X(s))ds$ with each θ_j in $L_{\infty 1}$. (Note that A is not a linear space.)

DEFINITION 4.1. $S \subseteq C^\nu[a, b]$ will be said to satisfy condition (P) if and only if for every $\lambda > 0$, $\lambda^{-1/2}X + \xi$ is in S for almost all (X, ξ) in $C_0^\nu[a, b] \times \boldsymbol{R}^\nu$. We will consider functions F defined on at least a subset S of $C^\nu[a, b]$ satisfying (P) and which are such that for every $\lambda > 0$, $F(\lambda^{-1/2}X + \xi)$ is a measurable function of (X, ξ). If F_1 and F_2 are two such functions, we will say that $F_1 \sim F_2$ if and only if there exists a set S satisfying (P) such that F_1 and F_2 are defined and agree on S. It is easy to see that $\sim$ is an equivalence relation. We will identify equivalent functions and will adopt the usual convention of using F to refer both to a function and to an equivalence class of functions.

Remark. The domain of any F in A is large enough to satisfy (P).

DEFINITION 4.2. Let $\{F_j\}$ be a sequence from A such that

$$\sum_1^\infty N_0(F_j) < \infty . \tag{4.16}$$

Let

$$F(\lambda^{-1/2}X + \xi) \equiv \sum_1^\infty F_j(\lambda^{-1/2}X + \xi) . \tag{4.17}$$

We define $\hat{A}$ to be the collection of equivalence classes of functions each of which contains a function F which arises as above from a sequence $\{F_j\}$ in A satisfying (4.16). For F in $\hat{A}$, we define $N^\#(F)$ as the infimum of the left side of (4.16) for all choices of sequences $\{F_j\}$ from A satisfying (4.17).

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Remark. Using Theorem 3.2 we see that if $\{F_j\}$ is a sequence from A satisfying (4.16), then, for every $\lambda > 0$ the series $\sum_1^\infty F_j(\lambda^{-1/2}X + \xi)$ converges absolutely for almost all (X, ξ) and so (4.17) makes sense.

Our next theorem is a generalization of Theorem 2.1 of [16, p. 141].

THEOREM 4.3. *$(\hat{A}, N^\#)$ is a commutative Banach algebra with identity.*

The next theorem follows easily from Definition 4.2, Theorem 3.2, and the bounds (2.27).

THEOREM 4.4. *Let F be in $\hat{A}$. Then $I_\lambda^{\text{an}}(F)$ and $J_q^{\text{an}}(F)$ exist for all λ in $\boldsymbol{C}^+$ and all real $q \neq 0$ and satisfy the bounds*

$$\|I_\lambda^{\text{an}}(F)\| \leq N^\#(F) \quad \text{and} \quad \|J_q^{\text{an}}(F)\| \leq N^\#(F) \; . \tag{4.18}$$

THEOREM 4.5. *Let F be in $\hat{A}$ and let f be a complex-valued function of a complex variable which is analytic in a circle about the origin with radius greater than $N^\#(F)$. Then the function G defined by $G(X) \equiv f(F(X))$ is in $\hat{A}$ and so $I_\lambda^{\text{an}}(G)$ and $J_q^{\text{an}}(G)$ exist for all λ in $\boldsymbol{C}^+$ and all real $q \neq 0$ respectively.*

Proof. This follows using some standard facts about Banach algebras [23, pp. 202–205].

In previous work [3, 4, 5, 11, 15] strong use was made of a sequential definition of the operator valued function space integrals in order to obtain the analytic continuation. (That is to say, $I_\lambda^{\text{seq}}(F)$ and $J_q^{\text{seq}}(F)$ were used to obtain $I_\lambda^{\text{an}}(F)$ and $J_q^{\text{an}}(F)$.) A counter-example given by Cameron and Storvick [6; pp. 358–60] for $p = 1$ can be generalized to show that for $1 \leq p < 2$, $I_\lambda^{\text{seq}}(F)$ need not exist under our present hypotheses. We will briefly outline the example.

Let $1 \leq p < 2$, let γ be given by (0.3) and let ν satisfy (0.4). Let $\theta\colon [a, b] \times \boldsymbol{R}^\nu \to \boldsymbol{C}$ be defined by

$$\theta(s, U) = \theta(U) = \theta((u_1, \cdots, u_\nu)) = \prod_1^\nu \frac{|u_j|^{-2/3\gamma}}{1 + |u_j|} \; .$$

Clearly θ is in $L_{\gamma\infty} \subseteq L_{\gamma r}$ for $1 \leq r \leq \infty$. Let

$$F(X) = \exp\left[\int_a^b \theta(s, X(s))ds\right] .$$

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We claim that $I_\lambda^{\text{seq}}(F)$ and $J_q^{\text{seq}}(F)$ fail to exist for all values of the parameters λ and q. For let σ be the partition $a = t_0 < t_1 < \cdots < t_n = b$ where $n > 1$. Now by [11; p. 50] whenever it exists

$$(4.19) \quad (I_\lambda^\sigma(F)\psi)(\xi) = \lambda^{n\nu/2}[(2\pi)^n(t_1 - a)\cdots(b - t_{n-1})]^{-\nu/2}\int_{R^\nu}(n)\int_{R^\nu}\psi(V_n)$$
$$\times \exp[(t_1 - a)\theta(\xi) + (t_2 - t_1)\theta(V_1) + \cdots + (b - t_{n-1})\theta(V_{n-1})]$$
$$\times \exp\left[\frac{-\lambda\|V_1 - \xi\|^2}{2(t_1 - a)} - \frac{\lambda\|V_2 - V_1\|^2}{2(t_2 - t_1)} - \cdots - \frac{\lambda\|V_n - V_{n-1}\|^2}{2(b - t_{n-1})}\right]$$
$$\times dV_1 \cdots dV_n .$$

But for all values of V_2 and ξ, the inner integral in (4.19) has absolute value ∞. Hence if ψ is non-trivial, $(I_\lambda^\sigma(F)\psi)(\xi)$ does not exist and so $I_\lambda^{\text{seq}}(F)\psi$, which by definition is the weak limit of $I_\lambda^\sigma(F)\psi$ as norm $\sigma \to 0$, does not exist.

