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THE NORMAL SUBGROUP STRUCTURE OF THE INFINITE GENERAL LINEAR GROUP

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1. Introduction and notation

The classification of the normal subgroups of the infinite general linear group $GL(\Omega, R)$ has received much attention and has been studied in, for example, (**6**), (**4**) and (**2**). The main theorem of (**6**) gives a complete classification of the normal subgroups of $GL(\Omega, R)$ when R is a division ring, while the results of (**2**) require that R satisfies certain finiteness conditions. The object of this paper is to produce a classification, along the lines of that given by Wilson in (**7**) or by Bass in (**3**) in the finite dimensional case, that does not require any finiteness assumptions. However, when R is Noetherian, the classification given here reduces to that given in (**2**).

R shall always denote a ring with identity and M shall denote the free R-module $R^{(\Omega)}$, for some infinite set Ω. Let $\{e_\lambda : \lambda \in \Omega\}$ be the canonical basis of M. For any two-sided ideal $\mathfrak{p}$ of R, we shall denote by $GL(\Omega, \mathfrak{p})$ the kernel of the natural group homomorphism induced by the projection $R \to R/\mathfrak{p}$, and by $GL'(\Omega, \mathfrak{p})$ the inverse image of the centre of $GL(\Omega, R/\mathfrak{p})$. Suppose $\Lambda \subset \Omega$, $\mu \in \Omega - \Lambda$ and $f: \Lambda \to R$. (We shall adopt the convention that f extends to a map $f: \Omega \to R$ by defining $f(\omega) = 0$ whenever $\omega \in \Omega - \Lambda$ and we shall use $\subset$ to denote proper subset inclusion.) Define $t(\Lambda, f, \mu)$ to be the R-automorphism of M

$$t(\Lambda, f, \mu)e_\rho = e_\rho + e_\mu f(\rho), \quad \text{for all} \quad \rho \in \Omega.$$

Clearly each $t(\Lambda, f, \mu) \in GL(\Omega, R)$ and since elements of $GL(\Omega, R)$ can be regarded as invertible $\Omega \times \Omega$ column finite matrices we shall call the $t(\Lambda, f, \mu)$ *elementary matrices*. Define $E(\Omega, R)$ to be the subgroup of $GL(\Omega, R)$ generated by $\{t(\Lambda, f, \mu): \Lambda \subset \Omega, \mu \in \Omega - \Lambda, f: \Lambda \to R\}$. For any right ideal $\mathfrak{p}$ of R we define $E(\Omega, \mathfrak{p})$ to be the normal closure of $\{t(\Lambda, f, \mu): \Lambda \subset \Omega, \mu \in \Omega - \Lambda, f: \Lambda \to \mathfrak{p}\}$ in $E(\Omega, R)$. Arguments similar to those of (**4**) show that $E(\Omega, R)$ and $E(\Omega, \mathfrak{p})$ are normal subgroups of $GL(\Omega, R)$. For any $\Lambda \subset \Omega$ we identify with each $a \in R$ the map $a: \Lambda \to R$ with $a(\lambda) = a$ and if $\Lambda = \{\lambda\}$ we shall abbreviate $t(\Lambda, a, \mu)$ to $t(\lambda, a, \mu)$. We shall define $EF(\Omega, R)$ to be the subgroup of $GL(\Omega, R)$ generated by $\{t(\lambda, a, \mu): \lambda, \mu \in \Omega, \lambda \neq \mu, a \in R\}$. For any right ideal $\mathfrak{p}$ of R, $EF(\Omega, \mathfrak{p})$ is defined to be the normal closure of $\{t(\lambda, a, \mu): \lambda, \mu \in \Omega, \lambda \neq \mu, a \in \mathfrak{p}\}$ in $EF(\Omega, R)$. Denoting the set of natural numbers $\{1, 2, 3, \ldots\}$ by $\boldsymbol{N}$ we see that $EF(\boldsymbol{N}, R)$ is just the subgroup $E(R)$ of the stable linear group of Bass (**3**) while, for any infinite set Ω, $EF(\Omega, R)$ was studied by Robertson in (**6**) when R is a simple ring.

