Bull. Aust. Math. Soc. **91** (2015), 129–133
doi:10.1017/S0004972714000744

ON THE MINIMAL PROPERTY OF DE LA VALLÉE POUSSIN'S OPERATOR

BEATA DEREGOWSKA✉ **and BARBARA LEWANDOWSKA**

(Received 15 April 2014; accepted 20 August 2014; first published online 8 October 2014)

Abstract

Let $X = C_0(2\pi)$ or $X = L_1[0, 2\pi]$. Denote by Π_n the space of all trigonometric polynomials of degree less than or equal to n. The aim of this paper is to prove the minimality of the norm of de la Vallée Poussin's operator in the set of generalised projections $\mathcal{P}_{\Pi_n}(X, \Pi_{2n-1}) = \{P \in \mathcal{L}(X, \Pi_{2n-1}) : P|_{\Pi_n} \equiv \text{id}\}$.

2010 *Mathematics subject classification*: primary 42A10; secondary 47A58.

Keywords and phrases: de la Vallée Poussin's operator, generalised projections, Dirichlet kernel.

1. Introduction

Let $C_0(2\pi)$ denote the space of all continuous, 2π-periodic functions equipped with the supremum norm. Let Π_n denote the space of all trigonometric polynomials of degree less than or equal to n. The Fourier projection $F_n : C_0(2\pi) \to \Pi_n$ is defined by

$$F_n(f)(t) = \frac{1}{2\pi} \int_0^{2\pi} f(s) D_n(t-s)\, ds,$$

where D_n is the Dirichlet kernel

$$D_n(t) = \sum_{k=-n}^{n} e^{ikt}.$$

It is well known by the classical result of Lozinski [8] that the Fourier operator F_n has the minimal norm among all projections from $C_0(2\pi)$ onto Π_n. If we replace $C_0(2\pi)$ by $L_1[0, 2\pi]$, the Lozinski theorem stays true. In 1969, Cheney *et al.* [2] proved that the Fourier projection is the unique minimal projection with respect to the operator norm in $\mathcal{L}(C_0(2\pi))$. In the same year, Lambert [5] proved the analogous result for $L_1[0, 2\pi]$. For other results concerning the minimality or the unique minimality of the Fourier-type operators see, for example, [6, 7, 11, 12].

In 1918, de la Vallée Poussin [15] introduced the following operator.

DEFINITION 1.1. De la Vallée Poussin's operator $H_n : X \to \Pi_{2n-1}$ is given by

$$H_n(f)(t) = \frac{1}{n}\sum_{k=n}^{2n-1} F_k(f)(t) = \frac{1}{2\pi}\int_0^{2\pi} f(s)V_n(t-s)\,ds,$$

where

$$V_n(t) = \frac{1}{n}\sum_{k=n}^{2n-1} D_n(t) = \frac{\sin^2(nt) - \sin^2(nt/2)}{n\sin^2(t/2)}.$$

Again, de la Vallée Poussin's operator has been extensively studied by many authors (see, for example, [1, 3, 4, 9, 10, 14]).

In our paper, we will show that de la Vallée Poussin's operator has the minimal norm in the set of generalised projections

$$\mathcal{P}_{\Pi_n}(X, \Pi_{2n-1}) = \{P \in \mathcal{L}(X, \Pi_{2n-1}) : P|_{\Pi_n} \equiv \mathrm{id}\},$$

where $X = C_0(2\pi)$ or $X = L_1[0, 2\pi]$. Our proof is surprisingly simple and it is based on the behaviour of the zeros of the kernel V_n and the classical theorem characterising the best approximation elements.

THEOREM 1.2 (See for example [13]). *Let X be a Banach space and $V \subset X$ be a linear subspace and let $x_0 \in X \setminus \mathrm{cl}(V)$. Then v_0 is a best approximation to x_0 in V if and only if there exists $f \in S(X^*)$ such that*

$$f(x_0 - v_0) = \|x_0 - v_0\| \quad \text{and} \quad f|_V \equiv 0.$$

2. Results

We will start with a lemma, which can be treated as a certain generalisation of the Lozinski theorem.

LEMMA 2.1. *Let X be $C_0(2\pi)$ or $L_1[0, 2\pi]$. For every $P \in \mathcal{P}_{\Pi_n}(X, \Pi_{2n-1})$, there exists $\widetilde{P} \in \mathcal{P}_{\Pi_n}(X, \Pi_{2n-1})$ such that*

$$\|\widetilde{P}\| \le \|P\| \quad \text{and} \quad \widetilde{P}(f)(t) = \frac{1}{2\pi}\int_0^{2\pi} f(s)\sum_{k=-2n+1}^{2n-1} a_k e^{ik(t-s)}\,ds$$

for some $a_k \in \mathbb{C}$.