However in the case $p = 2$ the following theorem can be obtained by generalizing the results of section 4 of [16; pp. 146–151].

THEOREM 4.6. *Let F be in $\hat{A}$. Then for all λ in C^+ and all real $q \neq 0$, $I_\lambda^{\text{seq}}(F)$ and $J_q^{\text{seq}}(F)$ exist and equal $I_\lambda^{\text{an}}(F)$ and $J_q^{\text{an}}(F)$ respectively.*

5. Application to the solution of an integral equation formally equivalent to Schroedinger's equation.

In this section we use our earlier results to give a solution to an integral equation which, in the case where $\lambda = -i$, is formally equivalent to Schroedinger's equation. The analysis also yields an approximation to the solution of the integral equation in terms of the sum of integrals over finite dimensional spaces.

The theorem below allows a wider variety of functions θ than were treated in the earlier papers [3, 4, 6, 11, 16, 21]; in addition the proof is more straightforward and somewhat shorter. Before stating the theorem it will be convenient to introduce some concepts and prove a lemma.

Let $1 < p \leq 2$ be fixed, let γ be given by (0.3) and let ν, r and δ satisfy (0.4), (0.5) and (0.7) respectively. Given a measurable function f on $[0, t]$, let

$$(5.1) \qquad N_t(f) \equiv \|s^\delta f(s)\|_\infty .$$

Let D_t denote the space of measurable functions f such that $N_t(f) < \infty$. It is not difficult to see that (D_t, N_t) is a Banach space; in fact, in the

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terminology of [27, Chapter 15], it is a Banach function space with the Fatou property. Note that when $p = 2$, $\delta = 0$ and (D_t, N_t) is the ordinary L_∞-space $(L_\infty([0, t]), \|\cdot\|_\infty)$.

Next given a measurable function $G(s, U)$ on $[0, t] \times \boldsymbol{R}^\nu$, let

$$\|G\|_t \equiv N_t(\|G(s, \cdot)\|_{p'}) \ . \tag{5.2}$$

Let Q_t denote the space of measurable functions such that $\|G\|_t < \infty$. $\|\cdot\|_t$ is an example of a product function norm as studied by Petersen [24] (Indeed, the spaces $L_{\gamma r}$ encountered earlier are also examples.). Since both $\|\cdot\|_{p'}$ and N_t are function norms with the Fatou property, it follows from results of Petersen that $\|\cdot\|_t$ is a function norm with the Fatou property and so $(Q_t, \|\cdot\|_t)$ is a Banach function space; in particular, it is a Banach space with the property that $\|G\|_t = 0$ if and only if $G = 0$ almost everywhere. The following lemma will be useful.

LEMMA 5.1. *Let* $\operatorname{Re}\lambda \geq 0$, $\lambda \neq 0$ *and let* θ *be in* $L_{\gamma r}$. *Given* G *in* Q_{t_0}, *let*

$$(T_\lambda G)(t, \xi) \equiv \int_0^t \left\{\left(\frac{\lambda}{2\pi(t-s)}\right)^{\nu/2} \times \int_{\boldsymbol{R}^\nu} \theta(s, U) G(s, U) \exp\left[\frac{-\lambda\|\xi - U\|^2}{2(t-s)}\right] dU\right\} ds. \tag{5.3}$$

Then for each t *in* $(0, t_0]$, T_λ *is in* $L(Q_t, L_{p'}(\boldsymbol{R}^\nu))$ *and*

$$\|T_\lambda\| \leq \left(\frac{|\lambda|}{2\pi}\right)^\delta t^{(1-2r'\delta)/r'} \|\theta\|_{\gamma r}\{\beta(1 - r'\delta, 1 - r'\delta)\}^{1/r'} \tag{5.4}$$

where again β *denotes the Beta function. In addition,* T_λ *is in* $L(Q_{t_0}, Q_{t_0})$ *and*

$$\|T_\lambda\| \leq \left(\frac{|\lambda|}{2\pi}\right)^\delta t_0^{(1-r'\delta)/r'} \|\theta\|_{\gamma r}\{\beta(1 - r'\delta, 1 - r'\delta)\}^{1/r'} \ . \tag{5.5}$$

Proof. First fix t in $(0, t_0]$. By Minkowski's inequality for integrals [25; p. 271], $\|(T_\lambda G)(t, \cdot)\|_{p'} \leq \int_0^t \|L_\lambda(s, \cdot)\|_{p'}\, ds$ where

$$L_\lambda(s, \xi) \equiv \left(\frac{\lambda}{2\pi(t-s)}\right)^{\nu/2} \int_{\boldsymbol{R}^\nu} \theta(s, U) G(s, U) \exp\left[\frac{-\lambda\|\xi - U\|^2}{2(t-s)}\right] dU \ . \tag{5.6}$$

But by Lemmas 1.1 and 1.3

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$$(5.7)\qquad \|L_\lambda(s,\cdot)\|_{p'} \le \left(\frac{|\lambda|}{2\pi(t-s)}\right)^\delta \|\theta(s,\cdot)\|_r \|G(s,\cdot)\|_{p'} .$$

Hence

$$\begin{aligned}\|T_\lambda G\|_{p'} &\le \left(\frac{|\lambda|}{2\pi}\right)^\delta \int_0^t \|\theta(s,\cdot)\|_r (s^\delta \|G(s,\cdot)\|_{p'})(s(t-s))^{-\delta} ds \\ &\le \left(\frac{|\lambda|}{2\pi}\right)^\delta \|G\|_t \int_0^t \|\theta(s,\cdot)\|_r (s(t-s))^{-\delta} ds \\ &\le \left(\frac{|\lambda|}{2\pi}\right)^\delta \|G\|_t \|\theta\|_{rr} \left[\int_0^t (s(t-s))^{-r'\delta} ds\right]^{1/r'} .\end{aligned}$$

Now using (2.10) we see that this last expression equals

$$\left(\frac{|\lambda|}{2\pi}\right)^\delta \|G\|_t \|\theta\|_{rr}\, t^{(1-2r'\delta)/r'} \{\beta(1-r'\delta, 1-r'\delta)\}^{1/r'}$$

which establishes (5.4). To establish (5.5) note that

$$\begin{aligned}\|T_\lambda G\|_{t_0} &= N_{t_0}(\|T_\lambda G(t,\cdot)\|_{p'}) = \|t^\delta\| (T_\lambda G)(t,\cdot)\|_{p'}\|_\infty \\ &\le \|t^\delta \|G\|_t \|\theta\|_{rr} \{\beta(1-r'\delta, 1-r'\delta)\}^{1/r'} t^{(1-2r'\delta)/r'}\|_\infty \\ &\le \|G\|_{t_0} \|\theta\|_{rr} t_0^{(1-r'\delta)/r'} \{\beta(1-r'\delta, 1-r'\delta)\}^{1/r'} .\end{aligned}$$

We are now ready for the main theorem.

