THE MODULUS OF NEAR SMOOTHNESS OF THE l^p PRODUCT OF A SEQUENCE OF BANACH SPACES

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1. Introduction. In the classical geometry of Banach spaces the notions of smoothness, uniform smoothness, strict and uniform convexity introduced by Day [**1**] and Clarkson [**2**] play a very important role and are used in many branches of functional analysis ([**3, 4, 5**], for example). In recent years a lot of papers have appeared containing interesting generalizations of these notions in terms of a measure of noncompactness. These new concepts investigated in this paper as near uniform smoothness, local near uniform smoothness and modulus of near smoothness have been introduced by Stachura and Sękowski [**6**] and Banaś [**7**] (see also [**8, 9**]).

The main aim of this paper is to provide an estimate of the modulus of near smoothness of the so-called l^p product of a sequence of Banach spaces. Further, we prove that the notions of local near uniform smoothness and convexity are hereditary in an l^p product of spaces. Apart from that we calculate the exact formula for the modulus of near smoothness for the space $l^p(l^{p_i})$.

2. Notation, definitions and auxiliary facts. Let E be a given real Banach space with norm $\|\cdot\|$ and the zero element θ. The dual space of E will be denoted by E^*. For brevity, the symbols B, B^*, S and S^* will denote the unit balls and the unit spheres in the spaces E and E^*, respectively.

For a given bounded subset X of E let $\chi_E(X)$ denote the Hausdorff measure of noncompactness, i.e. infimum of numbers $\varepsilon > 0$ such that X can be covered by a finite collection of balls having radius less than or equal to ε. Let $f \in S^*$ and $\varepsilon \in [0, 1]$. By the slice $F(f, \varepsilon)$ we will understand the set $\{x \in B : f(x) \geq 1 - \varepsilon\}$. Similarly, $F^*(x, \varepsilon)$ denotes the slice $\{f \in B^* : f(x) \geq 1 - \varepsilon\}$ in the space E^* provided $x \in S$. Now we recall the following definition [**9**].

Definition 1. The function $\Sigma : [0, 1] \to [0, 1]$ defined by the formula

$$\Sigma_E(\varepsilon) = \sup\{\chi(F^*(x, \varepsilon)) : x \in S\}$$

will be called *the modulus of near convexity of the space E.*

Obviously, we take $\chi = \chi_{E^*}$ in the above definition.

The function Σ_E is nondecreasing on $[0, 1]$. Moreover, for every space E the estimate $\Sigma_E(\varepsilon) \geq \varepsilon$ holds to be true. Moreover, $\Sigma_{c_0}(\varepsilon) = \varepsilon$, $\varepsilon \in [0, 1]$ [**9**]. In the space l^p the following formula is valid

$$\Sigma_{l^p}(\varepsilon) = (1 - (1 - \varepsilon)^q)^{1/q}, \; \varepsilon \in [0, 1], \tag{1}$$

where $1 < p < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Now we recall the second definition [**9**].

DEFINITION 2. A Banach space E is said to be *nearly uniformly smooth* (*NUS*) if

$$\lim_{\varepsilon\to 0} \Sigma_E(\varepsilon) = 0$$

i.e. if $\lim_{\varepsilon\to 0} \chi(F^*(x,\varepsilon)) = 0$ uniformly with respect to $x \in S$.

For example, the spaces l^p, $L^p(0,1)$, $p > 1$, c_0 are NUS. On the other hand the spaces l^∞, c and $C[0,1]$ fail to have this property.

Let us recall [7] that a space E is described as nearly smooth (NS) if for any $x \in S$ we have $\Sigma_E(0) = 0$, i.e. $\chi_{E^*}(\{f \in B^* : f(x) = 1\}) = 0$. Obviously, every NUS space is NS but not conversely.

In what follows we will need the following two definitions [7].

DEFINITION 3. A Banach space E is called *locally nearly uniformly smooth* (*LNUS*) if

$$\lim_{\varepsilon\to 0} \chi_{E^*}(F^*(x,\varepsilon)) = 0$$

for every $x \in S$.

DEFINITION 4. We say that a Banach space E is *locally nearly uniformly convex* (*LNUC*) if

$$\lim_{\varepsilon\to 0} \chi_E(F(f,\varepsilon)) = 0$$

for every $f \in S^*$.

The basic relation between the concepts of LNUC and LNUS spaces is contained in the following assertion:

$$E \text{ is LNUC if and only if } E^* \text{ is LNUS.} \tag{2}$$

3. An estimate of the modulus of near smoothness of the space $l^p(E_i)$. At the beginning let us introduce a few basic facts needed further on. Assume that $(E_i, \|\cdot\|_i)$ $(i = 1, 2, \ldots)$ is a sequence of Banach spaces. For a fixed number p, $1 < p < \infty$, let $l^p(E_i) = l^p(E_1, E_2, \ldots)$ denote the Banach space of all sequences $x = (x_i)$, $x_i \in E_i$ $(i = 1, 2, \ldots)$, such that $\sum_{i=1}^{\infty} \|x_i\|_i^p < \infty$, furnished with the norm

$$\|x\|_p = \left(\sum_{i=1}^{\infty} \|x_i\|_i^p\right)^{1/p}.$$

Recall [5] that $(l^p(E_i))^* = l^q(E_i^*)$, where $\frac{1}{p} + \frac{1}{q} = 1$.

