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# A NOTE ON THE LARGE VALUES OF $|\zeta^{(\ell)}(1+it)|$

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**Abstract**

We investigate the large values of the derivatives of the Riemann zeta function $\zeta(s)$ on the 1-line. We give a larger lower bound for $\max_{t\in[T,2T]} |\zeta^{(\ell)}(1+it)|$, which improves the previous result established by Yang ['Extreme values of derivatives of the Riemann zeta function', *Mathematika* **68** (2022), 486–510].

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## 1. Introduction

The study of the extreme values of the Riemann zeta function has a long history. Extreme values on the critical line $\sigma = 1/2$ were first considered by Titchmarsh [7], who showed that there exist arbitrarily large $t$ such that for any $\alpha < 1/2$, we have $|\zeta(1/2+it)| \geqslant \exp((\log t)^{\alpha})$. For the critical strip $1/2 < \sigma < 1$, it was also Titchmarsh [6] who first showed that for any $\varepsilon > 0$ and fixed $\sigma \in (1/2, 1)$, there exist arbitrarily large $t$ such that $|\zeta(\sigma+it)| \geqslant \exp\{(\log t)^{1-\sigma-\varepsilon}\}$. The study of the values on the 1-line dates back to 1925 when Littlewood [4] showed that there exist arbitrarily large $t$ for which

$$|\zeta(1+it)| \geqslant \{1+o(1)\}e^{\gamma} \log_2 t. \tag{1.1}$$

Here and throughout, we denote by $\log_j$ the $j$th iterated logarithm and by $\gamma$ the Euler constant. We refer to [2] for a detailed account of the historical developments.

We may also consider the extreme values of the derivatives of the Riemann zeta function. For any fixed $\ell \in \mathbb{N}$, denote

$$Z^{(\ell)}(T) := \max_{t\in[T,2T]} |\zeta^{(\ell)}(1+it)|.$$

In addition to other results, Yang [8] recently proved that if $T$ is sufficiently large, then uniformly for $\ell \leqslant (\log T)/(\log_2 T)$,

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$$Z^{(\ell)}(T) \geqslant \frac{e^{\gamma}\ell^{\ell}}{(\ell+1)^{\ell+1}}\{\log_2 T - \log_3 T + O(1)\}^{\ell+1}. \tag{1.2}$$

In this note, we aim to improve the constant $\ell^{\ell}/(\ell+1)^{\ell+1}$ in (1.2). We prove the following theorem.

THEOREM 1.1. *For $T \to \infty$ and $\ell \leqslant (\log T)/(\log_2 T)$,*

$$Z^{(\ell)}(T) \geqslant \frac{e^{\gamma}}{\ell+1}(\log_2 T)^{\ell+1}\{1+o(1)\}.$$

REMARK 1.2. Compared with (1.2), we improve the lower bound by a factor $(1+1/\ell)^{\ell}$, which tends to $e$ as $\ell \to \infty$. Further, in [2], we recently used the 'long resonance' method to show that

$$\max_{t\in[\sqrt{T},T]} |\zeta(1+it)| \geqslant e^{\gamma}(\log_2 T + \log_3 T + c),$$

where $c$ is a computable constant. Granville and Soundararajan [3] predicted that this is still true for $\max_{t\in[T,2T]} |\zeta(1+it)|$. These results seems stronger than Theorem 1.1 with $\ell = 0$. The reason is that after taking derivatives of the Riemann zeta function, we are no longer able to make use of the multiplicativity of its Dirichlet coefficients. Nevertheless, Theorem 1.1 remains a generalisation of Littlewood's initial bound (1.1).

Both Yang's proof and ours employ the resonance method used by Bondarenko and Seip [1]. For a large $x$ and a positive integer $b$, we take

$$\mathcal{P} := \prod_{p\leqslant x} p^{b-1} \quad \text{and} \quad \mathcal{M} := \{n \in \mathbb{N} : n \mid \mathcal{P}\}. \tag{1.3}$$

The key ingredient of the proof is a weighted reciprocal sum of the form

$$\mathcal{S}(x;\ell) := \sum_{m\in\mathcal{M}} \sum_{k\mid m} \frac{(\log k)^{\ell}}{k},$$

where $\ell \geqslant 0$ is an integer. In [8], Yang divided $\mathcal{M}$ as well as $\mathcal{P}$ into two subsets, according to whether $p \leqslant x^{\ell/(\ell+1)}$ or not. Our choice is to give a finer division. Specifically, let $J \geqslant 1$ be a positive integer. For $0 \leqslant j \leqslant J$, denote

$$\mathcal{M}_j := \Big\{m \in \mathbb{N} : m \mid \prod_{p\leqslant x^{j/J}} p^{b-1}\Big\}. \tag{1.4}$$

Thus, we divide the set $\mathcal{M}$ into $J$ subsets:

$$\mathcal{M} = \bigsqcup_{j=1}^{J} (\mathcal{M}_j \setminus \mathcal{M}_{j-1}).$$

By this trick, we are able to enlarge the estimate of $\mathcal{S}(x;\ell)$ with a factor $(1+1/\ell)^{\ell}$. We summarise it in the following proposition.

