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ON RELEVATION REDUNDANCY TO COHERENT SYSTEMS AT COMPONENT AND SYSTEM LEVELS

CHEN LI ,* *Tianjin University of Commerce*
XIAOHU LI ,** *Stevens Institute of Technology*

Abstract

Recently, the relevation transformation has received further attention from researchers, and some interesting results have been developed. It is well known that the active redundancy at component level results in a more reliable coherent system than that at system level. However, the lack of study of this problem with relevation redundancy prevents us from fully understanding such a generalization of the active redundancy. In this note we deal with relevation redundancy to coherent systems of homogeneous components. Typically, for a series system of independent components, we have proved that the lifetime of a system with relevation redundancy at component level is larger than that with relevation redundancy at system level in the sense of the usual stochastic order and the likelihood ratio order, respectively. For a coherent system with dependent components, we have developed a sufficient condition in terms of the domination function to the usual stochastic order between the system lifetime with redundancy at component level and that at system level.

Keywords: Coherent system; copula; domination function; hazard rate order; likelihood ratio order; series system; usual stochastic order

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Secondary 90B25

1. Introduction

A primary component is to be replaced on failure by one secondary component with the age as the failure time of the former one, and the time of the next failure is known as the relevation transform in engineering reliability. Since its introduction by Krakowski in 1973 [11], the relevation transform has been revisited from time to time in recent decades. In the context of the secondary component having an exponential distribution, the relevation transform reduces to the convolution owing to the lack-of-memory property; Grosswald *et al.* [8] derived the characterization of the exponential distribution in terms of the relevation. Subsequently, Lau and Rao [12] improved the characterization result due to [8], and Chukova and Dimitrov [6] further obtained a characterization of almost-lack-of-memory distributions. In addition, Baxter [3] discussed applications of the relevation transform in stochastic processes, and

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* Postal address: School of Science, Tianjin University of Commerce, Tianjin 300134, China. Email address: lichenxm@hotmail.com
** Postal address: Department of Mathematical Sciences, Stevens Institute of Technology, Hoboken, NJ 07030, USA. Email address: xli82@stevens.edu

Shanthikumar and Baxter [29] studied the closure of ageing properties under the formation of the relevation transform.

For several years at the beginning of this century, some researchers devoted their attention to the relevation transform from the viewpoint of stochastic processes. Pellerey *et al.* [25] generalized the non-homogeneous Poisson process to the so-called relevation counting process by virtue of the relevation transform. Subsequently, based on the relevation transform, Kapodistria and Psarrakos [10] constructed a sequence of stochastically increasing random variables, which represent a process describing the successive failures of a component replaced on failure by another component of equal age. In the context of the relevation replacement policy of [10], Sordo and Psarrakos [30] presented some stochastic comparisons on the failure times and inter-failure times of two systems. Typically, Belzunce *et al.* [4] conducted a comparison of stochastic processes arising from a policy based on the replacement of a failed unit by a new one, and from one in which the unit is being continuously subjected to a relevation policy. Meanwhile, Psarrakos and Di Crescenzo [26] studied an inaccuracy measure related to the relevation transform of two non-negative continuous random variables, and Sankaran and Kumar [27] investigated the reliability properties of a special case of relevation transform, namely proportional hazards relevation transform.

The relevation transform also represents the lifetime of a component equipped with one active redundancy given that the redundant one does survive the primary component. Apart from the above fruitful results on characterization, reliability properties, stochastic processes, and applications, the last few years have also witnessed research on relevation transform in the setting of engineering systems, which are usually composed of a number of components. If, due to the limiting resource, a relevation transform can only be accomplished at component level, then it is of genuine interest to study the optimal allocation of relevation for components of the system. As we know, relevation redundancy includes the minimal repair policy as a special case. Assume that the number of repairs that can be freely allocated to any component before the system starts to work is fixed; Arriaza *et al.* [1] studied stochastic comparisons of the coherent systems under different minimal repair policies, and proposed two methods to determine the optimal repair policies. Subsequently, Navarro *et al.* [20] compared the resulting system for three cases: minimal repair of the component that fails first, minimal repair of the component that causes the system failure, and minimal repair of a fixed component in the system. Recently, Belzunce *et al.* [5] discussed allocation of a relevation in redundancy problems, Wu *et al.* [32] examined the reliability improvement of a coherent system through one relevation and in the set-up of coherent systems, and Yan *et al.* [33] and Zhang and Zhang [34] further studied the allocation of relevation transforms of two components so as to increase the system reliability. On the other hand, some researchers devoted their attention to the optimal strategies for active and standby redundancies. For example, Zhao *et al.* [35] focused on the problem of how to optimally allocate one redundancy in a series systems, and Torrado *et al.* [31] studied the effect of redundancies on the reliability of coherent systems under redundancies at component level versus redundancies at module level.

According to the seminal theorem of Barlow and Proschan [2], the active redundancy at component level results in a stochastically larger system lifetime than the active redundancy at system level. Compared to the discussion of allocating relevation transform at component level, as yet no research has been reported to deal with the comparison of the reliability improvement between the relevation transform at system level and the relevation transforms at component level in the literature. Naturally, it is of both theoretical and practical interest to study whether allocating relevation redundancy to system components outperforms applying relevation redundancy to the system itself. This study aims to partially fill this gap. In particular, we start with the series system and end up with a general conclusion on coherent systems.