One of the most important problems considered in the geometry of Banach spaces is connected with the following question:

Which properties of the spaces E_i $(i = 1, 2, \ldots)$ are transmitted to the product $l^p(E_i)$?

Let us quote some positive results in this direction. First recollect that in [**12**, 8] it was shown that if E_i has the property H (or H^*) for $i = 1, 2, \ldots$ then $l^p(E_i)$ also has these properties. In the paper [**11**] Partington showed that if the E_i's are UKK and satisfy some additional conditions, then $l^p(E_i)$ is UKK. Let us also mention that in the paper [**9**] an estimate of the so-called modulus of UKK-ness was obtained.

In order to recall the next result obtained in the paper [9] let us denote by $r = r(\varepsilon)$ the function acting from the interval $[0,1]$ into itself and defined by the formula

$$r(\varepsilon) = \sup\{\Sigma_{E_i}(\varepsilon) : i \in \mathbf{N}\}.$$

Observe that the following inequality holds

$$\Sigma_{E_j}(\varepsilon) \leqslant r(\varepsilon) \leqslant \Sigma_{l^p(E_i)}(\varepsilon)$$

for $j \in \mathbf{N}$ and $\varepsilon \in [0,1]$. It was shown in [9] that if E_i is a reflexive NUS space $(i = 1, 2, \ldots)$ then $l^p(E_i)$ is NUS if and only if $\lim_{\varepsilon \to 0} r(\varepsilon) = 0$. The authors of [9] raised also the question if the assumption on reflexivity can be dropped in this result. In what follows we provide the affirmative answer to the above mentioned question. It is contained in the following theorem.

THEOREM 1. *Let $(E_i, \|\cdot\|_i)$ be a sequence of arbitrary Banach spaces. Then*

$$\Sigma_{l^p(E_i)}(\varepsilon) \leq 2(r^q(\sqrt{\varepsilon}) + 4(1 - (1 - \sqrt{\varepsilon})^q))^{1/q}. \tag{3}$$

Particularly, if $\lim_{\varepsilon \to 0} r(\varepsilon) = 0$ then $l^p(E_i)$ is NUS.

Before the proof let us introduce some notation. Namely, denote by B and B^* the closed unit balls in the spaces $l^p(E_i)$ and $(l^p(E_i))^*$, respectively. Moreover, for convenience we will write l^p instead of $l^p(E_i)$.

Proof. Let us fix a number $\varepsilon \in (0,1]$ and a number $\eta > 0$ small enough. We can choose $x = (x_i) \in S_{l^p}$ and a number γ such that

$$\chi(F^*(x, \varepsilon)) > \gamma \geq \Sigma_{l^p(E_i)}(\varepsilon) - \eta, \tag{4}$$

where $\chi = \chi_{l^q(E_i^*)}$. This implies that there exists a sequence $(f^n) \subset B^*$ such that

$$f^n(x) \geq 1 - \varepsilon \quad \text{and} \quad \|f^n - f^m\|_q > \gamma,$$

for n, $m \in \mathbf{N}$, $n \neq m$. Putting $f^n = (f_i^n)$, where $f_i^n \in E_i^*$ for every $i \in \mathbf{N}$ $(n = 1, 2, \ldots)$, we can rewrite the last inequalities in the form

$$\sum_{i=1}^{\infty} f_i^n(x_i) \geq 1 - \varepsilon \quad \text{and} \quad \sum_{i=1}^{\infty} \|f_i^n - f_i^m\|_i^q > \gamma^q, \tag{5}$$

for $n, m \in \mathbf{N}$, $n \neq m$.

Let us observe that applying the same argument as in [11] without loss of generality we may assume that

$$\|f_i^n\|_i \to a_i, \qquad f_i^n(x_i) \to b_i, \tag{6}$$

if $n \to \infty$, and

$$\|f_i^n - f_i^m\|_i \to c_i, \tag{7}$$

when $n, m \to \infty$ $(i = 1, 2, \ldots)$. For $\delta > 0$ there exists a number $n(\delta) \in \mathbf{N}$ such that

$$\lim_{\delta \to 0} n(\delta) = \infty \quad \text{and} \quad \left(\sum_{i=n(\delta)+1}^{\infty} \|x_i\|_i^p\right)^{1/p} \leq \delta. \tag{8}$$

Consider the sets S_δ, T_δ and W_δ defined as

$$S_\delta=\{i\in\mathbf{N}: i\le n(\delta),\, a_i>0 \text{ and } x_i\ne\theta\},$$
$$T_\delta=\{i\in\mathbf{N}: i\le n(\delta),\, a_i=0 \text{ and } x_i\ne\theta\},$$
$$W_\delta=\{i\in\mathbf{N}: i\le n(\delta) \text{ and } x_i=\theta\}.$$

Keeping in mind that $a_i=0$ for $i\in T_\delta$, we have for $m\in\mathbf{N}$ large enough the inequality $\sum_{i\in T_\delta} f_i^m(x_i)\le\delta$ which together with (5) and (8) gives

$$\begin{aligned}1-\varepsilon &\le \sum_{i\in S_\delta} f_i^m(x_i)+\sum_{i\in T_\delta} f_i^m(x_i)+\sum_{i=n(\delta)+1}^{\infty} f_i^m(x_i)\\ &\le \sum_{i\in S_\delta} f_i^m(x_i)+\delta+\left(\sum_{i=n(\delta)+1}^{\infty}\|f_i^m\|_i^q\right)^{1/q}\cdot\left(\sum_{i=n(\delta)+1}^{\infty}\|x_i\|_i^p\right)^{1/p}\\ &\le \sum_{i\in S_\delta} f_i^m(x_i)+2\delta.\end{aligned}$$