Whenever $\mathfrak{p}$ is a two-sided ideal of R it is possible to write $\mathfrak{p}$ as a sum of finitely generated right ideals; for example, $\mathfrak{p} = \sum_{x \in \mathfrak{p}} xR$. Of course, this decomposition may not be unique, however it is easy to see that whenever $\mathfrak{p}$ is a two-sided ideal of R and $\{\mathfrak{p}_\alpha : \alpha \in A\}$ and $\{\mathfrak{q}_\beta : \beta \in B\}$ are two families of finitely generated right ideals whose sums are $\mathfrak{p}$ then $\prod_{\alpha \in A} E(\Omega, \mathfrak{p}_\alpha) = \prod_{\beta \in B} E(\Omega, \mathfrak{q}_\beta)$. This observation allows us to define a further normal subgroup as follows. Let $\mathfrak{p}$ be a two-sided ideal of R and let $\{\mathfrak{p}_\alpha : \alpha \in A\}$ be a family of finitely generated right ideals of R such that $\sum_{\alpha \in A} \mathfrak{p}_\alpha = \mathfrak{p}$. Define $E[\Omega, \mathfrak{p}]$ to be the normal subgroup $\prod_{\alpha \in A} E(\Omega, \mathfrak{p}_\alpha)$. Our remarks above show that the group $E[\Omega, \mathfrak{p}]$ is independent of the choice of $\mathfrak{p}_\alpha$ and so $E[\Omega, \mathfrak{p}]$ is defined uniquely.

We shall find that each normal subgroup H of $GL(\Omega, R)$ determines uniquely a two-sided ideal $\mathfrak{p}$ of R such that $E[\Omega, \mathfrak{p}] \leqq H \leqq GL'(\Omega, \mathfrak{p})$. The analogous result of (**2**), which required that R had the maximal condition on right ideals, involved the groups $E(\Omega, \mathfrak{p})$ and $GL'(\Omega, \mathfrak{p})$. It is clear that, for any two-sided ideal $\mathfrak{p}$ of R, $E[\Omega, \mathfrak{p}] \leqq E(\Omega, \mathfrak{p})$ and that we have equality when $\mathfrak{p}$ is finitely generated as a right ideal; hence the result we shall give here generalizes the corresponding result of (**2**). Notice however that it is possible to have strict inclusion; for example, if $\mathfrak{k}$ is a field, if R is the commutative polynomial ring over $\mathfrak{k}$ in countably many indeterminates $x_1, x_2, \ldots$ we see that $E[\mathbf{Z}, \mathfrak{p}]$ is a proper subgroup of $E(\mathbf{Z}, \mathfrak{p})$ since $t(\mathbf{N}, f, 0) \notin E[\mathbf{Z}, \mathfrak{p}]$ where $f: \mathbf{N} \to R$ is given by $f(n) = x_n$ and $\mathfrak{p}$ is the ideal generated by $x_1, x_2, \ldots$.

We next give some definitions that will be required in the construction of the ideals $\mathfrak{p}$ mentioned above. For any $\Omega \times \Omega$ matrix X we define the *level* of X to be the two-sided ideal $J(X)$ generated by the matrix entries $X_{\alpha\beta}$, $X_{\alpha\alpha} - X_{\beta\beta}$, for all $\alpha, \beta \in \Omega$, $\alpha \neq \beta$. For any subgroup H of $GL(\Omega, R)$ we define the level of H to be the two-sided ideal $J(H) = \sum_{X \in H} J(X)$, (c.f. (**7**)). We also define the ideal $K(H)$ to be the two-sided ideal $K(H) = \sum J(X)$, where the summation is taken over all those $X \in H \cap E(\Omega, R)$ that have at least four trivial columns. (The φth column of X is said to be *trivial* if and only if $X(e_\varphi) = e_\varphi$.) Since matrices in $E(\Omega, R)$ differ from the $\Omega \times \Omega$ identity matrix in only finitely many rows we see that $K(H)$ is, in fact, the two-sided ideal of R generated by the matrix entries $X_{\alpha\beta}$, $X_{\alpha\alpha} - 1$, for all $\alpha, \beta \in \Omega$, $\alpha \neq \beta$ and all $X \in H \cap E(\Omega, R)$ that have at least four trivial columns. Clearly $K(H) \leqq J(H)$ and we shall see that when H is a normal subgroup of $GL(\Omega, R)$ we have equality. We say that a ring R is *d-finite* if each two-sided ideal of R is finitely generated as a right ideal. Thus, simple rings and Noetherian rings are d-finite.

Finally, we remark that we shall use $[x, y]$ to denote the commutator $x^{-1}y^{-1}xy$ and, for any group G, $Z(G)$ shall denote the centre of G.