THEOREM 5.1. *Let* $\theta(t,U)$ *be in* $L_{rr}([0,t_0]\times \boldsymbol{R}^\nu)$. *For* t *in* $(0,t_0]$ *let* $F_t(X) \equiv \exp\left[\int_0^t \theta(t-s, X(s))ds\right]$. *Let* ψ *be in* $L_p(\boldsymbol{R}^\nu)$. *Then for all* $\lambda \ne 0$ *such that* $\operatorname{Re}\lambda \ge 0$, *the function* $G(t,\xi,\lambda)$ *defined by*

$$(5.8)\qquad G(t,\xi,\lambda) \equiv \begin{cases}(I_\lambda^{\rm an}(F_t)\psi)(\xi), & \lambda \text{ in } \boldsymbol{C}^+ \\ (J_q^{\rm an}(F_t)\psi)(\xi), & \lambda = -iq, q \text{ real and nonzero}\end{cases}$$

exists, is in Q_{t_0} *and satisfies the integral equation*

$$\begin{aligned}(5.9)\qquad G(t,\xi,\lambda) = &\left(\frac{\lambda}{2\pi t}\right)^{\nu/2} \int_{\boldsymbol{R}^\nu} \psi(U) \exp\left[\frac{-\lambda\|\xi-U\|^2}{2t}\right] dU \\ &+ \int_0^t \left(\frac{\lambda}{2\pi(t-s)}\right)^{\nu/2} \\ &\times \int_{\boldsymbol{R}^\nu} \theta(s,U) G(s,U,\lambda) \exp\left[\frac{-\lambda\|\xi-U\|^2}{2(t-s)}\right] dU ds\end{aligned}$$

for each t *in* $(0,t_0]$. *In addition for each such* t,

$$(5.10)\qquad \|G(t,\cdot,\lambda)\|_{p'} \le t^{-\delta} \|\psi\|_p B(\lambda);$$

also

$$\|G(\cdot,\cdot,\lambda)\|_{t_0} \le \|\psi\|_p B(\lambda) \tag{5.11}$$

where

$$B(\lambda) = \sum_{m=0}^{\infty} \frac{|\lambda|^{(m+1)\delta}\|\theta\|_{rr}^m t_0^{m(1-r'\delta)/r'}\{\Gamma(1-r'\delta)\}^{(m+1)/r'}}{(2\pi)^{(m+1)\delta}(m!)^{1/r'}\{\Gamma[(m+1)(1-r'\delta)]\}^{1/r'}} . \tag{5.12}$$

The solution $G(t,\xi,\lambda)$ can be expressed by the series

$$G(t,\xi,\lambda) = \sum_{n=0}^{\infty}\int_0^t\int_0^{s_n}\cdots\int_0^{s_2}\lambda^{\nu/2}[2\pi(t-s_n)]^{-\nu/2}$$
$$\times\int_{R^\nu}\theta(s_n,V_n)\exp\left[\frac{-\lambda\|V_n-\xi\|^2}{2(t-s_n)}\right]$$
$$\lambda^{\nu/2}[2\pi(s_n-s_{n-1})]^{-\nu/2}\int_{R^\nu}\theta(s_{n-1},V_{n-1})\exp\left[\frac{-\lambda\|V_{n-1}-V_n\|^2}{2(s_n-s_{n-1})}\right] \tag{5.13}$$
$$\cdots$$
$$\lambda^{\nu/2}[2\pi(s_2-s_1)]^{-\nu/2}\int_{R^\nu}\theta(s_1,V_1)\exp\left[\frac{-\lambda\|V_1-V_2\|^2}{2(s_2-s_1)}\right]$$
$$\lambda^{\nu/2}[2\pi s_1]^{-\nu/2}\int_{R^\nu}\psi(V_0)\exp\left[\frac{-\lambda\|V_0-V_1\|^2}{2s_1}\right]dV_0\cdots dV_n ds_1\cdots ds_n .$$

Finally, if $G_N(t,\xi,\lambda)$ denotes the N^{th} partial sum of the above series, we get the error bounds

$$\|G(t,\cdot,\lambda) - G_N(t,\cdot,\lambda)\|_{p'} \le t^{-\delta}\|\psi\|_p B_N(\lambda) \tag{5.14}$$

and

$$\|G(\cdot,\cdot,\lambda) - G_N(\cdot,\cdot,\lambda)\|_{t_0} \le \|\psi\|_p B_N(\lambda) \tag{5.15}$$

where $B_N(\lambda)$ denotes the tail of the series from (5.12) *beginning with the term $m = N+1$.*

Remark. In case $p = 2$ and $r = 1$ (5.10), (5.11), (5.14) and (5.15) all become considerably simpler:

$$\|G(t,\cdot,\lambda)\|_2 \le \|\psi\|_2 \sum_{m=0}^{\infty}\frac{\|\theta\|_{\infty 1}^m}{m!} = \|\psi\|_2 \exp(\|\theta\|_{\infty 1}) \tag{5.10$'$}$$

$$\|G(\cdot,\cdot,\lambda)\|_{t_0} \le \|\psi\|_2 \sum_{m=0}^{\infty}\frac{\|\theta\|_{\infty 1}^m}{m!} = \|\psi\|_2 \exp(\|\theta\|_{\infty 1}) \tag{5.11$'$}$$