Hence

$$1-\varepsilon-2\delta\le\sum_{i\in S_\delta} f_i^m(x_i)\le\left(\sum_{i\in S_\delta}\|f_i^m\|_i^q\right)^{1/q}.$$

Putting $S=\{i\in\mathbf{N}: a_i>0 \text{ and } x_i\ne\theta\}$ we have that $\bigcup_{\delta>0} S_\delta=S$ and as $m\to\infty$ and $\delta\to0$ we deduce from the last inequality and (6) that

$$1-\varepsilon\le\sum_{i\in S} b_i\le\left(\sum_{i\in S} a_i^q\right)^{1/q}. \tag{9}$$

The Hölder's inequality, (8) and $\|(f_i^m)\|_q\le1$ imply

$$\sum_{i=n(\delta)+1}^{\infty} f_i^m(x_i)\le\delta.$$

Hence, in view of (5) we obtain

$$\begin{aligned}1-\varepsilon-\delta\le1-\varepsilon-\sum_{i=n(\delta)+1}^{\infty} f_i^m(x_i)\le\sum_{i\le n(\delta)} f_i^m(x_i)=\sum_{\substack{i\le n(\delta)\\ i\notin W_\delta}} f_i^m(x_i)&\le\left(\sum_{\substack{i\le n(\delta)\\ i\notin W_\delta}}\|f_i^m\|_i^q\right)^{1/q}\\ &=\left(\sum_{i\le n(\delta)}\|f_i^m\|_i^q-\sum_{i\in W_\delta}\|f_i^m\|_i^q\right)^{1/q}\\ &\le\left(1-\sum_{i\in W_\delta}\|f_i^m\|_i^q\right)^{1/q}\end{aligned}$$

which gives

$$\sum_{i\in W_\delta}\|f_i^m\|^q\le1-(1-\varepsilon-\delta)^q.$$

Now, let us put $W=\{i\in\mathbf{N}: x_i=\theta\}$, Since $W=\bigcup_{\delta>0} W_\delta$ then letting $m\to\infty$ and $\delta\to0$ we get from the last inequality and (6)

$$\sum_{i\in W} a_i^q\le1-(1-\varepsilon)^q.$$

Hence, keeping in mind that $c_i \le 2a_i$ we obtain

$$\sum_{i \in W} c_i^q \le 2^q . (1-(1-\varepsilon)^q). \tag{10}$$

Now, by Hölder's inequality and (8) we get

$$1-\varepsilon \le \sum_{1 \le n(\delta)} f_i^m(x_i) + \sum_{i=n(\delta)+1}^{\infty} f_i^m(x_i) \le \Big(\sum_{i \le n(\delta)} \|x_i\|_i^p\Big)^{1/p} . \Big(\sum_{i \le n(\delta)} \|f_i^m\|_i^q\Big)^{1/q}$$
$$+ \Big(\sum_{i=n(\delta)+1}^{\infty} \|x_i\|_i^p\Big)^{1/p} . \Big(\sum_{i=n(\delta)+1}^{\infty} \|f_i^m\|_i^q\Big)^{1/q} \le \Big(1-\sum_{i=n(\delta)+1}^{\infty} \|f_i^m\|_i^q\Big)^{1/q} + \delta.$$

Consequently

$$\sum_{i=n(\delta)+1}^{\infty} \|f_i^m\|_i^q \le 1-(1-\varepsilon-\delta)^q. \tag{11}$$

Combining (5), the inequality $(a+b)^q \le 2^q(a^q+b^q)$ and (11) we derive

$$\gamma^q < \sum_{i=1}^{n(\delta)} \|f_i^n - f_i^m\|_i^q + \sum_{i=n(\delta)+1}^{\infty} \|f_i^n - f_i^m\|_i^q \le \sum_{i=1}^{n(\delta)} \|f_i^n - f_i^m\|_i^q + 2^q \sum_{i=n(\delta)+1}^{\infty} (\|f_i^m\|_i^q + \|f_i^n\|_i^q)$$
$$\le \sum_{i=1}^{n(\delta)} \|f_i^n - f_i^m\|_i^q + 2^{q+1}(1-(1-\varepsilon-\delta)^q).$$

Hence

$$\sum_{i=1}^{n(\delta)} \|f_i^n - f_i^m\|_i^q > \gamma^q - 2^{q+1}(1-(1-\varepsilon-\delta)^q). \tag{12}$$

