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DENSE Q-SUBALGEBRAS OF BANACH AND C^*-ALGEBRAS AND UNBOUNDED DERIVATIONS OF BANACH AND C^*-ALGEBRAS

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The paper studies dense Q-subalgebras of Banach and C^*-algebras. It proves that the domain $D(\delta)$ of a closed unbounded derivation δ of a Banach unital algebra A automatically contains the identity and is a Q-subalgebra of A, so that $Sp_A(x)=Sp_{D(\delta)}(x)$ for all $x\in D(\delta)$. The paper shows that every finite-dimensional semisimple representation of a Q-subalgebra is continuous. It also shows that if π is an injective $*$-homomorphism of a dense locally normal Q^*-subalgebra B of a C^*-algebra, then $\|x\|\leqq\|\pi(x)\|$ for all $x\in B$. The paper studies the link between closed ideals of a Banach algebra A and of its dense subalgebra B. In particular, if A is a C^*-algebra and B is a locally normal $*$-subalgebra of A, then $I\to I\cap B$ is a one-to-one mapping of the set of all closed two-sided ideals in A onto the set of all closed two-sided ideals in B and $I=\overline{I\cap B}$.

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1. Introduction

The paper studies normed Q-algebras, their representations and the structure of their ideals. It establishes that the domains of unbounded derivations of Banach algebras are Q-algebras.

A topological algebra B with identity is said to be a *Q-algebra* if the group G_B of all invertible elements in B is open in B. Much work has been done on the theory of locally multiplicatively-convex Q-algebras (see, for example [**6, 7, 11**]). In this paper we mainly concentrate on the study of normed Q-algebras. If B is a normed algebra, then its completion is an algebra with continuous inverse. Therefore B is an algebra with continuous inverse if and only if B is a Q-algebra.

Section 2 investigates the domains $D(\delta)$ of closed unbounded derivations δ of Banach algebras A with identity. Bratteli and Robinson [**4**] proved that if A is a C^*-algebra and if δ is a $*$-derivation, then $1\in D(\delta)$. Theorem 4 shows that this holds for any closed derivation of a Banach algebra with identity. Theorem 5 establishes that the domains of closed unbounded derivations of Banach algebras A are Q-algebras and therefore $Sp_A(x)=Sp_{D(\delta)}(x)$ for all $x\in D(\delta)$ (the case when A is a C^*-algebra and δ is a closed $*$-derivation was considered in [**3**] and [**9**]).

In [**7**, Theorem 2.12] it was proved that any one-dimensional representation (multiplicative linear functional) of a topological Q-algebra with identity is continuous. In [**10**] it was shown that every finite-dimensional irreducible representation π of a normed Q-algebra is continuous. Theorem 6 extends this result to the case when π is a finite-dimensional semisimple representation of a locally multiplicatively-convex (lmc) Q-algebra.

If π is an injective $*$-homomorphism of a C^*-algebra A into a $*$-normed algebra, it is well-known (1.8.1 of [5]) that $\|x\| \leqq \|\pi(x)\|$ for all $x \in A$. Fragoulopoulou [6, Theorem 3.9] extended this result to the case when π is an injective $*$-homomorphism of a complete lmc C^*-algebra (pro-C^*-algebra) B into an lmc $*$-algebra $\mathscr{A}$. She proved that if every selfadjoint element of B has a compact spectrum and if the closure of $Im(\pi)$ is a Q^*-subalgebra of $\mathscr{A}$, then $\pi^{-1}|_{Im(\pi)}$ is continuous. Theorem 8 shows that if π is an injective $*$-homomorphism of a dense Q^*-subalgebra B of a C^*-algebra A and, in addition, B is a *locally normal* subalgebra of A, then $\|\pi(x)\| \geqq \|x\|$ for all $x \in B$.

Theorem 13 extends the result of Sonis [15] (cf. [13]) about the homeomorphism of the spaces of maximal ideals of commutative Banach algebras A and B which form a Wiener pair, to the case when A is a C^*-algebra and B is a dense locally normal subalgebra of A. It proves that the mapping $i_B: I \to I \cap B$ is a one-to-one mapping of the set of all closed two-sided ideals in A onto the set of all closed two-sided ideals in B. Furthermore i_B maps the set of all *maximal* ideals in A onto the set of all *maximal* ideals in B.

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2. Normed Q-algebras and the domains of unbounded derivations

Let B be a normed algebra with identity and let $x \in B$. If $Sp_B(x)$ is the spectrum of x in B and if $\lambda \in Sp_B(x)$, then $\lambda^n \in Sp_B(x^n)$. Therefore the spectral radius $r_B(x) = \sup_{\lambda \in Sp_B(x)} |\lambda|$ has the following property:

$$r_B(x)^n \leqq r_B(x^n). \tag{1}$$

If A is the completion of B with respect to the norm, then

$$r_A(x) = \lim_{k \to \infty} \sqrt[k]{\|x^k\|} \leqq \|x\| \text{ and } r_A(x)^n = r_A(x^n), \quad x \in A, \tag{2}$$

and

$$r_A(x) \leqq r_B(x), \quad x \in B.$$

The following theorem describes normed Q-algebras in terms of the spectrum and the spectral radii of their elements.

