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# STRONGLY PRIME NEAR-RINGS

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## 1. Introduction

Strongly prime rings were introduced by Handelman and Lawrence [5] and in [2] Groenewald and Heyman investigated the upper radical determined by the class of all strongly prime rings. In this paper we extend the concept of strongly prime to near-rings. We show that the class $M$ of distributively generated near-rings is a special class in the sense of Kaarli [6]. We also show that if $N$ is any distributively generated near-ring, then $UM(N)$, $UM$ denotes the upper radical determined by the class $M$, coincides with the intersection of all the strongly prime ideals of $N$.

## 2. Preliminaries

Unless otherwise stated, all near-rings are zero-symmetric right near-rings. For undefined terminologies, we refer to [9].

**Definition 1.** Let $N$ be a near-ring. $N$ is called (*right*) *strongly prime* if and only if for every $0 \neq a \in N$ there exists a finite subset $F$ of $N$ such that $r(aF) = \{n \in N : aFn = 0\} = 0$. $F$ is called an *insulator* of $a$ in $N$.

We now give the following alternative definition (c.f. [8] for corresponding definition for rings).

**Definition 2.** Let $N$ be a near-ring. $N$ is called (*right*) *strongly prime* if and only if every nonzero ideal $I$ of $N$ contains a finite subset $F$ such that $r(F) = 0$.

The two definitions of strongly prime agree for the class of zero-symmetric near-rings. The proof of this is based on the following observation and lemma.

**Observation.** Let $N$ be a zero-symmetric near-ring and $X \subset N$. The ideal generated by $X$ is the intersection of all ideals containing $X$ and can be obtained as follows:
Let $X_0 = X \cup XN$

$$X_0^+ \text{ be the normal subgroup of } N \text{ generated by } X_0.$$

$$[X_0^+] = \{z \in N \mid z = c(a+b) - ca, a, c \in N, b \in X_0^+\}$$

$$X_1 = [X_0^+] \cup X_0^+.$$



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Repeat this process to obtain successively $X_2, X_3, \ldots$ Clearly $X_0 \subset X_1 \subset X_2 \subset \ldots$ and it is straightforward to show that the ideal of $N$ generated by $X$ is precisely $\bigcup_0^\infty X_i$.

It is worth noting that an element of $X_i$ can be expressed by a formula involving only a finite number of terms involving expressions of the type $n+zxy-n$ where $n, z, y \in N, x \in X$.

**Lemma 2.1.** *If* $0 \neq a \in N$ *and* $aN=0$ *then* $\langle a \rangle N=0$.

**Proof.** $aN=0$ implies $axN=0$, $zaN=0$ for all $x, z \in N$. Hence $X_0N=0$. Suppose $X_iN=0$. Then $n \in N$ implies

$$\left(\sum(r+u-r)\right)n=\sum(rn+un-rn)=0 \text{ when } \pm u \in X_i$$

$$(r(a+u)-ra)n=r(an+un)-ran=0 \text{ when } \pm u \in X_i.$$

Hence $X_{i+1}N=0$. It follows that $\langle a \rangle N=0$.

**Corollary.** *If every ideal of* $N$ *contains a subset* $F$ *with* $r(F)=0$ *then for each* $a \in N$, $a \neq 0$, *there is a* $y \in N$ *with* $ay \neq 0$.

**Theorem 2.2.** *The following are equivalent:*

(i) $a \in \mathrm{N}$, $a \neq 0 \Rightarrow$ *there is a finite subset* $F$ *of* $N$ *with* $r(aF)=0$.

(ii) *Every non-zero ideal of* $N$ *contains a finite subset* $F$ *with* $r(F)=0$.

**Proof.** (i)$\Rightarrow$(ii) is obvious.

Suppose (ii) and let $a \in N, a \neq 0$. Then $\langle a \rangle$, the ideal of $N$ generated by $a$, is non-zero and hence by the corollary there exists $y \in N$ with $ay \neq 0$ and a finite subset $G \subset \langle ay \rangle$ with $r(G)=0$. Let $X_0=\{ay\} \cup ayN$. $G$ is finite to $G \subseteq X_j$ for some $j$. Hence each element is obtained by applying a finite number of operations of the type

$$\sum(r+u-r), u \in X_j \text{ or } r(a+u)-ra, u \in X_j.$$

Choose one such construction for each element of $G$. This set of constructions will involve a finite number of elements of $X_0$ of the form $ays_k$, $s_k \in N$.

Let $G'=\{ay, ays_k \mid \text{these occur in the chosen construction of an element of } G\}$.

Clearly $G'$ is finite and $r(G') \subseteq r(G)=0$.