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PROPOSITION 1.3. *With the previous notation,*

$$\frac{1}{|\mathcal{M}|}\mathcal{S}(x;\ell) \geqslant \frac{e^{\gamma}}{\ell+1}\Big\{1+O\Big(\frac{1}{J}+\frac{J\log_2 x}{b}+\frac{J^2}{\log x}\Big)\Big\}(\log x)^{\ell+1}$$

*uniformly for* $x \geqslant 3$, $b \geqslant 1$, $J \geqslant 1$, $\ell \geqslant 0$, *where the implied constant is absolute.*

It is worth noting that one cannot choose an extremely large $J$, since there are terms of the form $(J\log_2 x)/b$ and $J^2/\log x$. In fact, we will take $J$ to be the order of approximately $\log_2 x$, which seems to be the limit of this approach.

## 2. Proof of Proposition 1.3

The following asymptotic formula plays a key role in the proof of Proposition 1.3.

LEMMA 2.1. *We have*

$$\prod_{p\leqslant x}\sum_{\nu=0}^{b-1}\Big(1-\frac{\nu}{b}\Big)\frac{1}{p^{\nu}} = \Big\{1+O\Big(\frac{\log_2 x}{b}+\frac{1}{\log x}\Big)\Big\}e^{\gamma}\log x$$

*uniformly for* $x \geqslant 3$ *and* $b \geqslant 1$, *where the implied constants are absolute.*

PROOF. See also [8, (15)] and [1, page 129]. For a fixed prime $p$,

$$\begin{aligned}\sum_{\nu=0}^{b-1}\Big(1-\frac{\nu}{b}\Big)\frac{1}{p^{\nu}} &= \Big(\sum_{\nu\geqslant 0}-\sum_{\nu\geqslant b}\Big)\Big(1-\frac{\nu}{b}\Big)\frac{1}{p^{\nu}}\\ &= \Big(1-\frac{1}{b(p-1)}\Big)\Big(1-\frac{1}{p}\Big)^{-1}+O\Big(\frac{1}{p^b}\Big).\end{aligned}$$

Therefore,

$$\prod_{p\leqslant x}\sum_{\nu=0}^{b-1}\Big(1-\frac{\nu}{b}\Big)\frac{1}{p^{\nu}} = \Big\{1+O\Big(\frac{1}{b}\sum_{p\leqslant x}\frac{1}{p-1}\Big)\Big\}\prod_{p\leqslant x}\Big(1-\frac{1}{p}\Big)^{-1}.$$

The lemma follows from Mertens' formula

$$\prod_{p\leqslant x}\Big(1-\frac{1}{p}\Big)^{-1} = \Big\{1+O\Big(\frac{1}{\log x}\Big)\Big\}e^{\gamma}\log x,$$

and the fact that $\sum_{p\leqslant x}1/(p-1) \ll \log_2 x$. □

PROOF OF PROPOSITION 1.3. By the construction, the set $\mathcal{M}$ is divisor-closed which means $k \mid m$, $m \in \mathcal{M}$ implies $k \in \mathcal{M}$. By the definition of $\mathcal{M}_j$ in (1.4),

$$\sum_{m\in\mathcal{M}}\sum_{k\mid m}\frac{(\log k)^{\ell}}{k} = \sum_{j=1}^{J}\sum_{k\in\mathcal{M}_j\setminus\mathcal{M}_{j-1}}\frac{(\log k)^{\ell}}{k}\sum_{\substack{m\in\mathcal{M}\\ k\mid m}}1.$$

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Note that $k \in \mathcal{M}_j \setminus \mathcal{M}_{j-1}$ implies $k \geqslant x^{(j-1)/J}$. Therefore,

$$\sum_{m\in\mathcal{M}}\sum_{k|m}\frac{(\log k)^\ell}{k} \geqslant (\log x)^\ell \sum_{j=1}^{J}\Big(\frac{j-1}{J}\Big)^\ell \sum_{k\in\mathcal{M}_j\setminus\mathcal{M}_{j-1}} \frac{1}{k}\sum_{\substack{m\in\mathcal{M}\\ k|m}} 1. \tag{2.1}$$