PROOF. For any $s \in \mathbb{R}$, let us define an isometry $T_s : X \to X$ by

$$T_s(f)(t) = f((t+s)_{\mathrm{mod}\,2\pi}).$$

Now define an operator

$$Q := \frac{1}{2\pi}\int_0^{2\pi} T_s \circ P \circ T_{-s}\,ds.$$

It is obvious that $\|Q\| \le \|P\|$. Since $P \in \mathcal{P}_{\Pi_n}(X, \Pi_{2n-1})$, for $|k| \in \{0, \dots, n\}$ we have $P(e^{ikt}) = e^{ikt}$ and for $|k| \in \{n+1, \dots\}$ we have $P(e^{ikt}) = \sum_{l=-2n+1}^{2n-1} a_l^k e^{ilt}$ for some $a_l^k \in \mathbb{C}$.

https://doi.org/10.1017/S0004972714000744 Published online by Cambridge University Press

Define

$$\widetilde{P}(f)(t) = \frac{1}{2\pi}\int_0^{2\pi} f(s)\sum_{k=-2n+1}^{2n-1} a_k e^{ik(t-s)}\,ds, \tag{2.1}$$

where $a_k = 1$ for $|k| \in \{0,\dots,n\}$ and $a_k = a_k^k$ for $|k| \in \{n+1,\dots,2n-1\}$. We will show that $\widetilde{P}$ is a generalised projection and $\widetilde{P} \equiv Q$. Since the closure of the space generated by $\{e^{ikt} : k \in \mathbb{Z}\}$ is equal to X, it is enough to show that:

(1) $\widetilde{P}(e^{ikt}) = Q(e^{ikt}) = e^{ikt}$ for $|k| \in \{0,\dots,n\}$;
(2) $\widetilde{P}(e^{ikt}) = Q(e^{ikt}) \in \Pi_{2n-1}$ for $|k| \in \{n+1,\dots\}$.

Take $|k| \in \{0,\dots,n\}$. Then

$$\widetilde{P}(e^{ikt}) = \frac{1}{2\pi}\sum_{l=-2n+1}^{2n-1} a_l e^{ilt}\int_0^{2\pi} e^{i(k-l)s}\,ds = a_k e^{ikt} = e^{ikt}$$

and

$$Q(e^{ikt}) = \frac{1}{2\pi}\int_0^{2\pi} T_s(P(e^{ik(t-s)}))\,ds = \frac{1}{2\pi}\int_0^{2\pi} e^{-iks}T_s(e^{ikt})\,ds = \frac{1}{2\pi}\int_0^{2\pi} e^{ikt}\,ds = e^{ikt}.$$

Take $|k| \in \{n+1,\dots,2n-1\}$. Then

$$\widetilde{P}(e^{ikt}) = \frac{1}{2\pi}\sum_{l=-2n+1}^{2n-1} a_l e^{ilt}\int_0^{2\pi} e^{i(k-l)s}\,ds = a_k e^{ikt}$$

and

$$\begin{aligned} Q(e^{ikt}) &= \frac{1}{2\pi}\int_0^{2\pi} T_s(P(e^{ik(t-s)}))\,ds = \frac{1}{2\pi}\int_0^{2\pi} e^{-iks}T_s\Big(\sum_{l=-2n+1}^{2n-1} a_l^k e^{ilt}\Big)ds \\ &= \frac{1}{2\pi}\sum_{l=-2n+1}^{2n-1} a_l^k e^{ilt}\int_0^{2\pi} e^{i(k-l)s}\,ds = a_k^k e^{ikt} = a_k e^{ikt}. \end{aligned}$$

For $|k| \in \{2n,\dots\}$,

$$\widetilde{P}(e^{ikt}) = \frac{1}{2\pi}\sum_{l=-2n+1}^{2n-1} a_l e^{ilt}\int_0^{2\pi} e^{i(k-l)s}\,ds = 0$$

and

$$\begin{aligned} Q(e^{ikt}) &= \frac{1}{2\pi}\int_0^{2\pi} T_s(P(e^{ik(t-s)}))\,ds = \frac{1}{2\pi}\int_0^{2\pi} e^{-iks}T_s\Big(\sum_{l=-2n+1}^{2n-1} a_l^k e^{ilt}\Big)ds \\ &= \frac{1}{2\pi}\sum_{l=-2n+1}^{2n-1} a_l^k e^{ilt}\int_0^{2\pi} e^{i(k-l)s}\,ds = 0. \end{aligned}$$