Combining the above inequality and (7) we obtain

$$\sum_{i=1}^{\infty} c_i^q \ge \gamma^q - 2^{q+1}(1-(1-\varepsilon)^q).$$

This inequality together with (10) yields

$$\sum_{i \in S} c_i^q = \sum_{i=1}^{\infty} c_i^q - \sum_{i \in W} c_i^q \ge \gamma^q - 3 . 2^q(1-(1-\varepsilon)^q)$$

or, equivalently

$$\sum_{i \in S} c_i^q \ge \gamma^q - 3 . 2^q(1-(1-\varepsilon)^q). \tag{13}$$

Put $Y = \left\{ i \in S : \dfrac{b_i}{a_i . \|x_i\|_i} > 1-\sqrt{\varepsilon} \right\}$. Then, from (9) we get

$$1-\varepsilon \le \sum_{i \in S} b_i \le \sum_{i \in Y} b_i + \sum_{i \in S \setminus Y} \frac{b_i}{a_i . \|x_i\|_i} . a_i . \|x_i\|_i \le \sum_{i \in Y} b_i + (1-\sqrt{\varepsilon}) . \Big(1-\sum_{i \in Y} a_i . \|x_i\|_i\Big)$$
$$\le \sum_{i \in Y} b_i + (1-\sqrt{\varepsilon}) . \Big(1-\sum_{i \in Y} b_i\Big).$$

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Hence, after a simple calculation we obtain $1-\sqrt{\varepsilon}\leq\sum_{i\in Y} b_i$. Because for every set $K\subset\mathbf{N}$ we have that $\sum_{i\in K} b_i\leq(\sum_{i\in K} a_i^q)^{1/q}$ then we infer

$$1-\sqrt{\varepsilon}\leq\sum_{i\in Y} b_i\leq\Big(\sum_{i\in Y} a_i^q\Big)^{1/q}.$$

This implies

$$\sum_{i\in S\setminus Y} c_i^q\leq 2^q.\sum_{i\in S\setminus Y} a_i^q=2^q\Big(\sum_{i\in S} a_i^q-\sum_{i\in Y} a_i^q\Big)\leq 2^q.(1-(1-\sqrt{\varepsilon})^q)$$

and consequently

$$\sum_{i\in S\setminus Y} c_i^q\leq 2^q(1-(1-\sqrt{\varepsilon})^q).$$

Combining the above inequality and (13) we get

$$\begin{aligned}\sum_{i\in Y} c_i^q=\sum_{i\in S} c_i^q-\sum_{i\in S\setminus Y} c_i^q&\geq\gamma^q-3.2^q.(1-(1-\varepsilon)^q)-2^q.(1-(1-\sqrt{\varepsilon})^q)\\&\geq\gamma^q-2^{q+2}.(1-(1-\sqrt{\varepsilon})^q).\end{aligned}$$

This yields

$$\sum_{i\in Y} c_i^q\geq\gamma^q-2^{q+2}.(1-(1-\sqrt{\varepsilon})^q). \tag{14}$$

Now we consider two cases:

(i) $\gamma^q-2^{q+2}.(1-(1-\sqrt{\varepsilon})^q)\leq 0$,

(ii) $\gamma^q-2^{q+2}.(1-(1-\sqrt{\varepsilon})^q)>0$.

In the case (i), if $\eta\to 0$ we obtain from (4) that

$$\Sigma_{l^P(E_i)}(\varepsilon)\leq(2^{q+2}.(1-(1-\sqrt{\varepsilon})^q))^{1/q},$$

what means that (3) holds.

Now suppose that the case (ii) is satisfied. Take $\delta>0$ to be small enough and observe that there exists $i\in Y$ such that

$$\frac{c_i}{a_i}\geq(\gamma^q-2^{q+2}.(1-(1-\sqrt{\varepsilon})^q))^{1/q}-\delta>0. \tag{15}$$

Indeed, suppose that

$$c_i^q<a_i^q.((\gamma^q-2^{q+2}.(1-(1-\sqrt{\varepsilon})^q))^{1/q}-\delta)^q \text{ for all } i\in Y.$$

This implies

$$\begin{aligned}\sum_{i\in Y} c_i^q&<\sum_{i\in Y} a_i^q.((\gamma^q-2^{q+2}.(1-(1-\sqrt{\varepsilon})^q))^{1/q}-\delta)^q\\&<\gamma^q-2^{q+2}.(1-(1-\sqrt{\varepsilon})^q),\end{aligned}$$

but this contradicts (14).

For $i \in Y$ satisfying (15) let us put

$$\bar{x} = \frac{x_i}{\|x_i\|_i}, \qquad g^n = \frac{f_i^n}{\|f_i^n\|_i}.$$

From (6) and the definition of the set Y we infer that

$$\lim_{n\to\infty} g^n(\bar{x}) = \frac{b_i}{a_i \,.\, \|x_i\|_i} > 1 - \sqrt{\varepsilon}, \tag{16}$$

and from (6), (7) and (15) we have

$$\|g^n - g^m\|_i \to \frac{c_i}{a_i} \geq (\gamma^q - 2^{q+2} \,.\, (1 - (1 - \sqrt{\varepsilon})^q))^{1/q} - \delta, \tag{17}$$

when $n, m \to \infty$. The inequality (16) means that for $n \in \mathbf{N}$ large enough we have that $g_n \in F^*(\bar{x}, \sqrt{\varepsilon})$ which together with (17) implies

$$\tfrac{1}{2}((\gamma^q - 2^{q+2} \,.\, (1 - (1 - \sqrt{\varepsilon})^q))^{1/q} - \delta) \leq \chi_{E_i^*}(F^*(\bar{x}, \sqrt{\varepsilon})) \leq r(\sqrt{\varepsilon}).$$

The last inequality can be written in the form

$$\gamma^q \leq (\delta + 2r(\sqrt{\varepsilon}))^q + 2^{q+2} \,.\, (1 - (1 - \sqrt{\varepsilon})^q),$$

which in conjunction with (4) gives

$$\Sigma_{l^p(E_i)}(\varepsilon) \leq \gamma + \eta \leq ((\delta + 2r(\sqrt{\varepsilon}))^q + 2^{q+2} \,.\, (1 - (1 - \sqrt{\varepsilon})^q))^{1/q} + \eta.$$

The arbitrariness of η and δ allows us to deduce the inequality (3) which completes the proof of Theorem 1.

