SIMPLEXES SELF-POLAR FOR A SIMPLEX

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In this paper we prove the existence of simplexes self-polar for a given simplex in projective space of dimensions higher than two. And noting the reciprocity of the relationship between a simplex and any simplex self-polar for it, further, we consider a configuration in projective space associated with a pair of mutually self-polar simplexes. Finally the results obtained are related to the simplexes determined by Hadamard matrices in space of special dimensions.

1. Polarity with regard to a simplex

In projective space of n dimensions, Π_n, let S be the simplex of reference with vertex-vectors $\boldsymbol{e}_0, \boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$ and let $\boldsymbol{a} = a_0\boldsymbol{e}_0 + a_1\boldsymbol{e}_1 + \cdots + a_n\boldsymbol{e}_n = (a_0, a_1, \cdots, a_n)$ be the co-ordinate vector of an arbitrary point disjoint from S. The Π_{n-1} joining the face $x_i = x_j = 0$ of S to the point $\boldsymbol{a}$ meets the edge $\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle$ in the point $a_i\boldsymbol{e}_i + a_j\boldsymbol{e}_j$. The harmonic conjugate of this point with regard to the vertices $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ is $a_i\boldsymbol{e}_j - a_j\boldsymbol{e}_j$. S and $\boldsymbol{a}$ determine $\frac{1}{2}n(n+1)$ such points, all lying in the prime $\alpha \equiv \boldsymbol{a}^{*T}\boldsymbol{x} = 0$, where $a_i^* = a_i^{-1}$ and $\boldsymbol{a}^* = (a_0^*, a_1^* \cdots a_n^*)$.

Definition. The point $\boldsymbol{a}$ and the prime $\alpha \equiv \boldsymbol{a}^{*T}\boldsymbol{x} = 0$ are designated *pole* and *polar* with regard to S.

2. Self-polar simplexes

Definition. A simplex will be said to be *self-polar* for a given simplex, if and only if each prime face of it is the polar of the opposite vertex with regard to the given simplex.

Mutually self-polar tetrahedra and mutually self-polar simplexes have been considered by S. R. Mandan [1] and Ashgar Hameed [2].

Let A be a simplex with vertices $\boldsymbol{a}^0, \boldsymbol{a}^1, \cdots, \boldsymbol{a}^n$. If each of its faces is the polar for S of the corresponding vertex, then

$$(\boldsymbol{a}^i)^{*T}\boldsymbol{a}^j \begin{cases} = 0 & \text{if } i \neq j \\ = n+1 & \text{if } i = j \end{cases}$$

Write also A for the matrix of which the columns are the vectors $\boldsymbol{a}^i$, that is, $A = [\boldsymbol{a}^0, \boldsymbol{a}^1, \cdots, \boldsymbol{a}^n]$. Then from the definition it follows that

THEOREM 1. *The simplex A is self-polar for the simplex of reference S if and only if*

$$A^{*T}A = (n+1)I. \tag{1}$$

For $n = 1$,

$$A = \begin{bmatrix} a^0 & a^0 \\ a^1 & -a^1 \end{bmatrix},$$

and the simplex consists of a pair of points harmonically separating the two points of reference. For $n = 2$ there is no real matrix with the required property, (1) being seen to imply

$$\sum_{i=0}^{2}(a_k^i|a_j^i)^2 = 0, \qquad (k, j = 0, 1, 2;\ k \neq j)$$

which no set of real numbers can satisfy.

When $n \geqq 3$ we can find real matrices; and correspondingly there exist simplexes self-polar for a simplex. To prove this, let B be a simplex with vertices $\boldsymbol{b}^0, \boldsymbol{b}^1, \cdots, \boldsymbol{b}^n$; $\boldsymbol{b}^0 = (k, 1, 1, \cdots, 1)$, $\boldsymbol{b}^1 = (1, k, 1, \cdots, 1), \cdots$, $\boldsymbol{b}^n = (1, 1, \cdots, 1, k)$. The matrix $B = [\boldsymbol{b}^0, \boldsymbol{b}^1, \cdots, \boldsymbol{b}^n]$ will satisfy the condition (1) provided

$$k + \frac{1}{k} + (n-1) = 0 \tag{2}$$

If $n \geqq 3$ there is a real value of k which satisfies this equation. If k_n denote one of the roots of (2) and k_n^* its reciprocal then it is clear that B_n and B_n^*, obtained from B on putting $k = k_n$ of k_n^*, are real matrices satisfying (1), and the simplex with columns of B_n or B_n^* for co-ordinate vectors of the vertices will be self-polar for S. When $n = 3$, $k_n = k_n^*$ and $B_n = B_n^*$. Thus there is at least one simplex self-polar for a given simplex S, when $n > 2$.