Theorem 1. *Let B be a normed algebra with an identity* $\mathbf{1}$ *and let A be its completion. Then the following conditions are equivalent:*

(i) *B is a Q-algebra;*

(ii) *there exists $\alpha > 0$ such that $\|x\| < \alpha$ implies $\mathbf{1}+x$ has the inverse in B;*

(iii) *every element $\mathbf{1}+x$, $\|x\| < 1$, has the inverse in B;*

(iv) *$r_B(x) \leqq \|x\|$ for all $x \in B$;*

(v) *there exists $d > 0$ such that $r_B(x) \leqq d\|x\|$ for all $x \in B$;*

(vi) *if J is a left (right, two-sided) ideal in B, then its closure $\bar{J}$ is a left (right, two-sided) ideal in A;*

(vii) *if an element x in B has no left (right) inverse in B, then x has no left (right) inverse in A;*

(viii) *for every x in B, $Sp_A(x) = Sp_B(x)$, i.e., $G_B = B \cap G_A$.*

Proof. (iv)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Leftrightarrow$(i) follow easily.

(ii)$\Rightarrow$(v). If $x \in B$ and if $\alpha^{-1}\|x\| < \lambda$, then the element $x + \lambda\mathbf{1} = \lambda(\mathbf{1} + x/\lambda)$ has the inverse in B, since $\|x/\lambda\| < \alpha$. Therefore $r_B(x) \leqq \alpha^{-1}\|x\|$ and (v) holds for $d = \alpha^{-1}$.

(v)$\Rightarrow$(iv). Using (1), we obtain for x^n

$$r_B(x)^n \leqq r_B(x^n) \leqq d\|x^n\| \leqq d\|x\|^n.$$

Therefore $r_B(x) \leqq d^{1/n}\|x\|$ for all n. Hence $r_B(x) \leqq \|x\|$.

(iii)$\Rightarrow$(vi). In order to prove (vi) it is sufficient to show that $\mathbf{1}$ does not belong to $\bar{J}$. If $\mathbf{1} \in \bar{J}$, then there exist $\{x_n\}$ in J such that $x_n \to \mathbf{1}$. By (iii), all x_n such that $\|\mathbf{1} - x_n\| < 1$ are invertible in B. Therefore $\mathbf{1} \in J$. This contradiction proves (vi).

(vi)$\Rightarrow$(vii). If x has no left inverse in B, then x belongs to a left ideal J of B. By (vi), the closure $\bar{J}$ of J is a left ideal of A. Since $x \in \bar{J}$, it has no left inverse in A. Part (vii) is proved.

Part (i) follows immediately from (vii) and (vii)$\Leftrightarrow$(viii). The theorem is proved.

Remark. The conditions (i), (vi), (vii) and (viii) are equivalent for any topological algebra with continuous inverse ([13]). Part (iv) follows immediately from the general formula for the spectral radius in locally multiplicatively-convex Q-algebras [11].

Lemma 2. *Let B be a normed Q-algebra with identity and let A be the completion of B.*

(i) *If D is a Banach subalgebra of A and if $\mathbf{1} \in D$, then $D \cap B$ is a Q-algebra.*

(ii) *Let J be a two-sided ideal in B, let I be its closure in A and let ϕ be the canonical homomorphism of A onto the quotient algebra $\hat{A} = A/I$. Then $\phi(B)$ is a Q-algebra.*

Proof. Let $x \in D \cap B$ and let $\|x\| < 1$. By Theorem 1(iii), $\mathbf{1}+x$ has the inverse in B. Since D is a Banach algebra, $\mathbf{1}+x$ is also invertible in D. Therefore $(\mathbf{1}+x)^{-1} \in D \cap B$ and, by Theorem 1(iii), $D \cap B$ is a Q-algebra. Part (i) is proved.

The algebra $\phi(B)$ is dense in the Banach algebra $\hat{A}$. Let x be an element in B such that $\|\phi(x)\|<1$. Then there is an element y in I such that $\|x+y\|<1$. Since J is dense in I, there exists an element z in J such that $\|x+z\|<1$. Then $x+z\in B$ and, by Theorem 1 the element $\mathbf{1}+(x+z)$ is invertile in B. Therefore the element $\phi(\mathbf{1}+x+z)=\phi(\mathbf{1})+\phi(x)$ is invertible in $\phi(B)$. By Theorem 1, the $\phi(B)$ is a Q-algebra. The lemma is proved.

Let A and B ($A\supset B$) be Banach algebras with common identity and with norms $\|x\|$ and $\|x\|_1$ respectively. They form a *Wiener pair* (see [**13, 15, 16**]) if every element x in B which is invertible in A is also invertible in B, i.e. $Sp_A(x)=Sp_B(x)$ for every $x\in B$.

Lemma 3. *Let A and B ($A\supset B$) be Banach algebras with common identity and with norms $\|x\|$ and $\|x\|_1$ respectively. If there exist $d>0$ and a function $F(x)$ on B such that*

$$\|x^k\|_1\leqq F(x)d^k\|x\|^k,\quad x\in B,$$

then B is a Q-algebra with respect to the norm $\|\ \|$. If, in addition, B is dense in A with respect to the norm $\|\ \|$, then A and B form a Wiener pair.

Proof. The spectral radius of every element x in B does not depend on the norm on B. Since B is a Banach algebra with respect to the norm $\|\ \|_1$, it follows from (2) that

$$r_B(x)=\lim_{k\to\infty}\sqrt[k]{\|x^k\|_1}\leqq d\|x\|\lim_{k\to\infty}\sqrt[k]{F(x)}=d\|x\|.$$

From Theorem 1(v) it follows that B is a Q-algebra with respect to the norm $\|\ \|$. If B is dense in A, it follows from Theorem 1(viii) that A and B form a Wiener pair.

We shall now show that if δ is a closed derivation of a Banach algebra with an identity $\mathbf{1}$, then the domain $D(\delta)$ of δ contains $\mathbf{1}$ and is a Q-algebra. Applying the results of Theorem 1, we obtain that $Sp_A(x)=Sp_{D(\delta)}(x)$ for all $x\in D(\delta)$.