2. Statement and discussion of results

We shall prove

Theorem A. *Let R be a ring with identity, let Ω be an infinite set and let H be a subgroup of $GL(\Omega, R)$ that is normalized by $E(\Omega, R)$. There exists a unique two-sided ideal $\mathfrak{p}$ of R such that*

$$E[\Omega, \mathfrak{p}] \leqq H \leqq GL'(\Omega, \mathfrak{p}).$$

As we have seen above, whenever R has the maximal condition on right ideals the groups $E[\Omega, \mathfrak{p}]$ and $E(\Omega, \mathfrak{p})$ coincide. It follows that Theorem A extends the classification of the normal subgroups of $GL(\Omega, R)$ that was given in Theorem 1 of (**2**). We shall also prove

Theorem B. *For any ring R with identity and any infinite set Ω the following assertions are equivalent.*

(i) *R is d-finite.*

(ii) *whenever H is a subgroup of $GL(\Omega, R)$ normalized by $E(\Omega, R)$ there exists a unique two-sided ideal $\mathfrak{p}$ of R such that*

$$E(\Omega, \mathfrak{p}) \leqq H \leqq GL'(\Omega, \mathfrak{p}).$$

This theorem shows that in order to obtain the sandwiching of normal subgroups of $GL(\Omega, R)$ given in (**2**) or (**4**) it is necessary for R to be d-finite.

We shall see that the ideal $\mathfrak{p}$ of Theorems A and B is the level of H. The advantage of such "sandwiches" is that they depend only upon the level of H, no other knowledge of H is required. To construct $E[\Omega, J(H)]$ we restricted our attention to the finitely generated right ideals of R contained in $J(H)$. Although we know from Theorem B that we cannot use $E(\Omega, J(H))$ to sandwich the normal subgroups H of $GL(\Omega, R)$ in general, we can still ask whether or not H could be sandwiched by groups larger than $E[\Omega, J(H)]$ which still depend only on $J(H)$. To see that in some sense $E[\Omega, J(H)]$ is best possible consider the following example. Let $\mathfrak{k}$ be a field and let R be the commutative polynomial ring over $\mathfrak{k}$ in countably many indeterminates $x_1, x_2, \ldots$; put $\Omega = \boldsymbol{N}$. For each $i \geqq 1$, let $\mathfrak{p}_i$ be the ideal of R generated by $\{x_{2k}, x_{2j-1} \colon k \in \boldsymbol{N}, 1 \leqq j \leqq i\}$ and let $\mathfrak{q}_i$ be the ideal of R generated by $\{x_{2k-1}, x_{2j} \colon k \in \boldsymbol{N}, 1 \leqq j \leqq i\}$. If we let $\mathfrak{p}$ denote the ideal of R generated by all the indeterminates $x_1, x_2, \ldots$ we see that $\mathfrak{p} = \bigcup_i \mathfrak{p}_i = \bigcup_i \mathfrak{q}_i$; moreover $E_1 = \prod_i E(\Omega, \mathfrak{p}_i)$ and $E_2 = \prod_i E(\Omega, \mathfrak{q}_i)$ each have level $\mathfrak{p}$. The maximality of $E[\Omega, \mathfrak{p}]$ is demonstrated by noting that $E_1 \cap E_2 = E[\Omega, \mathfrak{p}]$.

3. Basic lemmas

It is easy to deduce from nothing more than matrix multiplication that the centre of $GL(\Omega, R)$ is just the centralizer of $EF(\Omega, R)$ in $GL(\Omega, R)$. This, together with the observation that $E(\Omega, R)$ has trivial centre, allows us to deduce

Lemma A. *If H is a subgroup of $GL(\Omega, R)$ that is normalized by $E(\Omega, R)$ then the following assertions are equivalent.*

(i) $H \leqq Z(GL(\Omega, R))$.

(ii) $[H, E(\Omega, R)] = 1$.

(iii) $H \cap E(\Omega, R) = 1$.