(5.14)′ $$\|G(t,\cdot,\lambda)-G_N(t,\cdot,\lambda)\|_2 \le \|\psi\|_2 \sum_{m=N+1}^{\infty} \frac{\|\theta\|_{\infty 1}^m}{m!}$$

(5.15)′ $$\|G(\cdot,\cdot,\lambda)-G_N(\cdot,\cdot,\lambda)\|_{t_0} \le \|\psi\|_2 \sum_{m=N+1}^{\infty} \frac{\|\theta\|_{\infty 1}^m}{m!}\,.$$

Proof. Let $f(z)=\exp(z)$, an entire function of growth (1,1). Let $\theta_t(s,U)\equiv\theta(t-s,U)$ and let $F_t(X)=f\left[\int_0^t \theta_t(s,X(s))ds\right]$. By Theorems 3.1 and 4.1 we know that $G(t,\xi,\lambda)$ exists as an $L_{p'}(R^\nu)$ function of ξ and satisfies (5.10). Clearly the series (5.12) converges for all λ. The in equality (5.11) is an immediate consequence of (5.10).

Now using (3.4), (3.5), (2.1) and letting τ_n be a permutation of the integers $\{1,\cdots,n\}$ we see that

$$G(t,\xi,\lambda)=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\tau_n}\int_{S(\tau_n)}\lambda^{\nu/2}[2\pi s_{\tau_n(1)}]^{-\nu/2}\int_{R^\nu}\theta_t(s_{\tau_n(1)},W_1)$$
$$\times\exp\left[\frac{-\lambda\|W_1-\xi\|^2}{2s_{\tau_n(1)}}\right]$$
$$\lambda^{\nu/2}[2\pi(s_{\tau_n(2)}-s_{\tau_n(1)})]^{-\nu/2}\int_{R^\nu}\theta_t(s_{\tau_n(2)},W_2)\exp\left[\frac{-\lambda\|W_2-W_1\|^2}{2(s_{\tau_n(2)}-s_{\tau_n(1)})}\right]$$
$$\cdots$$
$$\lambda^{\nu/2}[2\pi(s_{\tau_n(n)}-s_{\tau_n(n-1)})]^{-\nu/2}\int_{R^\nu}\theta_t(s_{\tau_n(n)},W_n)\exp\left[\frac{-\lambda\|W_n-W_{n-1}\|^2}{2(s_{\tau_n(n)}-s_{\tau_n(n-1)})}\right]$$
$$\lambda^{\nu/2}[2\pi(t-s_{\tau_n(n)})]^{-\nu/2}\int_{R^\nu}\psi(W_{n+1})\exp\left[\frac{-\lambda\|W_{n+1}-W_n\|^2}{2(t-s_{\tau_n(n)})}\right]$$
$$dW_{n+1}\cdots dW_1 dS$$

where the $n=0$ term is given by

$$\lambda^{\nu/2}[2\pi t]^{-\nu/2}\int_{R^\nu}\psi(W_1)\exp\left[\frac{-\lambda\|W_1-\xi\|^2}{2t}\right]dW_1\,.$$

Now since the θ_t's are the same, the integrals over $S(\tau_n)$ are equal (for fixed n) and so

$$G(t,\xi,\lambda)=\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^t\int_0^{s_n}\cdots\int_0^{s_2}\lambda^{\nu/2}[2\pi s_1]^{-\nu/2}\int_{R^\nu}\theta(t-s_1,W_1)$$
$$\times\exp\left[\frac{-\lambda\|W_1-\xi\|^2}{2s_1}\right]$$

$$\lambda^{\nu/2}[2\pi(s_2 - s_1)]^{-\nu/2} \int_{R^\nu} \theta(t - s_2, W_2) \exp\left[\frac{-\lambda \| W_2 - W_1 \|^2}{2(s_2 - s_1)}\right]$$

$$\cdots$$

$$\lambda^{\nu/2}[2\pi(s_n - s_{n-1})]^{-\nu/2} \int_{R^\nu} \theta(t - s_n, W_n) \exp\left[\frac{-\lambda \| W_n - W_{n-1} \|^2}{2(s_n - s_{n-1})}\right]$$

$$\lambda^{\nu/2}[2\pi(t - s_n)]^{-\nu/2} \int_{R^\nu} \psi(W_{n+1}) \exp\left[\frac{-\lambda \| W_{n+1} - W_n \|^2}{2(t - s_n)}\right]$$

$$dW_{n+1} \cdots dW_1 ds_1 \cdots ds_n .$$

Finally making the change of variables $r_1 = t - s_n$, $r_2 = t - s_{n-1}, \cdots, r_n = t - s_1$, and then appropriately changing the symbols for the variables of integration, we obtain (5.13).

The bound in (5.10) was obtained from the expansions (3.4) and (3.5) and the bound (2.2). The bound in (5.14) is obtained in the same way except that we just consider the terms $N + 1, N + 2, \cdots$. The inequality (5.15) follows immediately from (5.14).

It remains only to show that $G(t, \xi, \lambda)$ satisfies (5.9). Let $\Phi(t, \xi, \lambda)$ denote the second term on the right hand side of (5.9); that is to say let

$$\Phi(t, \xi, \lambda) = \int_0^t \left\{ \left(\frac{\lambda}{2\pi(t - s)} \right)^{\nu/2} \int_{R^\nu} \theta(s, U) G(s, U, \lambda) \times \exp\left[\frac{-\lambda \| \xi - U \|^2}{2(t - s)}\right] dU \right\} ds . \tag{5.16}$$