Let A be a Banach algebra. A closed derivation δ of A is a linear mapping from a dense subalgebra $D(\delta)$ of A into A such that

(i) $\delta(ab)=\delta(a)b+a\delta(b)$, $a,b\in D(\delta)$,

(ii) $a_n\in D(\delta)$, $a_n\to a$ and $\delta(a_n)\to b$ implies $a\in D(\delta)$ and $\delta(a)=b$.

If A is a Banach *-algebra and if, in addition, $x\in D(\delta)$ implies $x^*\in D(\delta)$ and $\delta(x^*)=\delta(x)^*$, then δ is a closed *-derivation. For every $n<\infty$, set

$$D(\delta^n)=\{x\in D(\delta):\delta^k(x)\in D(\delta)\text{ for }1\leqq k\leqq n-1\}.$$

and set

$$D(\delta^\infty)=\bigcap_{n=1}^{\infty}D(\delta^n).$$

It is easy to check that all $D(\delta^n)$ are subalgebras of A. Every $D(\delta^n)$, $1\leqq n<\infty$, is a Banach algebra with respect to the norm

$$\|x\|_n = \sum_{k=0}^{n} \|\delta^k(x)\|, \quad x \in D(\delta^n)$$

and $D(\delta^\infty)$ is a complete locally multiplicative-convex algebra.

If A contains an identity **1**, it is not "a priori" evident whether **1** is automatically included in $D(\delta)$. Bratteli and Robinson [4] proved that if δ is a closed *-derivation of a C^*-algebra A, then $\mathbf{1} \in D(\delta)$. In the following theorem we show that this holds for any closed derivation of a Banach algebra with identity.

Theorem 4. *Let A be a Banach algebra with an identity* **1** *and let δ be a closed derivation of A. Then $\mathbf{1} \in D(\delta)$.*

Proof. Let y be an element in $D(\delta)$ such that

$$\|\mathbf{1} - y\| = \varepsilon < 1.$$

Set

$$x_n = \mathbf{1} - (\mathbf{1} - y)^n = \sum_{k=1}^{n} C_n^k (-1)^{k+1} y^k,$$

where C_n^k denotes the usual binomial coefficient. Then $x_n \to \mathbf{1}$ and $x_n \in D(\delta)$. We shall show that $\delta(x_n) \to 0$. We have that

$$x_{n+1} = \mathbf{1} - (\mathbf{1} - y)(\mathbf{1} - x_n) = y + x_n - yx_n$$

and

$$\delta(x_{n+1}) = \delta(y) + \delta(x_n) - y\delta(x_n) - \delta(y)x_n = \delta(y)(\mathbf{1} - x_n) + (\mathbf{1} - y)\delta(x_n).$$

Therefore

$$\|\delta(x_{n+1})\| \leqq \|\delta(y)\| \, \|\mathbf{1} - x_n\| + \|\mathbf{1} - y\| \, \|\delta(x_n)\| \leqq \varepsilon^n \|\delta(y)\| + \varepsilon \|\delta(x_n)\|.$$

Set $t_n = \|\delta(x_n)\|$ and $C = \|\delta(y)\|$. Then we have that

$$t_{n+1} \leqq C\varepsilon^n + \varepsilon t_n.$$

By induction,

$$t_{n+1} \leqq n\varepsilon^n C + \varepsilon^n t_1 \to 0.$$

Therefore $\delta(x_n) \to 0$ and, since δ is closed, $\mathbf{1} \in D(\delta)$. The theorem is proved.

In [3] and [9] it was proved that if δ is a closed *-derivation of a C^*-algebra A with identity, then $Sp_A(x) = Sp_{D(\delta)}(x)$ for all $x \in D(\delta)$, so that A and $D(\delta)$ form a Wiener pair.

The following theorem extends this result to the case when A is a Banach algebra with identity and δ is a closed derivation of A.

Theorem 5. *Let A be a Banach algebra with an identity* **1**. *If δ is a closed derivation of A, then the algebras $D(\delta^n)$, $1 \leqq n \leqq \infty$, are Q-algebras. For all $x \in D(\delta)$, $Sp_A(x) = Sp_{D(\delta)}(x)$. For every $1 < n \leqq \infty$ such that $D(\delta^n)$ is dense in A, $Sp_A(x) = Sp_{D(\delta^n)}(x)$ for all $x \in D(\delta^n)$.*

Proof. By Theorem 4, $\mathbf{1} \in D(\delta)$. Therefore $\mathbf{1} \in D(\delta^n)$ for all n. Since

$$\delta(x^k) = \sum_{p=0}^{k} C_k^p x^p \delta(x) x^{k-p}, \tag{3}$$

we have that

$$\|\delta(x^k)\| \leqq \|x\|^k \|\delta(x)\| \sum_{p=0}^{k} C_k^p = 2^k \|x\|^k \|\delta(x)\|. \tag{4}$$

Therefore

$$\|x^k\|_1 = \|x^k\| + \|\delta(x^k)\| \leqq \|x\|^k (1 + 2^k \|\delta(x)\|).$$

Hence, by Lemma 3, $D(\delta)$ is a Q-algebra.