It may be now shown that if there is one simplex self-polar for a given simplex S, then there are infinitely many simplexes similarly self-polar for S. For, if B be a simplex with vertices $\boldsymbol{b}^0, \boldsymbol{b}^1, \cdots, \boldsymbol{b}^n$, $\boldsymbol{b}^i = (b_0^i, b_1^i, \cdots, b_n^i)$, self-polar for S, and if M be the matrix obtained from the matrix $B = [\boldsymbol{b}^0, \boldsymbol{b}^1, \cdots, \boldsymbol{b}^n]$ by multiplying the rows of B by non-zero real numbers $c_0, c_1, \cdots, c_n$ respectively so that $M = [\boldsymbol{m}^0, \boldsymbol{m}^1, \cdots, \boldsymbol{m}^n]$, where $\boldsymbol{m}^i = (c_0 b_0^i, c_1 b_1^i, \cdots, c_n b_n)$ then

$$M^{*T}M = B^{*T}B = (n+1)I$$

Hence the simplex M, that is the simplex with the columns of M for co-ordinate vectors of vertices, is self-polar for S. The numbers $c_0, c_1, \cdots, c_n$ being arbitrary, given any point not on any one of the faces of S, it is always possible to choose them so that $\boldsymbol{m}^i$ is this point. That means, given an arbitrary point in Π_n (not on any one of the faces of S) there is at least one simplex self-polar for S with this point for a vertex. Thus

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THEOREM 2. *In real projective space of 2 dimensions there is no simplex (triangle) self-polar for another simplex (triangle). But there exist infinitely many simplexes self-polar for a given simplex in space of every dimensions higher than 2.*

3. Related self-polar simplexes

Let A be a simplex self-polar for S. Also A be the matrix with the coordinate-vectors of the vertices of the simplex for columns. Then from condition (1), since $A^{*T} = A^{T*}$, we deduce

$$(3.1) \qquad (n+1)I = A^{*T}A = (A^*)^T(A^*)^* = (A^*)^{*T}A^*$$

$$(3.2) \qquad (n+1)I = AA^{*T} = (A^{T*})^T A^T = (A^T)^{*T}A^T.$$

So also

$$(3.3) \qquad (n+1)A^{-1} = A^{T*} \Rightarrow (A^{-1})^* = (n+1)A^T \Rightarrow (A^{-1})^{*T} = (n+1)A \Rightarrow (A^{-1})^{*T}A^{-1} = (n+1)I.$$

Hence

THEOREM 3. *If the columns of A are the coordinate-vectors of the vertices of a simplex self-polar for S, so also are the columns of A^*, A^T and A^{-1}.*

The coordinate-vectors of the vertices of S in relation to A as simplex of reference are the columns of A^{-1} so that the relation $(A^{-1})^{*T}A^{-1} = (n+1)I$ means that S is self-polar for A. So

THEOREM 4. *If A is self-polar for S, then S is self-polar for A.*

4. Simplexes in perspective

Let S be the simples of reference and $\boldsymbol{u} = (u_0, u_1, \cdots, u_n)$ be the coordinate-vector of any point disjoint from S. Any simplex T_k in perspective with S from the point $\boldsymbol{u}$ has vertices given by $\boldsymbol{t}_k^0 = (k_0u_0, u_1, \cdots, u_n)$, $\boldsymbol{t}_n^1 = (u_0, k_1u_1, u_2, \cdots, u_n)$, $\cdots$, $\boldsymbol{t}_k^n = (u_0, u_1, \cdots, k_nu_n)$. T_k is self-polar for S if and only if $T_k = [\boldsymbol{t}_k^0, \boldsymbol{t}_k^1, \cdots, \boldsymbol{t}_k^n]$ satisfies the condition (1), that is, $T_k^{*T}T_k = (n+1)I$. This requires that $k_0 = k_1 = \cdots = k_n = k$, where $k+1/k+(n-1) = 0$, that is, $\boldsymbol{k} = k(1, 1, \cdots, 1)$, $k = k_n$ or k_n^*. If $n > 3$, k_n and k_n^* are distinct, and if $n = 3$, they are equal. Thus

THEOREM 5. *Given any simplex S and any point $\boldsymbol{u}$ disjoint from S, there are two simplexes perspective with S from $\boldsymbol{u}$ and self-polar for S, if $n > 3$.*