Substituting (5.13) for G, replacing s by s_{m+1}, and using (5.15) and Lemma 5.1 to justify taking the sum outside the integral signs we obtain

$$\Phi(t, \xi, \lambda) = \sum_{m=0}^{\infty} \int_0^t \int_0^{s_{m+1}} \cdots \int_0^{s_2} \lambda^{\nu/2}[2\pi(t - s_{m+1})]^{-\nu/2}$$

$$\times \int_{R^\nu} \theta(s_{m+1}, V_{m+1}) \exp\left[\frac{-\lambda \| \xi - V_{m+1} \|^2}{2(t - s_{m+1})}\right]$$

$$\lambda^{\nu/2}[2\pi(s_{m+1} - s_m)]^{-\nu/2} \int_{R^\nu} \theta(s_m, V_m) \exp\left[\frac{-\lambda \| V_m - V_{m+1} \|^2}{2(s_{m+1} - s_m)}\right]$$

$$\cdots$$

$$\lambda^{\nu/2}[2\pi(s_2 - s_1)]^{-\nu/2} \int_{R^\nu} \theta(s_1, V_1) \exp\left[\frac{-\lambda \| V_2 - V_1 \|^2}{2(s_2 - s_1)}\right]$$

$$\lambda^{\nu/2}[2\pi s_1]^{-\nu/2} \int_{R^\nu} \psi(V_0) \exp\left[\frac{-\lambda \| V_0 - V_1 \|^2}{2s_1}\right]$$

$$dV_0 \cdots dV_m dV_{m+1} ds_1 \cdots ds_{m+1} .$$

Now letting $m = n - 1$ we obtain

$$\Phi(t, \xi, \lambda) = \sum_{n=1}^{\infty} \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} \lambda^{\nu/2}[2\pi(t - s_n)]^{-\nu/2}$$

$$\times \int_{R^\nu} \theta(s_n, V_n) \exp\left[\frac{-\lambda \|\xi - V_n\|^2}{2(t - s_n)}\right]$$

$$\lambda^{\nu/2}[2\pi(s_n - s_{n-1})]^{-\nu/2} \int_{R^\nu} \theta(s_{n-1}, V_{n-1}) \exp\left[\frac{-\lambda \|V_n - V_{n-1}\|^2}{2(s_n - s_{n-1})}\right]$$

$$\cdots$$

$$\lambda^{\nu/2}[2\pi(s_2 - s_1)]^{-\nu/2} \int_{R^\nu} \theta(s_1, V_1) \exp\left[\frac{-\lambda \|V_2 - V_1\|^2}{2(s_2 - s_1)}\right]$$

$$\lambda^{\nu/2}[2\pi s_1]^{-\nu/2} \int_{R^\nu} \psi(V_0) \exp\left[\frac{-\lambda \|V_1 - V_0\|^2}{2s_1}\right]$$

$$dV_0 \cdots dV_n ds_1 \cdots ds_n$$

$$= G(t, \xi, \lambda) - \left(\frac{\lambda}{2\pi t}\right)^{\nu/2} \int_{R^\nu} \psi(U) \exp\left[\frac{-\lambda \|U - \xi\|^2}{2t}\right] dU$$

which establishes (5.9) and completes the proof.

Next we give a uniqueness theorem. To understand the second assertion of the theorem, it will be useful to keep in mind that the function given by (5.8) satisfies both (5.9) and (5.10) for every t in $(0, t_0]$. Throughout the theorem we consider a nonzero λ such that $\operatorname{Re} \lambda \geq 0$ as given and fixed.

THEOREM 5.2. (i) *The function* $G(t, \xi, \lambda)$ *given by* (5.8) *is the unique solution to* (5.9) *which is in* Q_{t_0}. (ii) *Any function* $H(t, \xi)$ *on* $[0, t_0] \times R^\nu$ *which satisfies both* (5.9) *and* (5.10) *for every* t *in* $(0, t_0]$ *must have the property that for every* $t, H(t, \xi) = G(t, \xi, \lambda)$ *for almost every* ξ.

Proof. We will explicitly prove (i). Once this is done, it is easy to see that the same proof, except for some minor rewording, establishes (ii). Suppose that $E_1(t, \xi)$ and $E_2(t, \xi)$ are in Q_{t_0} and are solutions to (5.9). Let $E(t, \xi) = E_1(t, \xi) - E_2(t, \xi)$. We will show that $\|E\|_{t_0} = 0$ so that $E(t, \xi) = 0$ almost everywhere as desired.

First we consider the case $1 < p < 2$. In this case we will show that for almost every t in $(0, t_0]$

$$\|E(t, \cdot)\|_{p'} \leq t^{-\delta} \frac{|\lambda|^{n\delta} \|\theta\|_{rr}^n \{\Gamma(1 - r'\delta)\}^{(n+1)/r'} t^{n(1-r'\delta)/r'}}{(2\pi)^{n\delta} \{\Gamma[(n+1)(1 - r'\delta)]\}^{1/r'}} \cdot [\|E_1\|_{t_0} + \|E_2\|_{t_0}] \tag{5.17}$$

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for $n = 1, 2, \cdots$. But (5.17) will imply

$$\|E\|_{t_1} \leq \frac{|\lambda|^{n\delta}\|\theta\|_{\gamma r}^{n}\{\Gamma(1-r'\delta)\}^{(n+1)/r'}t_1^{n(1-r'\delta)/r'}}{(2\pi)^{n\delta}\{\Gamma[(n+1)(1-r'\delta)]\}^{1/r'}} \cdot [\|E_1\|_{t_0} + \|E_2\|_{t_0}] \tag{5.18}$$

for every t_1 in $(0, t_0]$. But the right hand side of (5.18) goes to zero as $n \to \infty$ and so we will be able to conclude in particular that $\|E\|_{t_0} = 0$ as desired.