For every $x, y \in D(\delta)$,

$$\delta(x^p y x^{k-p}) = \delta(x^p) y x^{k-p} + x^p \delta(y) x^{k-p} + x^p y \delta(x^{k-p}).$$

By (4), for every k

$$\|\delta(x^p y x^{k-p})\| \leqq 2^p \|x\|^p \|\delta(x)\| \, \|y\| \, \|x\|^{k-p} + \|x\|^k \|\delta(y)\|$$
$$+ \|x\|^p \|y\| 2^{k-p} \|x\|^{k-p} \|\delta(x)\| \leqq 2^k \|x\|^k G(x, y),$$

where $G(x, y) = 2\|\delta(x)\| \, \|y\| + \|\delta(y)\|$. Therefore, by (3),

$$\|\delta^2(x^k)\| = \|\delta(\delta(x^k))\| \leqq \sum_{p=0}^{k} C_k^p \|\delta(x^p \delta(x) x^{k-p})\|$$

$$\leqq \sum_{p=0}^{k} C_k^p 2^k \|x\|^k G(x, \delta(x)) = 4^k \|x\|^k G(x, \delta(x)).$$

Hence

$$\|x^k\|_2 = \|x^k\| + \|\delta(x^k)\| + \|\delta^2(x^k)\| \leqq \|x\|^k (1 + 2^k \|\delta(x)\|$$

$$+4^k G(x,\delta(x))) \leqq 4^k \|x\|^k (1+\|\delta(x)\| + G(x,\delta(x))).$$

It follows from Lemma 3 that $D(\delta^2)$ is a Q-algebra. Continuing this process we prove that all the algebras $D(\delta^n)$, $1 \leqq n < \infty$, are Q-algebras. From Theorem 1(iii) it follows that $D(\delta^\infty)$ is also a Q-algebra. From Theorem 1(viii) it follows that if $D(\delta^n)$ is dense in A, then $Sp_A(x) = Sp_{D(\delta^n)}(x)$ for all $x \in D(\delta^n)$. The proof is complete.

Remark. If A is a Banach algebra without the identity, it can be embedded in a canonical fashion in a larger Banach algebra $\hat{A} = A + \mathbb{C}\mathbf{1}$ with the identity. One may then extend a closed derivation δ of A to a closed derivation $\hat{\delta}$ of $\hat{A}$ by setting $D(\hat{\delta}) = D(\delta) + \mathbb{C}\mathbf{1}$ and

$$\hat{\delta}(x + t\mathbf{1}) = \delta(x), \quad x \in D(\delta),\ t \in \mathbb{C}.$$

By Theorem 5, the algebras $\hat{A}$ and $D(\hat{\delta})$ form a Wiener pair, so that $Sp_{\hat{A}}(x) = Sp_{D(\hat{\delta}^n)}(x)$ for all $x \in D(\hat{\delta}^n)$.

3. Representations of Q-subalgebras of Banach and C^*-algebras

Let B be a normed Q-algebra and let A be the completion of B. It follows from Theorem 2.2 of **[10]** that every finite-dimensional irreducible representation π of B is bounded and therefore extends to a representation of A. The following theorem gives a simple and different proof of this statement in the general case when π is a finite-dimensional semisimple representation of a metrizable locally multiplicatively-convex (lmc) Q-algebra.

Theorem 6. *Let B be a metrizable topological algebra and let π be a finite-dimensional semisimple representation of B on H, i.e., the algebra $\pi(B)$ is semisimple. If the spectral radius $r_B(x)$ is continuous at $x = 0$, then π is continuous. In particular, if B is a metrizable lmc Q-algebra, then π is continuous.*

Proof. By contradiction. Let there exist $\{x_n\}$ in B such that $x_n \to 0$ and $\pi(x_n)$ do not converge to 0. We can always assume that $\|\pi(x_n)\| \leqq 1$. Since H is finite-dimensional, the unit ball in $B(H)$ is compact and we can assume that $\pi(x_n)$ converge to $a \neq 0$ in $B(H)$. Since $\pi(B)$ is finite-dimensional, it is closed and there exists $y \in B$ such that $\pi(y) = a$. Then, for every z in B,

$$r_{B(H)}(\pi(z)\pi(x_n)) = r_{B(H)}(\pi(zx_n)) \leqq r_B(zx_n).$$

Since $x_n \to 0$, we have that $zx_n \to 0$. Since, by assumption, $r_B(x)$ is continuous at $x = 0$, $r_B(zx_n) \to 0$. Therefore $r_{B(H)}(\pi(z)\pi(x_n)) \to 0$. We also have that $\pi(z)\pi(x_n) \to \pi(z)\pi(y)$. In a finite-dimensional space the spectral radius of a matrix is the maximum of its eigenvalues. Therefore the spectral radius is a continuous function, so that

$$r_{B(H)}(\pi(z)\pi(y)) = \lim_{n\to\infty} r_{B(H)}(\pi(z)\pi(x_n)) = 0.$$

Hence $\mathbf{1}+\pi(z)\pi(y)$ is invertible for any z in B, so that $\pi(y)$ belongs to the radical of $\pi(B)$. Since $\pi(B)$ is semisimple, $\pi(y)=a=0$. This contradiction proves that π is continuous. In [**11**, Prop. III.6.2] it was shown that $r_B(x)$ is continuous at $x=0$ if B is an lmc Q-algebra. The proof is complete.

Assume now that A is a Banach $*$-algebra, that B is a dense $*$-subalgebra of A and that B is a Q-algebra. We shall call B a Q^*-subalgebra. It follows from Theorem 6 that every finite-dimensional semisimple representation of B on H extends to a bounded representation of A on H. However, B may have infinite dimensional irreducible representations which do not extend to A. An example of such a Q^*-algebra B was considered in [**9**] where A was a C^*-algebra of operators on a Hilbert space and where B was the domain $D(\delta)$ of a closed $*$-derivation δ of A implemented by a selfadjoint operator.

As far as $*$-homomorphisms of B into C^*-algebras are concerned, they all extend to A. This follows from the fact that

$$\|\pi(x)\|^2 = \|\pi(x^*x)\| = r(\pi(x^*x)) \leqq r_B(x^*x) \leqq \|x^*x\| \leqq \|x\|^2,$$

since, by Theorem 1(iv), $r_B(x^*x) \leqq \|x^*x\|$.