Then $H=\{z \mid az \in G'\}$ is an insulator of $a$ in $N$ and so (ii)$\Rightarrow$(i).

We define an ideal $I$ of the near-ring $N$ to be strongly prime if and only if for every $x \notin I$, there is a finite subset $F$ of $\langle x \rangle$ such that for all $a \in N, Fa \subseteq I$ implies $a \in I$.

It is now clear from the definition of a strongly prime near-ring and a strongly prime ideal that $N/I$ is a strongly prime near-ring if and only if $I$ is a strongly prime ideal. Furthermore, $N$ is strongly prime if (0) is a strongly prime ideal.

**Lemma 2.3.** *If $N$ is a strongly prime near-ring, then $N$ is a prime near-ring.*

**Proof.** Let $0 \neq A; B \lhd N$. We show $AB \neq 0$. Since $A \neq 0$, there exists a finite subset $F$ of $A$ such that $r(F)=0$. Hence for each $0 \neq b \in B$ we have $Fb \neq 0$. Therefore $AB \neq 0$.

We give the following example to show that in general a prime near-ring need not be a strongly prime near-ring.

**Example 1.** Consider the dihedral group $N$ with addition and multiplication defined as in Pilz ([9, p. 345, number 11]):

| · | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 |
| 2 | 0 | 2 | 0 | 2 | 0 | 2 | 2 | 0 |
| 3 | 0 | 3 | 0 | 3 | 0 | 3 | 3 | 0 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 5 | 4 | 5 | 4 | 5 | 4 | 5 | 5 | 4 |
| 6 | 4 | 6 | 4 | 6 | 4 | 6 | 6 | 4 |
| 7 | 4 | 7 | 4 | 7 | 4 | 7 | 7 | 4 |

$I=\{0,1,2,3\} \lhd N$. Furthermore $\{0,2\} \lhd I$ but $\{0,2\} \lhd N$. Clearly $N$ is prime. For each $F \subseteq I$, $F$ finite, we have $F \cdot 4=0$. Hence $r(F) \neq 0$ and $N$ not strongly prime.

In [7] Oswald defined a strictly prime near-ring as a near-ring $N$ such that if $A$ and $B$ are two $N$-subgroups of $N$ such that $AB=0$, then $A=0$ or $B=0$. When $N$ has a multiplicative identity or if $N$ is a d.g. near-ring, then $N$ is strictly prime if and only if for $a, b \in N$, $aNb=0$ implies $a=0$ or $b=0$.

**Proposition 2.4.** *If $N$ is distributively generated or has an identity then $N$ is strongly prime if $N$ is strictly prime with D.C.C. on right annihilators.*

**Proof.** Let $0 \neq I \lhd N$ and consider the collection of right annihilators $\{r(F)\}$ where $F$ runs over all finite subsets of $I$. From our assumption, there exists a minimal element $N=r(F_0)$. $M=(0)$, for if $M \neq 0$, then there is $0 \neq m \in M$ such that $F_0 m=0$. $N$ is strictly prime, hence there exists $0 \neq b \in N$ such that $mbm \neq 0$. Hence $bm \neq 0$. Let $S=r(F_i \cup \{b\})$. Now $m \in M$ but $m \notin S$. Consequently $S \nsubseteq M$. The fact that $S$ is smaller than $M$ implies that $M=(0)$. Hence $N$ is strongly prime.

**Theorem 2.5.** *Let $N$ be a zero symmetric near-ring. If $A$ is an ideal of $N$ and $P$ a strongly prime ideal of $N$, then $P \cap A$ is a strongly prime ideal of $A$.*

**Proof.** Let $p \notin P \cap A$. Since $P$ is a strongly prime ideal in $N$, there exists a finite subset $F=\{x_1, \ldots, x_n\} \subseteq N$ such that if $pFc \subseteq P$, then $c \in P$. Let $d \in A$ such that $d \notin P$. It is now possible to find $x_j \in F$ such that $px_j d \notin P$. For this $px_j d \notin P$ we can find a finite set $F_1 \subseteq N$ such that for all $t \in A, t \notin P$ we have $px_j dF_1 t \subseteq P$. Now $F_2=px_j dF_1$ is a finite subset of $\langle p\rangle_A$, ideal generated by $P$ in $A$, such that for all $a\in A, F_2 a\subseteq P\cap A$ implies $a\in P\cap A$. Hence $P\cap A$ is a strongly prime ideal in $A$.

**Corollary 2.6.** *Let $M$ be the class of all zero symmetric strongly prime near-rings. $M$ is hereditary.*

**Proof.** This follows from Theorem 2.5 if we take $P=(0)$.

An ideal $I$ of $N$ is called an essential ideal in $N$ if $I\cap K\neq\phi$ for any $0\neq K\lhd N$. In this case we shall write $I\lhd\cdot N$.