For each $i$ with $0 \leqslant i \leqslant J$, we rewrite uniquely $m = m_1 m_2$ where $P_+(m_1) \leqslant x^{i/J}$ and $P_-(m_2) > x^{i/J}$. Here, $P_+(\cdot)$ and $P_-(\cdot)$ denote the largest and smallest prime factors separately, with $P_+(1) = P_-(1) = \infty$ for convenience. Then,

$$\sum_{k\in\mathcal{M}_i}\frac{1}{k}\sum_{\substack{m\in\mathcal{M}\\ k|m}} 1 = \sum_{m_1\in\mathcal{M}_i}\sum_{k|m_1}\frac{1}{k}\sum_{\substack{m_2\in\mathcal{M}\\ P_-(m_2)>x^{i/J}}} 1.$$

For the sum over $m_1$,

$$\sum_{m_1\in\mathcal{M}_i}\sum_{k|m_1}\frac{1}{k} = \prod_{p\in\mathcal{M}_i}\sum_{\substack{m_1|p^{b-1}\\ k|m_1}}\frac{1}{k} = \prod_{p\leqslant x^{i/J}}\sum_{\nu=0}^{b-1}\frac{b-\nu}{p^\nu}.$$

For the sum over $m_2$,

$$\sum_{m_2\in\mathcal{M}\setminus\mathcal{M}_i} 1 = \prod_{x^{i/J}<p\leqslant x} b.$$

Therefore, we deduce that

$$\sum_{k\in\mathcal{M}_i}\frac{1}{k}\sum_{\substack{m\in\mathcal{M}\\ k|m}} 1 = b^{\pi(x)}\prod_{p\leqslant x^{i/J}}\sum_{\nu=0}^{b-1}\frac{1}{p^\nu}\Big(1-\frac{\nu}{b}\Big).$$

Note that $b^{\pi(x)} = |\mathcal{M}|$. By Lemma 2.1,

$$\sum_{k\in\mathcal{M}_i}\frac{1}{k}\sum_{\substack{m\in\mathcal{M}\\ k|m}} 1 = \frac{i}{J}|\mathcal{M}|\Big\{1 + O\Big(\frac{\log_2 x}{b} + \frac{J}{\log x}\Big)\Big\}e^\gamma \log x. \tag{2.2}$$

In view of (2.2), by taking the difference of $\mathcal{M}_{j-1}$ and $\mathcal{M}_j$, we obtain

$$\sum_{k\in\mathcal{M}_j\setminus\mathcal{M}_{j-1}}\frac{1}{k}\sum_{\substack{m\in\mathcal{M}\\ k|m}} 1 = \frac{|\mathcal{M}|}{J}\Big\{1 + O\Big(\frac{J\log_2 x}{b} + \frac{J^2}{\log x}\Big)\Big\}e^\gamma \log x.$$

Inserting this into (2.1),

$$\sum_{m\in\mathcal{M}}\sum_{k|m}\frac{(\log k)^\ell}{k} \geqslant \frac{|\mathcal{M}|}{J}\sum_{j=1}^{J}\Big(\frac{j-1}{J}\Big)^\ell\Big\{1 + O\Big(\frac{J\log_2 x}{b} + \frac{J^2}{\log x}\Big)\Big\}e^\gamma(\log x)^{\ell+1},$$

where the implied constant is absolute.

Now Proposition 1.3 follows given that

$$\frac{1}{J}\sum_{j=1}^{J}\Big(\frac{j-1}{J}\Big)^{\ell}=\frac{1}{\ell+1}+O\Big(\frac{1}{J}\Big),$$

which is an easy consequence of the inequalities

$$\frac{1}{J}\sum_{j=1}^{J}\Big(\frac{j-1}{J}\Big)^{\ell}\leqslant\int_0^1 u^{\ell}\,du\leqslant\frac{1}{J}\sum_{j=1}^{J}\Big(\frac{j-1}{J}\Big)^{\ell}+\frac{1}{J}.$$ □

## 3. Proof of Theorem 1.1

We start with the following lemma, which helps approximate the derivatives of the Riemann zeta function by Dirichlet polynomials.

LEMMA 3.1. *For $T\to\infty$, $T\leqslant t\leqslant 2T$ and $\ell\leqslant(\log T)/(\log_2 T)$,*

$$(-1)^{\ell}\zeta^{(\ell)}(1+it)=\sum_{n\leqslant T}\frac{(\log n)^{\ell}}{n^{1+it}}+O((\log_2 T)^{\ell}),$$

*where the implied constant is absolute.*

PROOF. This is [8, Lemma 1], where we have taken $\sigma=1$ and $\varepsilon=(\log_2 T)^{-1}$ as Yang did. See also [7, Theorem 4.11]. □

PROOF OF THEOREM 1.1. To employ the resonance method, we choose the same weight function $\phi(\cdot)$ as that used by Soundararajan [5, page 471]. Thus, let $\phi(t)$ be a smooth function compactly supported in $[1,2]$, such that $0\leqslant\phi(t)\leqslant 1$ always and $\phi(t)=1$ for $t\in(5/4,7/4)$. Then the Fourier transform of $\phi$ satisfies $\widehat{\phi}(u)\ll_{\alpha}|u|^{-\alpha}$ for any integer $\alpha\geqslant 1$.