In fact, Fragoulopoulou [**6**, Theorem 3.1] proved that every $*$-morphism of an lmc Q^*-algebra B into an lmc C^*-algebra (an involutive topological algebra whose topology is defined by a direct family of C^*-seminorms) is continuous.

If π is an injective $*$-homomorphism of a C^*-algebra into a $*$-normed algebra, then (see 1.8.1 of [**5**]) $\|x\| \leqq \|\pi(x)\|$ for all $x \in A$. Theorem 8 studies the case when π is an injective $*$-homomorphism of a dense Q^*-subalgebra B of a C^*-algebra. Using the approach of [**5**], it proves that if B is *locally normal* (all the domains $D(\delta)$ of closed $*$-derivations of C^*-algebras are locally normal), then $\|x\| \leqq \|\pi(x)\|$ for all $x \in B$. In order to prove this we need to consider normal families of functions on topological spaces.

Recall that a family F of functions on a topological space X is said to be *normal* (see, for example, [**10**, §15]) if for any disjoint closed subsets S and T in X, there exists a function $f \in F$ such that

$$f(x)=0 \text{ on } T \quad \text{and} \quad f(x)=1 \text{ on } S.$$

Lemma 7. *Let F be a normal family of functions on a normal topological space X. If T is a closed nontrivial subset of X, there are functions f and g in F such that*

$$f(x)g(x)=0 \textit{ for all } x \in X,\ f \neq 0 \textit{ and } g(x)=1 \textit{ for all } x \in T.$$

Proof. Let $x_0 \notin T$. Since X is normal, there exist open subsets W and V in X such that $x_0 \in W$, $T \subset V$ and that $W \cap V = \varnothing$. The complements W^c of W and V^c of V are closed subsets and $x_0 \notin W^c$ and $T \cap V^c = \emptyset$. Therefore there exist functions f and g in F such that

$$f(x_0)=1 \text{ and } f(x)=0 \text{ on } W^c, \; g(x)=0 \text{ on } V^c \text{ and } g(x)=1 \text{ on } T.$$

Then $f(x)g(x)=0$ for all $x \in X$, since $W^c \cup V^c = (W \cap V)^c = X$. The lemma is proved.

Definition. Let B be a dense subalgebra of a Banach algebra A with an identity $\mathbf{1}$ and let $\mathbf{1} \in B$.

(1) Let A be a commutative. The algebra B is said to be *normal* if the algebra of functions $\{x(s): x \in B\}$ is normal on the space S of all maximal ideals of A.

(2) The algebra B is said to be *locally normal* if for every $x \in B$, there is a commutative Banach subalgebra $A(x)$ in A which contains $\mathbf{1}$ and x and such that $B(x) = B \cap A(x)$ is a dense normal subalgebra of $A(x)$.

(3) If, in addition, A and B are $*$-algebras, then B is said to be *locally normal* if for every selfadjoint $x \in B$, there is a commutative Banach $*$-subalgebra $A(x)$ in A such that $\mathbf{1}$ and x belong to $A(x)$ and such that $B(x) = B \cap A(x)$ is a dense normal subalgebra of $A(x)$.

Remark. Commutative algebras can be normal without being Q-algebras and vice versa. Let $A = C([0,1])$ be the algebra of all continuous functions on $[0,1]$.

(1) The subalgebra B of all piecewise polynomial functions is a dense normal subalgebra of A but it is not a Q-algebra.

2. The subalgebra B of all rational functions is a Q-algebra but it is not normal.

Theorem 8. *Let B be a dense locally normal Q^*-subalgebra of a C^*-algebra A with an identity* $\mathbf{1}$. *If π is an injective $*$-homomorphism of B into a Banach $*$-algebra $\mathscr{A}$, then $\|x\| \leqq \|\pi(x)\|$ for all $x \in B$. If $\mathscr{A}$ is a C^*-algebra, then $\|x\| = \|\pi(x)\|$.*

Proof. Let x be a selfadjoint element in B and let $A(x)$ be a commutative C^*-subalgebra of A which contains $\mathbf{1}$ and x and such that $B(x) = B \cap A(x)$ is a dense normal subalgebra of $A(x)$. Then $\pi(B(x))$ is a commutative $*$-subalgebra of $\mathscr{A}$. Let R be a maximal commutative $*$-subalgebra of $\mathscr{A}$ such that $\pi(B(x)) \subseteq R$. Let S be the space of all maximal ideals of $A(x)$ and let T be the space of all maximal ideals of R. If $t \in T$, then $\pi(z)(t)$, $z \in B(x)$, is a one-dimensional representation of $B(x)$. Since B is a Q-algebra, it follows from Lemma 2 that $B(x)$ is also a Q-algebra. Hence, by Theorem 6, $\pi(z)(t)$ extends to a one-dimensional representation of $A(x)$. Therefore there exists a mapping φ of T into S such that for every $z \in B(x)$,

$$z(\varphi(t)) = \pi(z)(t), \quad t \in T \tag{5}$$

Since $\mathbf{1} \in B(x)$ and since $B(x)$ is dense in $A(x)$, the algebra of functions $Z_S = \{z(s): z \in B(x)\}$ on the compact space S separates points of S and there are no points in S

where all functions from Z_S vanish. It follows from [13, §2, 11, II] that the weak topology on S defined by Z_S coincides with the initial topology on S. By (5), all the functions $z(\varphi(t))$, $z \in B(x)$, are continuous on T. Therefore φ is a continuous mapping of T onto a compact subspace $\hat{S}$ of S.