We will establish (5.17) by induction on n. Note that $\|E\|_{t_0} \leq \|E_1\|_{t_0} + \|E_2\|_{t_0}$ and that, since E_1 and E_2 both satisfy (5.9) for almost every t in $(0, t_0], E$ satisfies

$$E(t, \xi) = \int_0^t \left\{\left(\frac{\lambda}{2\pi(t-s)}\right)^{\nu/2} \times \int_{R^\nu} \theta(s, U)E(s, U)\exp\left[\frac{-\lambda\|\xi - U\|^2}{2(t-s)}\right]dU\right\}ds \tag{5.19}$$

for almost every t in $(0, t_0]$. We apply Lemma 5.1 to (5.19) and obtain the $n = 1$ case of (5.17) as follows:

$$\begin{aligned}\|E(t, \cdot)\|_{p'} &\leq \left(\frac{|\lambda|}{2\pi}\right)^{\delta} t^{(1-2r'\delta)/r'}\{\beta(1-r'\delta, 1-r'\delta)\}^{1/r'}[\|E_1\|_t + \|E_2\|_t]\,\|\theta\|_{\gamma r} \\ &= t^{-\delta}\frac{|\lambda|\,\|\theta\|_{\gamma r}\{\Gamma(1-r'\delta)\}^{2/r'}t^{(1-r'\delta)/r'}}{(2\pi)^{\delta}\{\Gamma[2(1-r'\delta)]\}^{1/r'}}[\|E_1\|_{t_0} + \|E_2\|_{t_0}] \,.\end{aligned}$$

Now assume (5.17) holds for n. Applying Minkowski's inequality for integrals [25; p. 271] and then Lemmas 1.1 and 1.3 to (5.19) we can write

$$\begin{aligned}\|E(t, \cdot)\|_{p'} &\leq \int_0^t \left(\frac{|\lambda|}{2\pi(t-s)}\right)^{\delta}\|\theta(s, \cdot)\|_r\,\|E(s, \cdot)\|_{p'}\,ds \\ &\leq \frac{|\lambda|^{(n+1)\delta}\{\Gamma(1-r'\delta)\}^{(n+1)/r'}[\|E_1\|_{t_0} + \|E_2\|_{t_0}]\,\|\theta\|_{\gamma r}^{n}}{(2\pi)^{(n+1)\delta}\{\Gamma[(n+1)(1-r'\delta)]\}^{1/r'}} \\ &\quad \times \int_0^t (t-s)^{-\delta}\|\theta(s, \cdot)\|_r\, s^{-\delta}s^{n(1-r'\delta)/r'}ds \\ &\leq \frac{|\lambda|^{(n+1)\delta}\|\theta\|_{\gamma r}^{n+1}\{\Gamma(1-r'\delta)\}^{(n+1)/r'}[\|E_1\|_{t_0} + \|E_2\|_{t_0}]}{(2\pi)^{(n+1)\delta}\{\Gamma[(n+1)(1-r'\delta)]\}^{1/r'}} \\ &\quad \times \left[\int_0^t (t-s)^{-r'\delta}s^{n-(n+1)r'\delta}ds\right]^{1/r'} .\end{aligned}$$

Now applying (2.12) to the last integral and simplifying we obtain (5.17)

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with n replaced by $n+1$ as desired.

It remains to consider the case $p = 2$. As usual we take $r = 1$. We will show by induction that for almost every t in $(0, t_0]$

$$\|E(t, \cdot)\|_2 \leq \frac{\|E_1\|_{t_0} + \|E_2\|_{t_0}}{n!} \left[\int_0^t \|\theta(s, \cdot)\|_\infty \, ds\right]^n \tag{5.20}$$

for $n = 1, 2, \cdots$. But (5.20) will imply that

$$\|E\|_{t_1} \leq \frac{\|E_1\|_{t_0} + \|E_2\|_{t_0}}{n!} \left[\int_0^{t_1} \|\theta(s, \cdot)\|_\infty \, ds\right]^n \tag{5.21}$$

for every t_1 in $(0, t_0]$. But the right hand side of (5.21) goes to zero as $n \to \infty$ and so we will be able to conclude in particular that $\|E\|_{t_0} = 0$ as desired.

Now E satisfies (5.19) for almost every t in $(0, t_0]$. Using this and Lemma 5.1 we get the $n = 1$ case of (5.20). Next suppose that (5.20) holds for n. Then we can use (5.19), Minkowski's inequality for integrals, Lemma 1.1 and the induction hypothesis and write

$$\begin{aligned}
\|E(t, \cdot)\|_2 &\leq \int_0^t \|\theta(s_{n+1}, \cdot)E(s_{n+1}, \cdot)\|_2 \, ds_{n+1} \\
&\leq \int_0^t \|\theta(s_{n+1}, \cdot)\|_\infty \|E(s_{n+1}, \cdot)\|_2 \, ds_{n+1} \\
&\leq [\|E_1\|_{t_0} + \|E_2\|_{t_0}] \int_0^t \|\theta(s_{n+1}, \cdot)\|_\infty \frac{1}{n!} \\
&\qquad \cdot \left[\int_0^{s_{n+1}} \|\theta(w, \cdot)\|_\infty \, ds\right]^n ds_{n+1} \\
&= [\|E_1\|_{t_0} + \|E_2\|_{t_0}] \int_0^t \|\theta(s_{n+1}, \cdot)\|_\infty \\
&\qquad \times \int_0^{s_{n+1}} \int_0^{s_n} \cdots \int_0^{s_2} \prod_1^n \|\theta(s_j, \cdot)\|_\infty \, ds_1 \cdots ds_{n+1} \\
&= [\|E_1\|_{t_0} + \|E_2\|_{t_0}] \int_0^t \int_0^{s_{n+1}} \cdots \int_0^{s_2} \prod_1^{n+1} \|\theta(s_j, \cdot)\|_\infty \, ds_1 \cdots ds_{n+1} \\
&= \frac{[\|E_1\|_{t_0} + \|E_2\|_{t_0}]}{(n+1)!} \left[\int_0^t \|\theta(s, \cdot)\|_\infty \, ds\right]^{n+1}.
\end{aligned}$$

6. A counterexample.

First we shall give a counterexample which shows that for $p = 1$ the dimension restriction (0.4) is really needed. When $p = 1$, $\gamma = 1$ and so in this case the dimension restriction (0.4), namely $\nu < 2\gamma$, implies

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$\nu = 1$. Our examples will show that the theory doesn't hold for $\nu > 1$. We shall write out in detail the case $\nu = 3$ and then briefly discuss other values of ν. For notational simplicity we take $a = 0$ and $b = 1$.