This yields the desired conclusion. □

Theorem 2.2. *Let X be $C_0(2\pi)$ or $L_1[0,2\pi]$. Then de la Vallée Poussin's operator H_n is a minimal generalised projection in $\mathcal{P}_{\Pi_n}(X, \Pi_{2n-1})$.*

https://doi.org/10.1017/S0004972714000744 Published online by Cambridge University Press

Proof. By Lemma 2.1, it is enough to show that $\|H_n\| = \inf\{\|P\| : P \text{ satisfies } (2.1)\}$. Let $Y = \mathrm{span}\{e^{ikt} : k \in \mathbb{Z},\ n < |k| < 2n\} \subset L_1[0, 2\pi]$. Notice that if P is of the form (2.1), then there exists $y \in Y$ such that

$$P(f)(t) = \frac{1}{2\pi} \int_0^{2\pi} f(s)(V_n + y)(t - s)\, ds$$

and

$$\|P\| = \frac{1}{2\pi} \int_0^{2\pi} |(V_n + y)(s)|\, ds = \frac{1}{2\pi}\|V_n + y\|_1.$$

Now observe that the minimality of H_n is equivalent to the fact that 0 is a best approximation to V_n in Y. Notice that $(L_1[0, 2\pi])^* = L_\infty[0, 2\pi]$. We will show that, for any $y \in Y$, $\int_0^{2\pi} \mathrm{sgn}(V_n(t))y(t)\, dt = 0$, which, according to Theorem 1.2, gives us the desired conclusion. First observe that

$$\begin{aligned} V_n(t) &= \frac{\sin^2(nt) - \sin^2(nt/2)}{n \sin^2(t/2)} = \frac{(\sin(nt) - \sin(nt/2))(\sin(nt) + \sin(nt/2))}{n \sin^2(t/2)} \\ &= \frac{4 \sin(nt/4) \cos(nt/4) \sin(3nt/4) \cos(3nt/4)}{n \sin^2(t/2)} = \frac{\sin(nt/2) \sin(3nt/2)}{n \sin^2(t/2)}. \end{aligned}$$

Let $f(t) := \sin(nt/2) \sin(3nt/2)$. Then, for $t \in (0, 2\pi)$, $\mathrm{sgn}(V_n(t)) = \mathrm{sgn}(f(t))$. By the above, it is easy to see that $f(t) = 0$ if and only if $t = 2\pi b/3n$ for $b \in \{1, \ldots, 3n - 1\}$. Notice that it is only at double zeros of f (that is, $t = 2\pi a/n$ for $a \in \{1, \ldots, n - 1\}$) that the function does not change the sign. Denote $I_a := (2\pi(3a - 1)/3n, 2\pi a/n) \cup (2\pi a/n, 2\pi(3a + 1)/3n)$ for $a \in \{1, \ldots, n - 1\}$ and $J_a := (2\pi(3a + 1)/3n, 2\pi(3a + 2)/3n)$ for $a \in \{0, \ldots, n - 1\}$. Then, for any $a \in \{1, \ldots, n - 1\}$, $\mathrm{sgn}(f)|_{I_a} = \mathrm{sgn}(f)|_{(0,2\pi/3n)} = 1$ and, for any $a \in \{0, \ldots, n - 1\}$, $\mathrm{sgn}(f)|_{J_a} = \mathrm{sgn}(f)|_{(2\pi/3n,4\pi/3n)} = -1$. Now let $n < |k| < 2n$. Since $e^{(2\pi ik/n)} \neq 1$,

$$\begin{aligned} \int_0^{2\pi} \mathrm{sgn}(V_n(t))e^{ikt}\, dt &= \int_0^{2\pi/3n} e^{ikt}\, dt + \sum_{a=1}^{n-1} \int_{I_a} e^{ikt}\, dt - \sum_{a=0}^{n-1} \int_{J_a} e^{ikt}\, dt + \int_{2\pi-(2\pi/3n)}^{2\pi} e^{ikt}\, dt \\ &= (ik)^{-1}(e^{2\pi ik/3n} - 1) + (ik)^{-1} \sum_{a=1}^{n-1} (e^{2\pi ik(3a+1)/3n} - e^{2\pi ik(3a-1)/3n}) \\ &\quad + (ik)^{-1} \sum_{a=0}^{n-1} (e^{2\pi ik(3a+1)/3n} - e^{2\pi ik(3a+2)/3n}) \\ &\quad + (ik)^{-1}(1 - e^{2\pi ik(3n-1)/3n}) \\ &= 2(ik)^{-1} e^{2\pi ik/3n} \sum_{a=0}^{n-1} (e^{2\pi ik/n})^a - 2(ik)^{-1} e^{4\pi ik/3n} \sum_{a=0}^{n-1} (e^{2\pi ik/n})^a \\ &= 2(ik)^{-1}(e^{2\pi ik/3n} - e^{4\pi ik/3n}) \frac{1 - e^{2\pi ik}}{1 - e^{2\pi ik/n}} = 0. \end{aligned}$$