4. Local near uniform smoothness and convexity in the l^p product of Banach spaces. This section is devoted to the study of the concepts of local near uniform smoothness and convexity in the product space $l^p(E_i)$. The main result is contained in the following theorem.

THEOREM 2. *If E_i is LNUS $(i = 1, 2, \ldots)$ then $l^p(E_i)$ is LNUS.*

Proof. Suppose contrary, i.e. there exists $x = (x_i) \in S_{l^p}$ and a number $\gamma > 0$ such that

$$\lim_{\varepsilon\to 0} \chi(F^*(x, \varepsilon)) > \gamma > 0 \quad \text{i.e.} \quad \chi(F^*(x, \varepsilon)) > \gamma > 0 \quad \text{for} \quad \varepsilon \in (0, 1].$$

Take $\delta > 0$ such that for $\varepsilon \in [0, \delta]$

$$\gamma^q - 2^{q+1} \,.\, (1 - (1 - \varepsilon - \delta)^q) - 2^q \,.\, (1 - (1 - \varepsilon)^q) > \left(\frac{\gamma}{2}\right)^q. \tag{18}$$

Fix $\varepsilon \in (0, \delta]$. Similarly as in proof of Theorem 1 we can show that there exists a sequence $(f^n) \subset B^*$, $f^n = (f_i^n)$, $f_i^n \in E_i^*$ such that

$$\sum_{i=1}^{\infty} f_i^n(x_i) \geq 1 - \varepsilon, \qquad \sum_{i=1}^{\infty} \|f_i^n - f_i^m\|_i^q > \gamma^q,$$

and

$$\|f_i^n\|_i \to a_i, \qquad f_i^n(x_i) \to b_i, \qquad \|f_i^n - f_i^m\|_i \to c_i,$$

when $n \to \infty$ and $n, m \to \infty$ $(i = 1, 2, \ldots)$.

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In what follows we will need the following lemma.

LEMMA 1. *If* $x_i \neq \theta$, $a_i > 0$ *then*

$$\chi_{E_i^*}\left(F^*\left(\frac{x_i}{\|x_i\|_i}, \frac{2\varepsilon}{a_i \cdot \|x_i\|_i}\right)\right) \geq \frac{c_i}{2}.$$

Proof. Observe that

$$f_i^n(x_i) \geq \|f_i^n\|_i \cdot \|x_i\|_i - \varepsilon, \text{ for all } i \in \mathbf{N}. \tag{19}$$

Indeed, suppose the contrary, i.e. there is $j \in \mathbf{N}$ such that

$$f_j^n(x_j) < \|f_j^n\|_j \cdot \|x_j\|_j - \varepsilon.$$

Then

$$1 - \varepsilon \leq f_j^n(x_j) + \sum_{i \neq j} f_i^n(x_i) < -\varepsilon + \sum_{i=1}^{\infty} \|f_i^n\|_i \cdot \|x_i\|_i \leq 1 - \varepsilon$$

which gives a contradiction.

Assume now that $x_i \neq \theta$ and $a_i > 0$ and put $g^n = \dfrac{f_i^n}{\|f_i^n\|_i}$. From (19) we have

$$g^n\left(\frac{x_i}{\|x_i\|_i}\right) \geq 1 - \frac{\varepsilon}{\|f_i^n\|_i \cdot \|x_i\|_i},$$

and for $n \in \mathbf{N}$ large enough we have

$$g^n\left(\frac{x_i}{\|x_i\|_i}\right) \geq 1 - \frac{2\varepsilon}{a_i \cdot \|x_i\|_i}. \tag{20}$$

On the other hand

$$\|g^n - g^m\|_i \to \frac{c_i}{a_i} \geq c_i, \tag{21}$$

when $n, m \to \infty$. Combining (20) and (21) we obtain

$$\chi_{E_i^*}\left(F^*\left(\frac{x_i}{\|x_i\|_i}, \frac{2\varepsilon}{a_i \cdot \|x_i\|_i}\right)\right) \geq \chi_{E_i^*}(\{g^n : n \in \mathbf{N}\}) \geq \frac{c_i}{2}$$

and the proof of Lemma 1 is complete.