**Lemma 2.6.** *Let $M$ be any class of strongly prime near-rings. $M$ is closed under essential extensions.*

**Proof.** Let $I\lhd\cdot N$ where $I\in M$. Furthermore, let $J$ be any nonzero ideal of $N$. Since $0\neq I\cap J\lhd I$ and $I\in M$, there exists a finite subset $F\subseteq J\cap I$ such that $\{r\in I: Fr=0\}=0$. We have $l(I)\lhd N$ where $l(I)=\{n\in N: nI=0\}$, i.e. left annihilator of $I$ in $N$. Furthermore, $[l(I)\cap I]^2\subseteq l(I)\cdot I=0$. Now, $l(I)\cap I\lhd I$ and since $I$ is a prime near-ring, it follows that $l(I)\cap I=0$. From the fact that $I$ is an essential ideal in $N$, we have $l(I)=0$. Hence for all $0\neq n\in N$ there exist $0\neq p\in I$ such that $np\neq 0$ and consequently, $Fnp\neq 0$. From this we have $Fn\neq 0$. Hence $N$ is strongly prime which proves the lemma.

**Definition** (see [6, p. 57]). Let $X$ be a homomorphically closed class of near-rings. A class of $\sigma$ of near-rings is called $X$-special if the following conditions are satisfied.

$S1$ Each near-ring from $\sigma$ is prime.

$S2$ $N\in\sigma\cap X$ and $A\lhd N$ implies $A\in\sigma$.

$S3$ If $I\lhd A\lhd N\in X$ and $A/I\in\sigma$, then $I\lhd N$ and $N/(I:A)_N\in\sigma$ where $(I:A)_N=\{n\in N: nA\subseteq I\}$.

Let $D$ denote the class of all d.g. near-rings.

**Theorem 2.8.** *The class $M$ of strongly prime near-rings is $D$-special.*

**Proof.** From 2.3 and Corollary 2.6 it follows that conditions $S1$ and $S2$ are satisfied. We only have to show that $S3$ is also satisfied. Let $I\lhd A\lhd N\in D$ and $A/I\in M$. Since $A/I$ is also a prime near-ring, it follows from [6, Theorem 5.3] that $I\lhd N$. We show that $N/(I:A)_N\in M$, i.e. that $(I:A)_N$ is a strongly prime ideal in $N$. Let $x\notin(I:A)$. There exists $a\in A$ such that $xa\notin I$. Since $I$ is a strongly prime ideal in $A$, there exists a finite subset $F$ of $\langle xa\rangle_A\subseteq\langle x\rangle_N$ such that for all $z\in A$, $Fz\subseteq I$ implies $z\in I$. Let $t\in N$ be arbitrary such that $Ft\subseteq(I:A)$. Since $FtA\subseteq I$, we have $tA\subseteq I$. Hence $t\in(I:A)$. Therefore $(I:A)_N$ is a strongly prime ideal in $N$ and consequently $N/(I:A)_N\in M$.

## 3. The strongly radical prime

**Definition.** Let $N$ be any near-ring. As in the case of rings [2], we define the *strongly prime radical* $s(N)$ of $N$ as the intersection of all the strongly prime ideals in $N$.

If $M$ is the class of all strongly prime near-rings, let $UM$ be the upper radical class determined by $M$. The following result is easy to prove.

**Proposition 3.1.** *If $M$ denotes the class of all strongly prime near-rings, then $UM=\{N:N$ is a near-ring such that $N=s(N)\}$.*

We also need the following result which follows from [1] and [10].

**Proposition 3.2.** *Let $M$ be a regular and essential closed class of near-rings satisfying the following condition:*

$$I\lhd K\lhd N \text{ and } I/K\in M \text{ imply } K\lhd N. \tag{F}$$

*Then $UM(N)=\cap\{I_\alpha\lhd N:N/I_\alpha\in M\}$.*

**Proof.** This follows from [10, Proposition 11] and [1, Theorem 1].

**Theorem 3.3.** *If $M$ is the class of all strongly prime near-rings, then for every $N\in D$ and $I\lhd N$ we have $UM(I)=I\cap UM(N)$ and $s(N)=UM(N)$.*

**Proof.** From Theorem 2.8 and Lemma 2.7 it follows that the conditions of [1, Theorem 1] are satisfied. From [1, Theorem 1] we have $UM$ hereditary and from [1, Proposition 3] it follows that $SUM$ is hereditary. Hence for every $N\in D$ and every ideal $I$ of $N$ we have $UM(I)=UM(N)\cap I$.