The rest of the proof of the theorem is the same as the proof of 1.8.1 of [5] and we shall bring it here in order to be self contained. Suppose that $\hat{S} \neq S$. Since $B(x)$ is normal, it follows from Lemma 6 that there are elements $u \neq 0$ and $v \neq 0$ in $B(x)$ such that the corresponding functions $u(s)$ and $v(s)$ satisfy the conditions:

$$u(s)v(s) = 0 \text{ on } S \quad \text{and} \quad v(s) = 1 \text{ on } \hat{S}.$$

Therefore uv is quasinilpotent. Since $A(x)$ is a C^*-algebra, $uv = 0$ and, by (5), $\pi(v)(t) = v(\varphi(t)) = 1$ for all $t \in T$, since $\varphi(t) \in \hat{S}$. Then $\pi(u)\pi(v) = \pi(uv) = 0$ and $\pi(v)$ has the inverse in R. Therefore $\pi(u) = 0$ which contradicts the assumption that π is injective. Thus $\hat{S} = S$. From this and from (5) it follows that

$$\|x\| = \sup_{s \in S} |x(s)| = \sup_{t \in T} |\pi(x)(t)| \leqq \|\pi(x)\|.$$

Then for every x in B,

$$\|x\|^2 = \|x^*x\| \leqq \|\pi(x^*x)\| = \|\pi(x)^*\pi(x)\| \leqq \|\pi(x)\|^2.$$

If, in addition, $\mathscr{A}$ is a C^*-algebra, then $\|\pi(x)\| \leqq \|x\|$, so that $\|x\| = \|\pi(x)\|$. The proof is complete.

Remarks. (1) The condition that B is locally normal cannot be omitted. If $A = C([0, 1])$ and if B is the subalgebra of all rational functions, then B is a Q-algebra but it is not normal. The mapping

$$\pi: f \in B \to f|_{[0, 1/2]} \text{ (the restriction of } f \text{ to } [0, 1/2])$$

is an injective homomorphism of B into $C([0, 1/2])$. However, the condition: $\|\pi(f)\| \geqq \|f\|$ for all $f \in B$, does not hold.

(2) Similarly, the condition that B is a Q^*-subalgebra of A can also not be omitted. Let $A = C([0, 1])$, let B be the normal $*$-subalgebra of piecewise polynomial functions (which is not a Q^*-subalgebra of A) and let $\mathscr{A}$ be the C^*-algebra of bounded functions on $[0, 1)$. For $f \in B$ and $t \in [0, 1)$, let P_{f_t} be the unique polynomial which coincides with f on $(t, t + \varepsilon)$ for some $\varepsilon > 0$. Define $\pi: B \to A$ by:

$$\pi(f)(t) = P_{f_t}(t - 1).$$

This is an injective $*$-homomorphism. If $f(t) = (t + 1)^2$, then $\pi(f)(t) = t^2$ and $\|\pi(f)\| = 1 \leqq \|f\| = 4$.

(3) Although the condition that B is a Q^*-subalgebra of A cannot be omitted in Theorem 8, it can be exchanged for the condition that *B is closed under square roots of strictly positive elements*. The only place where we use the assumption that B is a Q^*-subalgebra is to deduce that a hermitian multiplicative linear functional $\phi(z)=\pi(z)(t)$ on $B(x)$ is continuous. Suppose $y \in B(x)$, $\|y\| \leqq 1$, and $\varepsilon > 0$. If B is closed under square roots of strictly positive elements, then $z=((1+\varepsilon)\mathbf{1}-y^*y)^{1/2} \in B(x)$. Therefore

$$|\phi(y)|^2=\phi(y^*y)=1+\varepsilon-\phi(z)^2 \leqq 1+\varepsilon.$$

Thus ϕ is continuous on $B(x)$ and the result of Theorem 8 holds in this case.

The result of Theorem 8, in a weaker form, can be extended to arbitrary normed algebras B. By $Q(B)$ we denote the set of all quasinilpotent elements in B, i.e., $Q(B)=\{z \in B: \lim_{k\to\infty} \sqrt[k]{\|z^k\|}=0\}$. An element $x \in B$ is said to be *regular* if it is contained in a commutative normal Q-subalgebra $B(x)$ with identity such that $B(x) \cap Q(B)=\{0\}$. Repeating the argument of Theorem 8, we obtain the following theorem.

Theorem 9. *Let π be an injective homomorphism of a normed algebra B with identity into a Banach algebra $\mathscr{A}$. If x is a regular element in B, then $Sp_B(x)=Sp_{\mathscr{A}}(\pi(x))$.*

From Theorem 8 we obtain the following corollary.

Corollary 10. *Let B be a dense locally normal Q^*-subalgebra of a C^*-algebra A with an identity* $\mathbf{1}$. *If π is an injective $*$-homomorphism of B into a Banach $*$-algebra, then $\pi(B)$ is a Q^*-algebra.*

Proof. By Theorem 8, $\|x\| \leqq \|\pi(x)\|$ for all $x \in B$. Since B and $\pi(B)$ are isomorphic algebras, $r_B(x)=r_{\pi(B)}(\pi(x))$ for all $x \in B$. By Theorem 1(iv), $r_B(x) \leqq \|x\|$. Therefore $r_{\pi(B)}(\pi(x)) \leqq \|\pi(x)\|$. It follows from Theorem 1, that $\pi(B)$ is a Q^*-algebra.

4. Closed ideals of some dense subalgebras of Banach and C^*-algebras

Definition. Let B be a normed (not necessarily Banach) algebra. An ideal J is *closed* in B if $\{x_n\} \in J$ and $x_n \to x \in B$ implies $x \in J$.