The system S, $\boldsymbol{u}$, T_k, $\boldsymbol{k} = k(1, 1, \cdots, 1)$, determines a central collineation (homology) with centre $\boldsymbol{u}$ and axial prime $\boldsymbol{u}^{*T}\boldsymbol{x} = 0$ which maps S to T. Let $\boldsymbol{u}$, $\boldsymbol{a}^1, \boldsymbol{a}^2, \cdots, \boldsymbol{a}^n$ be the vertices of a simplex self-polar for S so that, in particular, each point $\boldsymbol{a}^i$ lies in the axial plane $\boldsymbol{u}^{*T}\boldsymbol{x} = 0$ and each prime $(\boldsymbol{a}^i)^{*T}\boldsymbol{x} = 0$ passes through the centre $\boldsymbol{u}$. Under this homology S maps to the simplex T_k and polar

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pairs for S to the polar pairs for T_k. All points $\boldsymbol{a}^i$ are invariant since they lie in the axial prime and all primes $(\boldsymbol{a}^i)^{*T}\boldsymbol{x} = 0$ are invariant because they pass through the centre. Thus

THEOREM 6. *If two mutually self-polar simplexes S and T are in perspective from a point* $\boldsymbol{u}$, *then, if any simplex of which* $\boldsymbol{u}$ *is a vertex is self-polar for S, it is self-polar for T also.*

5. A configuration

Let S be the simplex of reference and A be any simplex self-polar for S with vertices $\boldsymbol{a}^0, \boldsymbol{a}^1, \cdots, \boldsymbol{a}^n$, $\boldsymbol{a}^i = (a_0^i, a_1^i, \cdots, a_n^i)$. Let $A^j(k)$ denote the simplex with vertices $\boldsymbol{c}_0^j(k) = (ka_0^j, a_1^j, \cdots, a_n^j)$, $\boldsymbol{c}_1^j(k) = (a_0^j, ka_1^j, a_2^j, \cdots, a_n^j), \cdots, \boldsymbol{c}_n^j = (a_0^j, a_1^j, \cdots, ka_n^j)$. $A^j(k)$ is a simplex perspective with S from the point $\boldsymbol{a}^j$ which is a vertex of A. If $k = k_n$ or k_n^*, $A^j(k)$ will be also self-polar for S. Then since A is self-polar for S and has the point $\boldsymbol{a}^j$ for a vertex, A will be self-polar for $A^j(k)$ also. Thus S, A, $A^j(k)$ form a set of three mutually self-polar simplexes. And $A^j(k)$, $(j = 0, 1, \cdots, n)$ form a set of $(n+1)$ simplexes each self-polar for both A and S.

Similarly let $S_i(k)$ denote the simplex with $\boldsymbol{c}_i^0(k)$, $\boldsymbol{c}_i^1(k) \cdots \boldsymbol{c}_i^n(k)$ for vertices. Then $S_i(k)$ will be a simplex perspective with A from the point $\boldsymbol{e}_i^i$ which is a vertex of the simplex S. S and $S_i(k)$ are mutually self-polar. Writing down the co-ordinate vectors of the vertices of $S_i(k)$ in relation to A as simplex of reference and using the condition (1), it is seen that $S_i(k)$ is self-polar for A also, if, and only if, $k = k_n$ or k_n^*. Thus $S_i(k)$, $(i = 0, 1, \cdots, n)$ are simplexes self-polar for both S and A, provided $k = k_n$ or k_n^*.

The $(n+1)^2$ points which form the vertices of the $(n+1)$ simplexes $A^j(k)$ are the same as the $(n+1)^2$ points which form the vertices of the $(n+1)$ simplexes $S_i(k)$. The simplex $A^j(k)$ has one vertex in common with each of the simplexes $S_0(k), S_1(k), \cdots, S_n(k)$. And the simplex $S_i(k)$ has one vertex in common with each of the simplexes $A^j(k)$.

Since when $k = k_n$ or k_n^* the simplexes $A^j(k)$ and $S_i(k)$ are self-polar for both S and A the polar of each point $\boldsymbol{c}_i^j(k)$ is the same for both S and A. As each point $\boldsymbol{c}_i^j(k)$ is the vertex common to the two simplexes $A^j(k)$ and $S_i(k)$, so also the polar of $\boldsymbol{c}_i^j(k)$ for S and A will be a prime face common to the two simplexes.