Lemma B. *Let H be a subgroup of $GL(\Omega, R)$ that is normalized by $EF(\Omega, R)$. If $A \in H$ then, for all $x \in R$ and all $\mu, \rho \in \Omega$, $\mu \neq \rho$, $K(H)$ contains $A_{\mu\rho}x$ and $(A_{\mu\mu} - A_{\rho\rho})x$.*

Proof. Let μ, ρ and x be as in the statement of the lemma. We shall show that

$A_{\mu\rho}x \in K(H)$; the case of $(A_{\mu\mu} - A_{\rho\rho})x$ is similar. If $A_{\mu\rho}x = 0$ then the result is obvious so assume that $A_{\mu\rho}x \neq 0$. Pick $\lambda \in \Omega$, $\lambda \neq \mu$, ρ and pick distinct $\varphi_i \in \Omega$, $i = 1, \ldots, 4$ such that $\varphi_i \neq \lambda$ and $A^{-1}_{\lambda\varphi_i} = 0$; this is possible since A^{-1} is column finite. Put $t = t(\lambda, x, \rho)$. If $[t, A] = 1$ then $A_{\alpha\rho}x = 0$ for all $\alpha \neq \rho$; in particular, $A_{\mu\rho}x = 0$ contrary to hypothesis. We deduce that t and A do not commute and hence neither do t and A^{-1}. Moreover, $[t, A^{-1}] \in E(\Omega, R) \cap H$ and by the choice of the φ_i, $[t, A^{-1}]e_{\varphi_i} = e_{\varphi_i}$, $i = 1, \ldots, 4$. This shows that for all $\alpha, \beta \in \Omega$, $\alpha \neq \beta$, $[t, A^{-1}]_{\alpha\beta}$ and $[t, A^{-1}]_{\alpha\alpha} - 1$ lie in $K(H)$. However, for all $\beta \neq \mu$, $[t, A^{-1}]_{\mu\beta} = A_{\mu\rho}xA^{-1}_{\lambda\beta}$ and $[t, A^{-1}]_{\mu\mu} - 1 = A_{\mu\rho}xA^{-1}_{\lambda\mu}$ and hence $K(H)$ contains the sum $\sum_{\beta} A_{\mu\rho}xA^{-1}_{\lambda\beta}A_{\beta\lambda}$; but this sum is just $A_{\mu\rho}x$ and this observation completes the proof of the lemma.

We shall also need

Lemma C. *Let H be a subgroup of $GL(\Omega, R)$ that is normalized by $EF(\Omega, R)$ and let a be a generator of $K(H)$. For all $\rho, \sigma \in \Omega$, $\rho \neq \sigma$, H contains $t(\rho, a, \sigma)$.*

Proof. We shall consider the case $a = X_{\alpha\beta}$, for some $\alpha, \beta \in \Omega$, $\alpha \neq \beta$, and some $X \in H \cap E(\Omega, R)$ that has at least four trivial columns; the case $a = X_{\alpha\alpha} - 1$ is similar. Let Λ index the non-zero entries of the αth row of X and define $f: \Lambda \to R$ by $f(\lambda) = X_{\alpha\lambda}$ if $\lambda \neq \alpha$ and $f(\alpha) = X_{\alpha\alpha} - 1$. There exists $\varphi \in \Omega$, $\varphi \neq \alpha, \rho, \sigma$ such that $X(e_\varphi) = e_\varphi$. Put $t = t(\alpha, 1, \varphi)$. Then $t_1 = [t, X] = t(\Lambda, f, \varphi)$ and $t_1 \in H$ by hypothesis. Further similar conjugations now show that H contains $t(\rho, f(\beta), \sigma)$ and this completes the proof of the lemma since $f(\beta) = x_{\alpha\beta}$.

Corollary A. *If H is a subgroup of $GL(\Omega, R)$ that is normalized by $EF(\Omega, R)$ then H contains $EF(\Omega, J(H))$.*

Proof. Lemma C shows that H contains $t(\lambda, a, \mu)$ for all $\lambda, \mu \in \Omega$, $\lambda \neq \mu$ and all generators a of $K(H)$, while Lemma B shows that $K(H) = J(H)$, since we already know that $K(H) \leqq J(H)$. It follows that H contains $\{t(\lambda, y, \mu)$: $\lambda, \mu \in \Omega, \lambda \neq \mu, y \in J(H)\}$ and hence $EF(\Omega, J(H)) \leqq H$ since H is normalized by $EF(\Omega, R)$.

We complete this section with

Lemma D. *If H is a subgroup of $GL(\Omega, R)$ that is normalized by $E(\Omega, R)$ and if H contains $EF(\Omega, \mathfrak{q})$, for some two-sided ideal $\mathfrak{q}$ of R, then H contains $E(\Omega, \mathfrak{p})$, for any finitely generated right ideal $\mathfrak{p}$ of R contained in $\mathfrak{q}$.*

Proof. It will be sufficient to show that H contains any generator $t(\Lambda, f, \mu)$ of $E(\Omega, \mathfrak{p})$ since H is normalized by $E(\Omega, R)$. Since $\mathfrak{p}$ is finitely generated we can write $\mathfrak{p} = x_1R + \ldots + x_sR$, for $x_i \in \mathfrak{p}$, $i = 1, \ldots, s$. The result now follows from the proof of the Lemma of **(2)** since, by hypothesis, H contains $t(\lambda, x_i, \mu)$, for all $\lambda, \mu \in \Omega$, $\lambda \neq \mu$, $i = 1, \ldots, s$.