Since $M\in D$ is a regular and essentially closed class of near-rings satisfying condition $(F)$ we have $UM(N)=s(N)$ for each $N\in D$.

**Remark.** In general, $s(N)\cap I\neq s(I)$ where $I\lhd N$. Take $N$ and $I$ as in Example 1. $I$ is a strongly prime ideal in $N$: For every $x\notin I$ take $F=\{4\}\subseteq\langle x\rangle=N$. Now we have $Fy\not\subseteq I$ for each $y\notin I$. Since (0) is not a strongly prime ideal we have $s(N)=I=\{0,1,2,3\}$. We show $K=\{0,2\}$ is a strongly prime ideal in $I$. For each $x\notin K$, take $F=\{1\}\subseteq\langle x\rangle=I$. Now $Fy\not\subseteq K$ for each $y\notin K$. Hence $K$ is a strongly prime ideal in $I$. Since $I$ is not a prime near-ring it is also not a strongly prime near-ring. Hence $s(I)=K$ and $K=s(I)\not\subseteq s(N)\cap I=I$.

Regarding the position of strongly prime radical among the well-known radicals, we have: If $P(N)$ denotes the prime radical of the near-ring $N$, then we have from Lemma 2.3 that $P(N)\subseteq s(N)$. This inclusion can be strict, for the near-ring in Example 1 is prime. Hence $P(N)=0\not\subseteq I=s(N)$. In [4] the completely prime radical of a near-ring $N, C(N)$, was defined as the intersection of all the completely prime ideals, i.e. all ideals $I\lhd N$ such that $a,b,\notin I$ implies $ab\notin I$. Every completely prime ideal is strongly prime for if $I$ is completely prime then for every $x\notin I$ take $F=\{x\}$. Clearly $Fy\not\subseteq I$ for every $y\notin I$. Hence $s(N)\subseteq C(N)$.

We have the following example to show that this inclusion can be strict.

**Example 2.** Let $N$ be the ring of $2\times 2$ matrices over the two element field $Z_2=[0,1]$. Every nonzero ideal of $N$ contains the one element subset $F=\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$ such that $Fz=0$

implies $z=0, z \in N$. Hence $N$ is strongly prime and therefore $s(N)=(0)$. $N$ is not completely prime for $0 \neq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $0 \neq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence $C(N)=N$ and consequently $s(N) \nsubseteq C(N)$.

In [3] it was proved that the Levitzki radical $L(N)$ of a near-ring $N$ is the intersection of all the $l$-prime ideals of $N$. $I \lhd N$ is a $l$-prime ideal if for every $a \notin I$ there exists a finite number of elements $a_1, a_1, \ldots, a_{n(a)} \in \langle a \rangle$ such that the following condition is satisfied. If $a, b \notin I$ then for every $n > 1$ there exists a product of $N \geqq n$ factors, consisting of $a_i$'s and $b_j$'s which is not in $I$.

**Lemma 3.4.** *If $N$ is a near-ring, then $L(N) \subseteq s(N)$.*

**Proof.** We show that any strongly prime ideal is a $l$-prime ideal. Suppose $I \lhd N$ is a strongly prime ideal in $N$. For every $a, b, \notin I$ there exists finite sets $F_1 \subseteq \langle a \rangle$ and $F_2 \subseteq \langle b \rangle$ such that for any $x \notin I$ we have $F_1 x \subseteq I$ and $F_2 x \subseteq I$. It is now easy to show that for every $n > 1$ there exists a product of $N \geqq n$ factors consisting of elements from $F_1$ and $F_2$ which is not in $I$. Hence $I$ is an $l$-prime ideal and therefore $L(N) \subseteq s(N)$.

For a characterization of $s(N)$ by using certain systems (as is the case for the prime radical and $m$-systems) we use the approach of [11] for rings.

**Definition.** An *sp-system* in $N$ is a pair $(G, P)$ where $P$ is an ideal in $N$ and $G$ is a subset of $N$ such that $G \cap P$ contains no nonzero elements of $N$ and for any $g \in G$, there is a finite subset $F \subseteq \langle g \rangle$ such that $Fz \cap G \neq \phi$ for all $z \notin P$.

Now $I \lhd N$ is a strongly prime ideal if and only if $(N \backslash I, I)$ is an $sp$-system.

**Proposition.** *For any near-ring $N$, we have $s(N)=\{x \in N$: if $x \in G$ where $(G, I)$ is an sp-system for some ideal $I$ in $N$, then $0 \in G\}$.*

**Proof.** The same as for rings (cf. [11, Proposition 2.3]).

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