Let A be the completion of a normed algebra B with identity. The mapping

$$i_B: I \to I \cap B$$

maps the set of all closed ideals (left, right, two-sided) of A *onto* the set of all closed ideals (left, right, two-sided) of B. The algebra B may have no closed ideals apart from $\{0\}$, even though A may have other closed ideals and even though B may have nonclosed ideals. However, if B is a Q-algebra and J is a nonclosed ideal in B, it follows from Theorem 1, that $\bar{J} \cap B$ is a closed ideal in B. Moreover every maximal ideal J in B

is closed and there exists a maximal closed ideal I in A such that $i_B(I)=J$. Nevertheless, A may have maximal ideals I such that the ideals $i_B(I)$ are not maximal in B.

If B is a commutative normed Q-algebra, then repeating the argument of [**15**], [**16**] and [**13**, §11, subsection 7] and using Theorem 1, one can easily prove the following theorem.

Theorem 11 (cf. [**13, 15, 16**]). *Let B be a commutative normed Q-algebra and let A be the completion of B.*

(i) *The mapping $I \to I \cap B$ is a homeomorphism of the spaces of maximal ideals of A and B and $I=\overline{I \cap B}$.*

(ii) *If J is a maximal ideal in B, then J is closed in B, and the quotient algebra B/J is isomorphic to the field of complex numbers.*

(iii) *Let J be a closed ideal in B and $\bar{J}$ be its closure in A. If $\bar{J}$ is the intersection of maximal ideals in A which contain it, then J is the intersection of maximal ideals in B which contain it.*

If A is commutative, then it follows from Theorem 11(i) that $I=\overline{I \cap B}$ for every maximal ideal I in A. If, however, I is not maximal, then the situation is different. The following example shows that even if A is a commutative symmetric algebra, i.e., $1+x^*x$ is invertible for every $x \in A$, there may still exist closed ideals I in A such that $I \neq \overline{I \cap B}$.

Example. Let A be Fourier–Wiener algebra of all absolutely convergent series $f(t)=\sum_{n=-\infty}^{\infty} c_n e^{int}$ with the norm $\|f\|=\sum_{n=-\infty}^{\infty}|c_n|$. Then (see [**13**, §14]) A is symmetric. Let B be the *-subalgebra of A which consists of all continuously differentiable functions on $T=[0,2\pi]$. Then B is a dense normal Q^*-subalgebra of A. For every closed ideal I in A we denote by $\mathrm{Null}(I)$ the subset of T such that $f(t)=0$ for all $f \in I$ and for all $t \in \mathrm{Null}(I)$, i.e., $\mathrm{Null}(I)=\bigcap_{f \in I} f^{-1}(0)$. By Malliavin's theorem (see [**8**]), the correspondence $I \to \mathrm{Null}(I)$ is not injective. For every compact K in T there are the largest closed ideal

$$I_{\max}(K)=\{f \in A : K \subseteq f^{-1}(0)\}$$

and the smallest closed ideal

$$I_{\min}(K)=\mathrm{Closure}\,\{f \in A : K \subset \mathrm{int}\,(f^{-1}(0))\}$$

such that $K=\mathrm{Null}\,(I_{\max}(K))=\mathrm{Null}\,(I_{\min}(K))$ and such that, generally speaking, $I_{\max}(K) \neq I_{\min}(K)$. By the Beurling–Pollard theorem (see [**8**]), $I_{\min}(K) \cap B = I_{\max}(K) \cap B$. Therefore if I is a closed ideal in A, then

$$I_{\min}(\mathrm{Null}\,(I)) \subseteq I \subseteq I_{\max}(\mathrm{Null}\,(I))$$

and

$$\overline{I \cap B} \neq I \quad \text{if} \quad I \neq I_{\min}(\text{Null}(I)).$$

Although, in the above example, $\overline{I \cap B}$ does not necessarily equal I, nevertheless $I \cap B \neq \{0\}$ for all closed ideals I in A. In the general case the following question arises: under what conditions on A, B and I we have that $I \cap B \neq \{0\}$? Theorem 12 considers some sufficient conditions for this.

Theorem 12. *Let A be a Banach algebra with identity and such that the spectral radius is a continuous function on A. Let B be a dense locally normal Q-subalgebra of A and let $B \cap Q(A) = \{0\}$, where $Q(A)$ is the set of all quasinilpotent elements in A. If I is a closed ideal in A and if I is not contained in $Q(A)$ then $I \cap B \neq \{0\}$.*

Proof. Let ϕ be the canonical homomorphism of A onto A/I. If $I \cap B = \{0\}$, then the restriction of ϕ to B is injective. Let $x \in B$. Since B is locally normal, there is a commutative Banach subalgebra $A(x)$ in A which contains $\mathbf{1}$ and x and such that $B(x) = B \cap A(x)$ is a dense normal subalgebra of $A(x)$. By Lemma 2, $B(x)$ is a Q-subalgebra of $A(x)$. Since B has no quasinilpotent elements, the element x is regular. Therefore, by Theorem 9, $Sp_B(x) = Sp_{A/I}(\phi(x))$. Hence

$$r_B(x) = r_{A/I}(\phi(x)) \leqq \|\phi(x)\|.$$

Let $y \in I$. Since B is dense in A, we can choose $x_n \to y$. Then

$$r_B(x_n) \leqq \|\phi(x_n)\| \leqq \|x_n - y\| \to 0.$$

Since, by the assumption, the spectral radius is continuous on A,

$$r_B(y) = \lim r_B(x_n) = 0$$

and y is quasinilpotent. Thus $I \subseteq Q(A)$. The contradiction proves the theorem.