Taking (12) with $n, m \to \infty$ we have

$$\sum_{i=1}^{n(\delta)} c_i^q \geq \gamma^q - 2^{q+1} \cdot (1 - (1 - \varepsilon - \delta)^q). \tag{22}$$

Let us put $P = \{i \in \mathbf{N} : i \leq n(\delta) \text{ and } x_i \neq \theta\}$. Then from (10) we obtain

$$\sum_{\substack{i \leq n(\delta) \\ i \notin P}} c_i^q \leq 2^q(1 - (1 - \varepsilon)^q). \tag{23}$$

Now, combining (22), (23) and (18) we can infer that

$$\sum_{i \in P} c_i^q = \sum_{i=1}^{n(\delta)} c_i^q - \sum_{\substack{i \leq n(\delta) \\ i \notin P}} c_i^q \geq \gamma^q - 2^{q+1} \cdot (1 - (1 - \varepsilon - \delta)^q) - 2^q(1 - (1 - \varepsilon)^q)$$
$$> \left(\frac{\gamma}{2}\right)^q, \quad \text{i.e.} \quad \sum_{i \in P} c_i^q > \left(\frac{\gamma}{2}\right)^q.$$

Putting $\gamma_0 = \dfrac{\gamma}{2(n(\delta))^{1/q}}$ we conclude from the above inequality that there exists $i \in P$ such that $c_i > \gamma_0 > 0$. The number i depends on ε, similarly as c_i, then we will write $i(\varepsilon)$ and $c_{i,\varepsilon}$. Because the set P is finite there exist the sequence (ε_k) converging to 0 and $j \in P$ such that $j = i(\varepsilon_k)$ and $c_{j,\varepsilon_k} > \gamma_0$ for $k = 1, 2, \ldots$ Applying Lemma 1 we obtain

$$\chi_{E_j^*}\left(F^*\left(\frac{x_j}{\|x_j\|_j}, \frac{2\varepsilon_k}{a_j \cdot \|x_j\|_j}\right)\right) \geq \frac{c_{j,\varepsilon_k}}{2} > \frac{\gamma_0}{2} > 0,$$

which implies as $k \to \infty$ that

$$\lim_{\varepsilon \to 0} \chi_{E_j^*}\left(F^*\left(\frac{x_j}{\|x_j\|_j}, \varepsilon\right)\right) \geq \frac{\gamma_0}{2} > 0,$$

but this contradicts the fact that E_j is LNUS. Thus the proof is completed.

As a simple consequence of Theorem 2 we have the following result.

THEOREM 3. *If E_i is LNUC $(i = 1, 2, \ldots)$ then $l^p(E_i)$ is LNUC.*

Proof. If E_i is LNUC then by (2) E_i^* is LNUS and by Theorem 2 $l^q(E_i^*)$ is LNUS, but $l^q(E_i^*) = (l^p(E_i))^*$ and again by (2) we have that $l^p(E_i)$ is LNUC which ends the proof.

REMARK 1. Applying similar methods as in the proofs of Theorem 1 and 2 we can obtain analogous estimates of moduli of near smoothness of the product spaces $c_0(E_i)$. We omit details.

5. Modulus of near smoothness of $l^p(l^{p_i})$. In this section we will calculate the modulus of near smoothness of the product space $l^p(l^{p_i})$, $p_i > 1$. Let us define $\bar{q} = \sup\{q, q_i : i = 1, 2, \ldots\}$, where $\dfrac{1}{p_i} + \dfrac{1}{q_i} = 1$. We will show that if $\bar{q} < \infty$ then

$$\Sigma_{l^p(l^{p_i})}(\varepsilon) = (1 - (1 - \varepsilon)^{\bar{q}})^{1/\bar{q}}, \tag{24}$$

and if $\bar{q} = \infty$ then

$$\Sigma_{l^p(l^{p_i})}(\varepsilon) = \begin{cases} 0 & \text{for} \quad \varepsilon = 0, \\ 1 & \text{for} \quad \varepsilon \in (0, 1]. \end{cases} \tag{25}$$

We start with the following fact which may be found in **[10]**. Let E be a space with the Schauder basis (e_n). Denote by R_n the n-remainder operator

$$R_n\left(\sum_{i=1}^{\infty} \beta_i e_i\right) = \sum_{i=n+1}^{\infty} \beta_i e_i.$$

In the case when $\||R_n\|| = 1$ for $n = 1, 2, \ldots$ we have

$$\chi(X) = \limsup_{n \to \infty} (\sup\{\|R_n x\| : x \in X\}), \tag{26}$$

for $X \subset E$. Recall that $(l^p(l^{p_i}))^* = l^q(l^{q_i})$, where $\frac{1}{p}+\frac{1}{q}=1$ and $\frac{1}{p_i}+\frac{1}{q_i}=1$. The norm in $l^q(l^{q_i})$ will be denoted by $\|\cdot\|_q$. If $f \in l^q(l^{q_i})$ we will write $f=(f^i)=(f^i_j)$, where $f^i \in l^{q_i}$ and $f^i_j \in \mathbf{R}$. Further, let $e_{m,k}$ denote the natural basis in $l^q(l^{q_i})$, i.e.

$$(e_{m,k})^i_j = \begin{cases} 1 & \text{for} \quad i=m \quad \text{and} \quad j=k, \\ 0 & \text{for} \quad i \neq m \quad \text{or} \quad j \neq k, \end{cases}$$

and let h denote a one-to-one mapping between $\mathbf{N}$ and $\mathbf{N}\times\mathbf{N}$. Put

$$e_n = e_{h(n)}. \tag{27}$$

Observe that (e_n) is the Schauder basis in $l^q(l^{q_i})$ and $|\!|\!| R_n |\!|\!| = 1$.

In what follows we shall need the following lemma.