Let

$$\theta(s, U) = \theta(U) = \theta((u_1, \cdots, u_\nu)) = \prod_{i=1}^{\nu} |u_i|^{-3/4} \chi_{(0,1)}(u_i) , \tag{6.1}$$

$$\psi(U) = \psi((u_1, \cdots, u_\nu)) = \prod_{i=1}^{\nu} |u_i|^{-11/12} \chi_{(0,1)}(u_i) . \tag{6.2}$$

and

$$F(X) = \int_0^1 \theta(s, X(s)) ds . \tag{6.3}$$

Clearly θ is in $L_{1\infty}([0,1] \times \boldsymbol{R}^\nu)$ and ψ is in $L_1(\boldsymbol{R}^\nu)$ for all values of ν.

Next we fix $\nu = 3$ and proceed to show that for $0 < \lambda < \infty$, the function $I_\lambda(F)\psi$ is not in $L_1((0,1)^3)$. This will imply that $I_\lambda(F)\psi$ is not in $L_\alpha(\boldsymbol{R}^3)$ for any α such that $1 \leq \alpha \leq \infty$, and so, in particular, $I_\lambda(F)$ is not in $L(L_1, L_\infty)$ as required by the theory. We shall write out the details for $\lambda = 1$; by a change of scale clearly the same results hold for $0 < \lambda < \infty$.

First note that $(I_1(F)\psi)(\xi) \geq 0$ for all ξ. We need only show $\int_{(0,1)^3} (I_1(F)\psi)(\xi) d\xi = \infty$. Using the Fubini Theorem and a fundamental Wiener integration formula we obtain

$$\begin{aligned}
\int_{(0,1)^3} (I_1(F)\psi)(\xi) d\xi &= \int_{(0,1)^3} \int_0^1 [(2\pi)^2 s(1-s)]^{-3/2} \int_{\boldsymbol{R}^3} \int_{\boldsymbol{R}^3} \theta(U_1)\psi(U_2) \\
&\quad \times \exp\left[\frac{-\|U_1 - \xi\|^2}{2s} - \frac{\|U_2 - U_1\|^2}{2(1-s)}\right] dU_2 dU_1 ds d\xi \\
&\geq \int_0^1 [2\pi(1-s)]^{-3/2} \int_{(0,1/2)^3} \psi(U_2) \int_{(0,1)^3} \theta(U_1) \exp\left[\frac{-\|U_2 - U_1\|^2}{2(1-s)}\right] \\
&\quad \times \prod_{i=1}^{3} \left\{(2\pi s)^{-1/2} \int_0^1 \exp\left[\frac{-(\xi_i - u_{1i})^2}{2s}\right] d\xi_i\right\} dU_1 dU_2 ds .
\end{aligned}$$

But for all $(s, U_1) = (s, (u_{11}, u_{12}, u_{13}))$ in $(0,1) \times (0,1)^3$,

$$\begin{aligned}
(2\pi s)^{-1/2} \int_0^1 \exp\left[\frac{-(\xi_i - u_{1i})^2}{2s}\right] d\xi_i &= (2\pi)^{-1/2} \int_{-u_{1i}/\sqrt{s}}^{(1-u_{1i})/\sqrt{s}} e^{-w^2/2} dw \\
&\geq (2\pi)^{-1/2} \int_0^1 e^{-w^2/2} dw \equiv K_1 > 0
\end{aligned}$$

since for all $(s, u_{1i}) \in (0,1) \times (0,1), \left(\frac{-u_{1i}}{\sqrt{s}}, \frac{1-u_{1i}}{\sqrt{s}}\right)$ is an interval of length at least 1 which contains the origin. Thus

$$\int_{(0,1)^3} (I_1(F)\psi)(\xi)d\xi \geq K_1^3 \int_0^1 [2\pi(1-s)]^{-3/2} \int_{(0,1/2)^3} \psi(U_2)$$
$$\times \int_{(0,1)^3} \theta(U_1) \exp\left[\frac{-\|U_2 - U_1\|^2}{2(1-s)}\right] dU_1 dU_2 ds$$
$$= K_1^3 \int_0^1 (2\pi s)^{-3/2} \int_{(0,1/2)^3} \psi(U_2) \int_{(0,1)^3} \theta(U_1) \exp\left[\frac{-\|U_2 - U_1\|^2}{2s}\right] dU_1 dU_2 ds$$
$$= K_1^3 (2\pi)^{-3/2} \int_0^1 \int_{(0,1/2)^3} \psi(U_2) \int_{-u_{21}/\sqrt{s}}^{(1-u_{21})/\sqrt{s}} \int_{-u_{22}/\sqrt{s}}^{(1-u_{22})/\sqrt{s}} \int_{-u_{23}/\sqrt{s}}^{(1-u_{23})/\sqrt{s}}$$
$$\times \exp\left[\frac{-(w_1^2 + w_2^2 + w_3^2)}{2}\right]$$
$$\times \theta((\sqrt{s}\,w_1 + u_{21}, \sqrt{s}\,w_2 + u_{22}, \sqrt{s}\,w_3 + u_{23}))dw_1 dw_2 dw_3 dU_2 ds$$
$$\geq K_1^3 (2\pi)^{-3/2} \int_0^1 \int_{(0,1/2)^3} \psi(U_2)$$
$$\times \prod_{i=1}^3 \left\{\int_0^{1/2} \exp\left(\frac{-w_i^2}{2}\right) |\sqrt{s}\,w_i + u_{2i}|^{-3/4} \chi_{(0,1)}(\sqrt{s}\,w_i + u_{2i}) dw_i\right\} dU_2 ds$$

since $0 < u_{2i} < 1/2$ implies that $(0, 1/2) \subseteq \left(\frac{-u_{2i}}{\sqrt{s}}, \frac{1-u_{2i}}{\sqrt{s}}\right)$ for all s in $(0,1)$ and $i = 1,2,3$.