For sufficiently large $T$, we set

$$x=\frac{\log T}{3\log_2 T}\quad\text{and}\quad b=\lfloor\log_2 T\rfloor.$$

Furthermore, we take $\mathcal{P}$ and $\mathcal{M}$ as in (1.3). Note that $\mathcal{P}\leqslant\sqrt{T}$ by the prime number theorem. Then we define the resonator

$$R(t):=\sum_{m\in\mathcal{M}}m^{it}.$$

Denote

$$M_1(R,T):=\int_{\mathbb{R}}|R(t)|^2\phi\Big(\frac{t}{T}\Big)\,dt,$$

$$M_2(R,T):=\int_{\mathbb{R}}(-1)^{\ell}\zeta^{(\ell)}(1+it)|R(t)|^2\phi\Big(\frac{t}{T}\Big)\,dt.$$

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Since $\mathrm{supp}(\phi) \subset [1,2]$,

$$Z^{(\ell)}(T) \geqslant \frac{|M_2(R,T)|}{M_1(R,T)}. \tag{3.1}$$

For $M_1(R,T)$,

$$M_1(R,T) = \sum_{m,n\in\mathcal{M}} \int_{\mathbb{R}} \left(\frac{m}{n}\right)^{it} \phi\left(\frac{t}{T}\right) dt = T \sum_{m,n\in\mathcal{M}} \widehat{\phi}(T\log(n/m)).$$

When $m \neq n$, the choice of $\mathcal{P}$ guarantees that $|\log(n/m)| \gg 1/\sqrt{T}$, and consequently

$$\widehat{\phi}(T\log(n/m)) \ll \frac{1}{T^2}. \tag{3.2}$$

Thus, the off-diagonal terms contribute

$$T \sum_{\substack{m,n\in\mathcal{M}\\ m\neq n}} \widehat{\phi}(T\log(n/m)) \ll \frac{1}{T}|\mathcal{M}|^2.$$

Therefore,

$$M_1(R,T) = T\widehat{\phi}(0)|\mathcal{M}| + O\left(\frac{1}{T}|\mathcal{M}|^2\right). \tag{3.3}$$

For $M_2(R,T)$, by Lemma 3.1,

$$\begin{aligned} M_2(R,T) &= \int_{\mathbb{R}} \left(\sum_{k\leqslant T} \frac{(\log k)^\ell}{k^{1+it}}\right)|R(t)|^2 \phi\left(\frac{t}{T}\right) dt + O((\log_2 T)^\ell M_1(R,T)) \\ &= T \sum_{k\leqslant T} \frac{(\log k)^\ell}{k} \sum_{m,n\in\mathcal{M}} \widehat{\phi}(T\log(kn/m)) + O((\log_2 T)^\ell M_1(R,T)). \end{aligned} \tag{3.4}$$

Similar to (3.2), for $kn \neq m$,

$$\widehat{\phi}(T\log(kn/m)) \ll \frac{1}{T^2}.$$

Consequently,

$$\sum_{k\leqslant T} \frac{(\log k)^\ell}{k} \sum_{\substack{m,n\in\mathcal{M}\\ kn\neq m}} \widehat{\phi}(T\log(kn/m)) \ll \frac{(\log T)^{\ell+1}}{T^2}|\mathcal{M}|^2.$$

Inserting this into (3.4),

$$M_2(R,T) = T\widehat{\phi}(0) \sum_{m\in\mathcal{M}} \sum_{k|m} \frac{(\log k)^\ell}{k} + O\left(\frac{(\log T)^{\ell+1}}{T}|\mathcal{M}|^2\right) + O((\log_2 T)^\ell M_1(R,T)).$$

Combining this with (3.1) and (3.3),

$$Z^{(\ell)}(T) \geqslant \frac{1}{|\mathcal{M}|} \sum_{m\in\mathcal{M}} \sum_{k|m} \frac{(\log k)^\ell}{k} + O((\log_2 T)^\ell).$$

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By taking $J = \lfloor \frac{1}{2}\log_3 T \rfloor$ in Proposition 1.3, we deduce that

$$\frac{1}{|\mathcal{M}|}\sum_{m\in\mathcal{M}}\sum_{k|m}\frac{(\log k)^\ell}{k} \geqslant \frac{e^\gamma}{\ell+1}\{1+o(1)\}(\log x)^{\ell+1}.$$

The theorem follows by recalling that $x = (\log T)/(3\log_2 T)$. □

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