Hence, for all $y \in Y$, $\int_0^{2\pi} \mathrm{sgn}(V_n(t))y(t)\, dt = 0$, as required. □

It is worth mentioning that in the paper [9] Mehta showed that for any $n \in \mathbb{N}$ the norm of de la Vallée Poussin's operator H_n is equal to $1/3 + 2\sqrt{3}/\pi$.

https://doi.org/10.1017/S0004972714000744 Published online by Cambridge University Press

Acknowledgement

The authors would like to thank Professor Grzegorz Lewicki for many fruitful discussions concerning the subject of this paper.

References

[1] A. S. Cavaretta and T. Hanchin, 'A generalization of a result of G. Polya and its application to a continuous extension of the de la Vallée Poussin means', *J. Approx. Theory* **158**(2) (2009), 184–193.

[2] E. W. Cheney, C. R. Hobby, P. D. Morris, F. Schurer and D. E. Wulbert, 'On the minimal property of the Fourier projection', *Trans. Amer. Math. Soc.* **143** (1969), 249–258.

[3] E. W. Cheney, P. D. Morris and K. H. Price, 'On an approximation operator of de la Vallée Poussin', *J. Approx. Theory* **13** (1975), 375–391; collection of articles dedicated to G. G. Lorentz on the occasion of his sixty-fifth birthday, IV.

[4] Á. Horváth, 'Near-best approximation by a de la Vallée Poussin-type interpolatory operator', *Studia Sci. Math. Hungar.* **49**(1) (2012), 1–18.

[5] P. V. Lambert, 'On the minimum norm property of the Fourier projection in L_1-spaces', *Bull. Soc. Math. Belg.* **21** (1969), 370–391.

[6] G. Lewicki, G. Marino and P. Pietramala, 'Fourier-type minimal extensions in real L_1-spaces', *Rocky Mountain J. Math.* **30**(3) (2000), 1025–1037.

[7] G. Lewicki and A. Micek, 'Uniqueness of minimal Fourier-type extensions in L_1-spaces', *Monatsh. Math.* **170**(2) (2013), 161–178.

[8] S. M. Lozinski, 'On a class of linear operators', *Dokl. Akad. Nauk SSSR* **61**(2) (1948), 193–196.

[9] H. Mehta, 'The L_1 norm of the generalized de la Vallée Poussin kernel', (2013), 1–12; arXiv:1311.1407 [math.CA].

[10] F. Schurer and F. W. Steutel, 'Degree of approximation by the operators of de la Vallée Poussin', *Monatsh. Math.* **87**(1) (1979), 53–64.

[11] B. Shekhtman and L. Skrzypek, 'Norming points and unique minimality of orthogonal projections', *Abstr. Appl. Anal.* **2006** (2006), Article ID 42305, 17 pages.

[12] B. Shekhtman and L. Skrzypek, 'On the uniqueness of the Fourier projection in L_p spaces', *J. Concr. Appl. Math.* **8**(3) (2010), 439–447.

[13] I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces* (Springer, Berlin, 1970).

[14] S. A. Telyakovskii, 'Approximation to functions differentiable in Weyl's sense by de la Vallée Poussin sums', *Dokl. Akad. Nauk SSSR* **131** 259–262; (in Russian), transl. *Sov. Math. Dokl.* **1** (1960), 204–243.

[15] C. de la Vallée Poussin, 'Sur la meilleure approximation des fonctions d'une variable réelle par des expressions d'ordre donné', *C. R. Acad. Sci. Paris* **166** (1918), 799–802.

BEATA DEREGOWSKA,
Faculty of Mathematics and Computer Science,
Jagiellonian University,
Lojasiewicza 6, 30-048 Krakow, Poland
e-mail: beata.deregowska@im.uj.edu.pl

BARBARA LEWANDOWSKA,
Faculty of Mathematics and Computer Science,
Jagiellonian University,
Lojasiewicza 6, 30-048 Krakow, Poland
e-mail: barbara.lewandowska@im.uj.edu.pl