LEMMA 2. *If $\bar{q}<\infty$ and (e_n) is the basis in $l^q(l^{q_i})$ defined in (27) then*

$$\|R_n f\|_q^{\bar{q}} + \|(I-R_n)f\|_q^{\bar{q}} \le \|f\|_q^{\bar{q}}, \tag{28}$$

for $f \in l^q(l^{q_i})$ and $n \in \mathbf{N}$.

Proof. Repeating the argument from the proof of the Minkowski inequality we obtain that

$$\left(\sum_{k=1}^{\infty} s_k^{1/\alpha}\right)^{\alpha} + \left(\sum_{k=1}^{\infty} t_k^{1/\alpha}\right)^{\alpha} \le \left(\sum_{k=1}^{\infty} (s_k+t_k)^{1/\alpha}\right)^{\alpha} \tag{29}$$

for $\alpha \ge 1$ and $s_k,\ t_k \ge 0$, $k=1,2,\dots$. Fix $n \in \mathbf{N}$ and $f=(f^i)=(f^i_j) \in l^q(l^{q_i})$. Putting $J_k=\{i \in \mathbf{N} : (k,i) \in h(\{1,2,\dots,n\})\}$ we get

$$\|R_n f\|_q = \left(\sum_{k=1}^{\infty}\left(\sum_{i \in \mathbf{N}\setminus J_k} |f^k_i|^{q_k}\right)^{q/q_k}\right)^{1/q},$$

$$\|(I-R_n)f\|_q = \left(\sum_{k=1}^{\infty}\left(\sum_{i \in J_k} |f^k_i|^{q_k}\right)^{q/q_k}\right)^{1/q},$$

$$\|f\|_q = \left(\sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} |f^k_i|^{q_k}\right)^{q/q_k}\right)^{1/q}.$$

Applying the inequality (29) for

$$\alpha=\frac{\bar{q}}{q}, \qquad s_k=\left(\sum_{i \in \mathbf{N}\setminus J_k} |f^k_i|^{q_k}\right)^{\bar{q}/q_k}, \qquad t_k=\left(\sum_{i \in J_k} |f^k_i|^{q_k}\right)^{\bar{q}/q_k}$$

we infer that

$$\|R_n f\|_q^{\bar{q}} + \|(I-R)f\|_q^{\bar{q}} \le \left(\sum_{k=1}^{\infty}\left(\left(\sum_{i \in \mathbf{N}\setminus J_k} |f^k_i|^{q_k}\right)^{\bar{q}/q_k} + \left(\sum_{i \in J_k} |f^k_i|^{q_k}\right)^{\bar{q}/q_k}\right)^{q/\bar{q}}\right)^{\bar{q}/q}. \tag{30}$$

Applying again (29) for $\alpha=\dfrac{\bar{q}}{q_k}$,

$$s_i = \begin{cases} 0 & \text{for } i \in J_k, \\ |f^k_i|^{q} & \text{for} \quad i \in \mathbf{N}\setminus J_k, \end{cases} \qquad t_i = \begin{cases} |f^k_i|^{\bar{q}} & \text{for} \quad i \in J_k, \\ 0 & \text{for} \quad i \in \mathbf{N}\setminus J_k, \end{cases}$$

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we get

$$\left(\sum_{i\in\mathbf{N}\setminus J_k}|f_i^k|^{q_k}\right)^{\bar{q}/q_k}+\left(\sum_{i\in J_k}|f_i^k|^{q_k}\right)^{\bar{q}/q_k}\le\left(\sum_{i\in\mathbf{N}}|f_i^k|^{q_k}\right)^{\bar{q}/q_k}$$

which together with (30) gives (28). This completes the proof of our lemma.

Now assume that $\bar{q}<\infty$. Let $x=(x^i)=(x_j^i)\in l^p(l^{p_i})$ and $\|x\|_p=1$. For $\eta>0$ there is $r\in\mathbf{N}$ such that

$$\left(\sum_{i=r+1}^{\infty}\|x^i\|_i^p\right)^{1/p}\le\eta. \tag{31}$$

Further, we can find $m\in\mathbf{N}$ such that

$$\left(\sum_{j=m+1}^{\infty}|x_j^i|^{p_i}\right)^{1/p_i}\le\frac{\eta}{r^{1/p}}\quad\text{for } i=1,2,\ldots,r. \tag{32}$$

Taking $f=(f^i)=(f_j^i)\in F^*(x,\varepsilon)$ we obtain from (31):

$$\begin{aligned}1-\varepsilon&\le\sum_{i=1}^{\infty}f^i(x^i)=\sum_{i=1}^{r}f^i(x^i)+\left(\sum_{i=r+1}^{\infty}\|f^i\|_i^q\right)^{1/q}\cdot\left(\sum_{i=r+1}^{\infty}\|x^i\|_i^p\right)^{1/p}\\&\le\sum_{i=1}^{r}f^i(x^i)+\eta,\end{aligned}$$

i.e.