4. The proof of the theorems

We begin with the proof of Theorem A. Let H be a subgroup of $GL(\Omega, R)$ that is normalized by $E(\Omega, R)$. If H is central then $J(H) = 0$, since central matrices are of the

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form rI, where r is a central unit of R, and we see that $E(\Omega, 0) \leqq H \leqq GL'(\Omega, 0)$. We may thus suppose that H is not central. Corollary A shows that H contains $EF(\Omega, J(H))$ and hence Lemma D shows that $E(\Omega, \mathfrak{p}) \leqq H$, for every finitely generated right ideal $\mathfrak{p}$ of R contained in $J(H)$. It follows immediately that $E[\Omega, J(H)] \leqq H$. We remark that $J(H) \neq 0$, for if not $J(H \cap E(\Omega, R)) = 0$ and so $H \cap E(\Omega, R) = 1$; hence, by Lemma A, H is central contrary to hypothesis. Let $\bar{}$ denote images under the homomorphism induced by the projection $R \to R/J(H)$. We see that $J(\bar{H}) = 0$ and that $\overline{H \cap E(\Omega, R)} = 1$. It follows from Lemma A that $\bar{H}$ is central and we deduce that $E[\Omega, J(H)] \leqq H \leqq GL'(\Omega, J(H))$. Thus, we may take $\mathfrak{p} = J(H)$ in the statement of the theorem; the uniqueness of $\mathfrak{p}$ follows by noting that $\mathfrak{p}$ is maximal with respect to the first inclusion and minimal with respect to the second. This completes the proof of Theorem A.

That (i) implies (ii) in Theorem B is immediate from Theorem A since, whenever R is d-finite, we have seen that $E[\Omega, \mathfrak{p}] = E(\Omega, \mathfrak{p})$, for any two-sided ideal $\mathfrak{p}$ of R. It remains to prove that (ii) implies (i). For any $X \in GL(\Omega, R)$, $X \neq 1$, we shall say that X has *finite* $\mathfrak{p}$-*support* if there exists a proper two-sided ideal $\mathfrak{p}$ of R and a finitely generated right ideal $\mathfrak{q}$ of R contained in $\mathfrak{p}$ such that all the entries of $X-1$ lie in $\mathfrak{q}$. Since all $X \in GL(\Omega, R)$ are column finite, it follows that whenever $\mathfrak{q}$ is a finitely generated right ideal contained in a proper two-sided ideal $\mathfrak{p}$ of R every $X \in E(\Omega, \mathfrak{q})$, $X \neq 1$, has finite $\mathfrak{p}$-support. Suppose that (ii) holds yet R is not d-finite. There exists a proper two-sided ideal $\mathfrak{p}$ of R which is not finitely generated as a right ideal. Moreover, there exists a collection $\{a_i : i \in \boldsymbol{N}\}$ of elements of $\mathfrak{p}$ that is not contained in any finitely generated right ideal of R that is contained in $\mathfrak{p}$. Let $\mathfrak{p}_j$ denote the right ideal of R that is generated by $\{a_i : 1 \leqq i \leqq j\}$ and put $H = \bigcup_j E(\Omega, \mathfrak{p}_j)$; then H is a normal subgroup of $E(\Omega, R)$, every non-trivial element of which has finite $\mathfrak{p}$-support. By (ii) there exists a unique two-sided ideal $\mathfrak{n}$ of R such that $E(\Omega, \mathfrak{n}) \leqq H$ and clearly $\mathfrak{p}_i \leqq \mathfrak{n}$, for all $i \in \boldsymbol{N}$. Thus $E\left(\Omega, \bigcup_i \mathfrak{p}_i\right) \leqq H$. Let Ω_0 be a countable proper subset of Ω and $f: \Omega_0 \to \{a_i : i \in \boldsymbol{N}\}$ be a bijection. Pick $\omega \in \Omega - \Omega_0$ and define $t = t(\Omega_0, f, \omega)$. Then $t \in H$ yet, by the choice of the a_i, t does not have finite $\mathfrak{p}$-support. This contradiction completes the proof of Theorem B.

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