Let $p_i^j(k)$ denote the polar of $\boldsymbol{c}_i^j(k)$ for S and A. The $(n+1)^2$ points $\boldsymbol{c}_i^j(k)$ and the $(n+1)^2$ primes $p_i^j(k)$ now form a configuration with each point $\boldsymbol{c}_i^j(k)$ lying in $2n$ primes and each prime containing $2n$ points. The point $\boldsymbol{c}_i^j(k)$ lies in every one of the prime faces of the two simplexes $A^j(k)$ and $S_i(k)$ except the common face $p_i^j(k)$; and the prime $p_i^j(k)$ contains every vertex of the simplexes except the common vertex $\boldsymbol{c}_i^j(k)$.

For each value of $k = k_n, k_n^*$, there is one such configuration. And the whole configuration for each value, made up of $(n+1)^2$ points and $(n+1)^2$ primes is self-polar for both S and A.

6. In 2^m-1 dimensions

Consider the space Π_n where n is a Mersenne number, $n = 2^m-1$. The 2^n points $(1, \pm 1, \cdots, \pm 1)$ in Π_n form a set A called a *set of associated points*. The simplex of reference is called the *diagonal simplex of the set*. We prove that the points of the set fall into the vertices of a number of simplexes each of which is self-polar for the diagonal simplex.

Let

$$U_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad U_2 = \begin{bmatrix} U_1 & U_1 \\ U_1 & -U_1 \end{bmatrix}, \cdots$$

$$U_m = \begin{bmatrix} U_{m-1} & U_{m-1} \\ U_{m-1} & -U_{m-1} \end{bmatrix}.$$

Then U_m is a $2^m \times 2^m$ matrix with the coordinates of some point of the set A in each column.

It is easily verified that the simplex U_m, that is, the simplex with the columns of U_m as coordinate-vectors of vertices, is self-polar for the diagonal simplex S. (It may be observed that in each of the matrices $U_1, U_2, \cdots, U_m$, the elements of any column multiplied by the corresponding elements of another column give a column different from the first column of the matrix and that the sum of the elements of every column other than the first column is zero; this, if true in U_r, being easily seen to be true in U_{r+1}.) If the rows of U_m are multiplied respectively by $a_0, a_1, \cdots, a_n$ where a_i are all ± 1 or the coordinates of any one of the points of set A, we will have a simplex which is self-polar for S and has all its vertices in the set A.

The set G of all $(n+1)$-tuples $(a_0, a_1, \cdots, a_n)$ where $a_0 = 1$, $a_i = \pm 1$, $(i = 1, 2, \cdots, n)$, with the law of composition defined on it by $(a_0, a_1, \cdots, a_n)$. $(b_0, b_1, \cdots, b_n) = (a_0 b_0, a_1 b_1, \cdots, a_n b_n)$ is a group with $(1, 1, \cdots, 1)$ for unit element. And the subset H of the $(n+1)$ elements $(u_0, u_1, \cdots, u_n)$ where $u_0, u_1, \cdots$ u_n are the elements of a column of U_m is a subgroup of G, the product of any two elements of the set H being always an element of the set. Let $H, H_1, \cdots, H_k$ be the distinct cosets of H in G.

Now there is a one-to-one correspondence between G and the set A of points. To H corresponds the set of points which form the vertices of the simplex U_m; and to each H_i corresponds a simplex U_m^i obtained by multiplying the rows of U_m respectively by the coordinates of one of the points of A. As U_m is self-polar for S, so each simplex U_m^i is self polar for S. And as G is partitioned into the classes $H, H_1, \cdots, H_k$ so that each element of G belongs to one and only one of these classes so also the points of the set A are partitioned into the vertices of the simplexes $U_m, U_m^1, U_m^2, \cdots, U_m^k$. Thus the points of the set A fall into a number of simplexes each of which is self-polar for S. The number of these simplexes is evidently $2^n/(n+1) = 2^{n-m}$.

If the elements of the $(j+1)$-th row of U_m^i are all multiplied by (-1), we will have a simplex self-polar for S and one among the simplexes into which the points of the set A fall. This simplex will be prespective with U_m^i from the $(j+1)$-th vertex of S, i.e. the point e_j. As j ranges over $0, 1, 2, \cdots, n$, each of the $(n+1)$ simplexes we get is prespective with U_m^i from one of the vertices of S. Thus the simplexes $U_m, U_m^1, \cdots, U_m^k$ are such that each one is perspective with $(n+1)$ others, the centres of perspective being the different vertices of S.

In particular, when $n = 3$ the set of associated points fall into two tetrahedra

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

which are both self-polar for the tetrahedron of reference. They are also self-polar for each other. The desmic system thus consists a set of three tetrahedra which are mutually self-polar.

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