$$1-\varepsilon-\eta\le\sum_{i=1}^{r}f^i(x^i).$$

Hence, in virtue of (32) we obtain

$$\begin{aligned}1-\varepsilon-\eta&\le\sum_{i=1}^{r}\left(\sum_{j=1}^{m}f_j^i\cdot x_j^i\right)+\sum_{i=1}^{r}\left(\sum_{j=m+1}^{\infty}f_j^i\cdot x_j^i\right)\\&\le\sum_{i=1}^{r}\left(\sum_{j=1}^{m}|f_j^i|^{q_i}\right)^{1/q_i}\cdot\left(\sum_{j=1}^{m}|x_j^i|^{p_i}\right)^{1/p_i}+\sum_{i=1}^{r}\left(\sum_{j=m+1}^{\infty}|f_j^i|^{q_i}\right)^{1/q_i}\cdot\left(\sum_{j=m+1}^{\infty}|x_j^i|^{p_i}\right)^{1/p_i}\\&\le\left(\sum_{i=1}^{r}\left(\sum_{j=1}^{m}|f_j^i|^{q_i}\right)^{q/q_i}\right)^{1/q}\cdot\left(\sum_{i=1}^{r}\left(\sum_{j=1}^{m}|x_j^i|^{p_i}\right)^{p/p_i}\right)^{1/p}\\&\quad+\left(\sum_{i=1}^{r}\left(\sum_{j=m+1}^{\infty}|f_j^i|^{q_i}\right)^{q/q_i}\right)^{1/q}\cdot\left(\sum_{i=1}^{r}\left(\sum_{j=m+1}^{\infty}|x_j^i|^{p_i}\right)^{p/p_i}\right)^{1/p}\\&\le\left(\sum_{i=1}^{r}\left(\sum_{j=1}^{m}|f_j^i|^{q_i}\right)^{q/q_i}\right)^{1/q}+\eta,\end{aligned}$$

i.e.

$$1-\varepsilon-2\eta\le\left(\sum_{i=1}^{r}\left(\sum_{j=1}^{m}|f_j^i|^{q_i}\right)^{q/q_i}\right)^{1/q}.$$

Next, take $s\in\mathbf{N}$ so large that

$$h(\{1,2,\ldots,s\})\supset\{1,2,\ldots,r\}\times\{1,2,\ldots,m\}.$$

For $n \geq s$ we have

$$1 - \varepsilon - 2\eta \leq \left(\sum_{i=1}^{r} \left(\sum_{j=1}^{m} |f_j^i|^{q_i}\right)^{q/q_i}\right)^{1/q} \leq \|(I - R_n)f\|_q.$$

Now, combining Lemma 2 and above inequality we derive that

$$\|R_n f\|_q \leq (1 - (1 - \varepsilon - 2\eta)^{\bar{q}})^{1/\bar{q}}$$

for $f \in F^*(x, \varepsilon)$ and $n \geq s$. This inequality together with (26) and the arbitrariness of η allows us to deduce that $\chi(F^*(x, \varepsilon)) \leq (1 - (1 - \varepsilon)^{\bar{q}})^{1/\bar{q}}$ for every $x \in S$. This implies

$$\Sigma_{l^p(l^{p_i})}(\varepsilon) \leq (1 - (1 - \varepsilon)^{\bar{q}})^{1/\bar{q}}.$$

On the other hand

$$\begin{aligned}\Sigma_{l^p(l^{p_i})}(\varepsilon) &\geq \sup\{\Sigma_{l^p}(\varepsilon), \Sigma_{l^{p_i}}(\varepsilon) : i \in \mathbf{N}\}\\ &= \sup\{(1 - (1 - \varepsilon)^q)^{1/q}, (1 - (1 - \varepsilon)^{q_i})^{1/q_i} : i \in \mathbf{N}\}\\ &= (1 - (1 - \varepsilon)^{\bar{q}})^{1/\bar{q}}\end{aligned}$$

which gives (24). Now suppose that $\bar{q} = \infty$. Taking into account the formula (1) we get

$$\Sigma_{l^p(l^{p_i})}(\varepsilon) \geq \sup\{\Sigma_{l^{p_i}}(\varepsilon) = (1 - (1 - \varepsilon)^{q_i})^{1/q_i} : i \in \mathbf{N}\} = 1$$

for $\varepsilon \in (0, 1]$. Keeping in mind that l^{p_i} is NS and the property NS is transmitted to the l^p-product of spaces, we obtain that $l^p(l^{p_i})$ is NS which means $\Sigma_{l^p(l^{p_i})}(0) = 0$ and (25) is proved.

Remark 2. Let us recall the definitions of the so-called moduli of near convexity of the space E introduced in [**7**].

$$\Delta_E(\varepsilon) = \inf\{1 - \operatorname{dist}(\theta, X) : X \subset B_E, X = \operatorname{Conv} X, \chi_E(X) \geq \varepsilon\},$$

$$\beta_E(\varepsilon) = \sup\{\chi_E(X) : X \subset B_E, X = \operatorname{Conv} X, \operatorname{dist}(\theta, X) \geq 1 - \varepsilon\}$$

for $\varepsilon \in [0, 1]$. It is easy to check that in the case of reflexivity of the space E we have $\Sigma_{E^*} = \beta_E$. This gives immediately that

$$\beta_{l^p(l^{p_i})}(\varepsilon) = (1 - (1 - \varepsilon)^{\bar{p}})^{1/\bar{p}}$$

where $\bar{p} = \sup\{p, p_i : i = 1, 2, \ldots\}$. Moreover, using the relations between Δ_E and β_E ([**7**]) we obtain the following formula

$$\Delta_{l^p(l^{p_i})}(\varepsilon) = (1 - (1 - \varepsilon^{\bar{p}})^{1/\bar{p}}.$$

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