Remark. (1) If A is a commutative Banach algebra with identity, then the spectral radius is continuous on A and $Q(A)$ is the radical $R(A)$ of A. Let B be a dense normal Q-subalgebra of A. From Theorem 12 it follows that if $B \cap R(A) = \{0\}$ and if I is not contained in $R(A)$, then $B \cap I \neq \{0\}$.

(2) Let A be a Banach $*$-algebra and let $\mathscr{P}(A)$ be the set of all positive functionals on A. Set $P = \bigcap_{f \in \mathscr{P}(A)} I_f$ where $I_f = \{x \in A : f(x^*x) = 0\}$. The algebra A is said to be *P-commutative* if $xy - yx \in P$ for all $x, y \in A$. Tiller [17] showed that the spectral radius is continuous on P-commutative algebras. Therefore for P-commutative algebras, Theorem 12 holds.

If A is a C^*-algebra and B is a dense locally normal $*$-subalgebra of A, then the link between closed ideals in A and B becomes clear and simple. If A is a C^*-subalgebra of the algebra of all bounded operators on a Hilbert space H and if $A \cap C(H) \neq \{0\}$ where

$C(H)$ is the ideal of all compact operators, then Bratteli and Robinson [3] proved that $D(\delta)\cap C(H)\neq\{0\}$ for any closed *-derivation δ of A. Batty [1] generalized their result and showed that if δ is a closed *-derivation of a C^*-algebra A, then, for every closed ideal I in A, $I\cap D(\delta)$ is dense in I. Theorem 13 extends this result to the case when A is a C^*-algebra and B is a dense locally normal *-subalgebra of A. The condition that B is a Q^*-subalgebra of A is not necessary in this case at all.

Theorem 13. *Let B be a dense locally normal *-subalgebra of a C^*-algebra A with an identity $\mathbf{1}$ and let $\mathbf{1}\in B$.*

(i) *If I is a closed two-sided ideal in A, then $I=\overline{I\cap B}$.*

(ii) *The mapping $i_B: I\to I\cap B$ is a one-to-one mapping of the set of all closed two-sided ideals in A onto the set of all closed two-sided ideals in B. It maps the set of all maximal two-sided ideals in A onto the set of all maximal two-sided ideals in B.*

(iii) *Every closed two-sided ideal J in B is selfadjoint and the mapping $J\to\bar{J}$ is inverse to i_B.*

Proof. Let y be a selfadjoint member of I, $\|y\|=1$, and $\varepsilon>0$. Choose a selfadjoint element x in B such that $\|y-x\|<\varepsilon$. Let $A(x)$ be a commutative C^*-subalgebra containing $\mathbf{1}$ and x such that $B(x)=B\cap A(x)$ is normal. Let S be the maximal ideal space of $A(x)$, and

$$T_1=\{s\in S:|x(s)|\leqq\varepsilon\},\ T_2=\{s\in S:|x(s)|\geqq 2\varepsilon\}.$$

Let z be a selfadjoint element of $B(x)$ such that $z(s)=0$ for $s\in T_1$ and $z(s)=1$ for $s\in T_2$. Replacing z by $p(z)$ for a suitable polynomial p, we may arrange that $\|z\|\leqq 2$. Let $u=xz\in B(x)$.

Let ϕ be the canonical homomorphism of A onto A/I. Then $\phi(A(x))$ is isomorphic to the quotient C^*-algebra $A(x)/(I\cap A(x))$. Therefore the distance $d(x,I\cap A(x))=d(x,I)<\varepsilon$. If ε is small, then, since $1-\varepsilon\leqq\|x\|$, it becomes evident that $I\cap A(x)\neq\{0\}$. Hence there is a closed subset S_0 of S such that

$$I\cap A(x)=\{v\in A(x): v(s)=0 \text{ for all } s\in S_0\}.$$

Since $d(x,I\cap A(x))<\varepsilon$, we have that $|x(s)|<\varepsilon$ for all $s\in S_0$. Thus S_0 is contained in T_1. Hence $z\in I\cap A(x)$, so $u\in I\cap B(x)$. Moreover,

$$\|x-u\|=\sup\{|x(s)(1-z(s))|:s\in S\}=\sup\{|x(s)(1-z(s))|:s\in S\backslash T_2\}\leqq 6\varepsilon,$$

so $\|y-u\|<7\varepsilon$ and part (i) is proved. The proof of the rest of the theorem is standard.

Remark. (1) Any dense *-subalgebra of a C^*-algebra which is closed under C^∞-functional calculus is a locally normal Q^*-algebra. The converse is not true. The

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Fourier–Wiener algebra is a locally normal Q^*-algebra, but it is not closed under C^∞-functional calculus. A theorem of Katznelson (see [**8**, p. 82]) shows that it is closed under composition only with analytic functions.

(2) Let δ be a closed *-derivation of a C^*-algebra A with identity. Let x be a selfadjoint element in $D(\delta^n)$ and let $[a,b]$ be a closed interval containing $Sp_A(x)$. Powers [**14**] and Bratteli, Elliott and Jorgensen [**2**] (see also [**12**]) proved that if a function $f(t)$ has $n+1$ continuous derivatives on $[a,b]$, then $f(x)\in D(\delta^n)$. From this it follows that all the algebras $D(\delta^n)$, $1\leqq n\leqq\infty$, are locally normal. Therefore, for every n such that $D(\delta^n)$ is dense in A, the results of Theorem 13 hold for $D(\delta^n)$.

(3) The condition set in Theorem 13 that A must be a C^*-algebra is essential. The example of Fourier–Wiener algebra given after Theorem 11, shows that if A is not a C^*-algebra, then the results of Theorem 13 will no longer hold.

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