But for $0 < u_{2i} < 1/2$, $0 < s < 1$ and $i = 1,2,3$,

$$\int_0^{1/2} \exp\left(\frac{-w_i^2}{2}\right) |\sqrt{s}\,w_i + u_{2i}|^{-3/4} \chi_{(0,1)}(\sqrt{s}\,w_i + u_{2i}) dw_i$$
$$\geq e^{-1/8} \int_0^{1/2} (\sqrt{s}\,w_i + u_{2i})^{-3/4} dw_i$$
$$= \frac{4e^{-1/8}}{\sqrt{s}} \left[\left(\frac{\sqrt{s}}{2} + u_{2i}\right)^{1/4} - u_{2i}^{1/4}\right].$$

Hence

$$\int_{(0,1)^3} (I_1(F)\psi)(\xi)d\xi \geq 64 K_1^3 (2\pi)^{-3/2} e^{-3/8} \int_0^1 s^{-3/2} \int_0^{1/2} \int_0^{1/2} \int_0^{1/2} \psi((u_{21}, u_{22}, u_{23}))$$
$$\times \prod_{i=1}^3 \left[\left(\frac{\sqrt{s}}{2} + u_{2i}\right)^{1/4} - u_{2i}^{1/4}\right] du_{21} du_{22} du_{23} ds$$
$$= 64 K_1^3 (2\pi)^{-3/2} e^{-3/8} \int_0^1 s^{-3/2} \prod_{i=1}^3 \left[\int_0^{1/2} \frac{\left(\frac{\sqrt{s}}{2} + u_{2i}\right)^{1/4} - u_{2i}^{1/4}}{u_{2i}^{11/12}} du_{2i}\right] ds$$

$$\geq 64K_1^3(2\pi)^{-3/2}e^{-3/8}\int_0^1 s^{-3/2}\prod_{i=1}^{3}\left[\int_0^{\sqrt{s}/2}\frac{\left(\frac{\sqrt{s}}{2}+u_{2i}\right)^{1/4}-u_{2i}^{1/4}}{u_{2i}^{11/12}}du_{2i}\right]ds$$

$$\geq 64K_1^3(2\pi)^{-3/2}e^{-3/8}\int_0^1 s^{-3/2}\prod_{i=1}^{3}\left\{\left[(\sqrt{s})^{1/4}-\left(\frac{\sqrt{s}}{2}\right)^{1/4}\right]\int_0^{\sqrt{s}/2}u_{2i}^{-11/12}du_{2i}\right\}ds$$

$$= 64K_1^3(2\pi)^{-3/2}e^{-3/8}[1-(1/2)^{1/4}]^3 12^3(1/2)^{1/4}\int_0^1 s^{-3/2}s^{1/2}ds$$

$$= \infty .$$

Thus we have that $I_\lambda(F)\psi$ is not in $L_1((0,1)^3)$.

Proceeding as above one can show that if θ, ψ and F are given by (6.1)–(6.3) then $I_\lambda(F)\psi$ is not in $L_1((0,1)^\nu)$ for $\nu = 4, 5, \cdots$. Furthermore for $\nu = 3, 4, \cdots$ it then is clear that the function $I_\lambda(G)$ is not in $L_1((0,1)^\nu)$ where

$$G(X) = \exp\left[\int_0^1 \theta(s, X(s))ds\right] \tag{6.4}$$

and θ and ψ are given by (6.1) and (6.2).

Now we want to discuss the case $\nu = 2$. Here one can show that for any $r \in (2, \infty)$ there exists θ in $L_{1r}([0,1] \times \boldsymbol{R}^2)$ and ψ in $L_1(\boldsymbol{R}^2)$ such that $I_\lambda(F)\psi$ will not be in $L_1((0,1)^2)$ where F is given by (6.3). To obtain this counterexample simply fix $r \in (2, \infty)$ and choose $r_0 \in (r, \infty)$. Then let

$$\theta(s, U) = \theta(s, (u_1, u_2)) = s^{-1/r_0}\prod_{i=1}^{2}|u_i|^{1/2r_0-1}\chi_{(0,1)}(u_i) \tag{6.5}$$

and

$$\psi(U) = \psi((u_1, u_2)) = \prod_{i=1}^{2}|u_i|^{1/2r_0-1}\chi_{(0,1)}(u_i) . \tag{6.6}$$

Proceeding as in the example above one obtains

$$\int_{(0,1)^2}(I_1(F)\psi)(\xi)d\xi \geq K\int_0^1 s^{-1}s^{-1/r_0}(\sqrt{s})^{2/r_0}ds = +\infty .$$

Next we want to note that the techniques of the above example can be generalized to apply to $1 < p < \frac{3}{2}$. More specifically for fixed p in $\left(1, \frac{3}{2}\right)$ there exists $\nu > 1$, $\theta \in L_{r\infty}([0,1] \times \boldsymbol{R}^\nu)$ and ψ in $L_p(\boldsymbol{R}^\nu)$ such that $I_\lambda(F)\psi$ is not in $L_1((0,1)^\nu)$ for $0 < \lambda < \infty$ where F is given by (6.3). In particular this implies that $I_\lambda(F)$ is not in $L(L_p(R^\nu), L_{p'}(R^\nu))$ for $0 < \lambda < \infty$. We will outline the example.

Fix p in $\left(1, \frac{3}{2}\right)$. Now fix p_0 so that $p < p_0 < \frac{3}{2}$. Then $\gamma = \frac{p}{2-p}$ and we let $\gamma_0 = \frac{p_0}{2 - p_0}$. Let ν be a positive integer such that $\nu \geq \frac{2p_0}{3 - 2p_0}$. Let

$$\theta(s, U) = \theta(U) = \theta((u_1, \cdots, u_\nu)) = \prod_{i=1}^{\nu} |u_i|^{-1/\gamma_0} \chi_{(0,1)}(u_i)$$

and

$$\psi(U) = \psi((u_1, \cdots, u_\nu)) = \prod_{i=1}^{\nu} |u_i|^{-1/p_0} \chi_{(0,1)}(u_i) .$$

Let F be given by (6.3). Note that θ is in $L_{\gamma\infty}$ and ψ is in $L_p(R^\nu)$. Then proceeding as in the above examples one can show that

$$\int_{(0,1)^\nu} (I_1(F)\psi)(\xi) d\xi \geq K \int_0^1 s^{-\nu/2} s^{(3p_0-3)\nu/2p_0} ds = +\infty$$

since $\left(\frac{3p_0 - 3}{2p_0}\right)\nu \leq \frac{\nu}{2} - 1$.

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