The remainder of this note is set out as follows. Section 2 recalls several important concepts, including stochastic order, copulas and domination functions, and generalized domination functions of coherent systems. In Section 3 we formalize the relevation redundancy model at component and system levels. Section 4 deals with the series system of independent components and presents the usual stochastic order and likelihood ratio order, respectively, between the system lifetime with redundancy at component level and that at system level. In Section 5 we discuss the coherent system with relevation redundancy at both component level and system level in the setting of statistically dependent component lifetimes. In Section 6 we close this note by making some concluding remarks.

2. Some preliminaries

Various stochastic orders are found of nice applications in reliability, actuarial risk, and economics, etc. The following typical orders will be employed to compare system or component lifetimes.

Let X and Y be random variables with distribution functions F and G, survival functions $\bar{F}$ and $\bar{G}$, and probability densities f and g, respectively. The X is said to be smaller than Y in the

(i) *usual stochastic order* (denoted $X \le_{\rm st} Y$) if $\bar{F}(x) \le \bar{G}(x)$ for all x,

(ii) *hazard rate order* (denoted $X \le_{\rm hr} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in x,

(iii) *reversed hazard rate order* (denoted $X \le_{\rm rh} Y$) if $G(x)/F(x)$ is increasing in x,

(iv) *likelihood ratio order* (denoted $X \le_{\rm lr} Y$) if $g(x)/f(x)$ is increasing in x.

For comprehensive discussions and more recent advances on stochastic orders one may refer to [15] and [28].

In engineering practice, system components usually operate under the same circumstances and thus their lifetimes are statistically dependent. Next we recall copula functions, which are to be utilized in modeling the dependence among component lifetimes.

For a random vector $(X_1, \ldots, X_n)$ with distribution function F, survival function $\bar{F}$, and univariate marginal distribution functions $F_1, \ldots, F_n$, if there exists some

$$C(u_1, \ldots, u_n) \colon [0, 1]^n \mapsto [0, 1] \quad \text{and} \quad \widehat{C}(u_1, \ldots, u_n) \colon [0, 1]^n \mapsto [0, 1]$$

such that for all x_i, $i \in \mathcal{I}_n$,

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)), \qquad \bar{F}(x_1, \ldots, x_n) = \widehat{C}(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)),$$

then $C(u_1, \ldots, u_n)$ and $\widehat{C}(u_1, \ldots, u_n)$ are called the *copula* and *survival copula* of $(X_1, \ldots, X_n)$, respectively. In particular, the copula and survival copula corresponding to the independence are known as

$$C_I(u_1, \ldots, u_n) = u_1 \cdots u_n = \widehat{C}_I(u_1, \ldots, u_n)$$

for $u_i \in [0, 1]$, $i = 1, \ldots, n$.

For an n-monotone function $\psi \colon [0, +\infty) \mapsto [0, 1]$ with $\psi(0) = 1$ and $\psi(+\infty) = 0$, the mapping

$$C_\psi(u_1, \ldots, u_n) = \psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_n)),$$

for all $u_i \in [0, 1]$, $i = 1, \ldots, n$, is called an *Archimedean* copula with generator ψ, where ψ^{-1} denotes the right continuous inverse. This family comprises Clayton copulas, Frank copulas, Gumbel copulas, and other typical members, which are widely used in survival analysis, dependence modeling, biostatistics, and other related areas due to the flexibility in statistics and tractability in mathematics. For more on Archimedean copulas, readers may refer to [24].

A reliability system is said to be *coherent* if (i) the functioning system never fails upon replacing a failed component with a working one, and (ii) each component is relevant to the system, i.e. the component status affects the system functionality. The series system, the parallel system, and the k-out-of-n system are typical examples of coherent systems. Due to mathematical complexity, researchers consider only the scenario of statistically independent component lifetimes in most of the existing references. However, in real engineering practice, system components usually operate under the same environment or suffer from some common stresses and thus could have interdependent lifetimes.

For a coherent system with components having lifetimes $X_1, \ldots, X_n$, the system lifetime T, which is determined as the so-called *structure function* $\phi(X_1, \ldots, X_n)$, has reliability function

$$\bar{H}(t) = \mathbb{P}(T > t) = \mathbb{P}(\phi(X_1, \ldots, X_n) > t) \quad \text{for all } t \geq 0.$$

Take, for example, $\min\{x_1, \ldots, x_n\}$ for the series system, $\max\{x_1, \ldots, x_n\}$ for the parallel system, and $x_{k:n}$ (the kth smallest of $x_1, \ldots, x_n$) for the $(n - k + 1)$-out-of-n system. Evidently, the system structure, the component lifetimes, and the statistical dependence shape the system reliability function.

Lastly, we review two new ways to characterize coherent systems, which play a crucial role in developing the preservation of ageing properties. Let $\bar{F}_1, \ldots, \bar{F}_n$ denote reliability functions corresponding to component lifetimes $X_1, \ldots, X_n$, respectively. Then the system reliability simply comes down to

$$\bar{H}(t) = \varphi(\bar{F}_1(t), \ldots, \bar{F}_n(t)) \quad \text{for all } t \geq 0,$$

where the multivariate $\varphi(u_1, \ldots, u_n)$ is called the *generalized domination function* (GDF). Naturally, $\varphi(u_1, \ldots, u_n)$ is increasing on $[0, 1]^n$ and such that $\varphi(0, \ldots, 0) = 0$, $\varphi(1, \ldots, 1) = 1$. In the context of components with homogeneous lifetimes, i.e. $\bar{F}_1 = \cdots = \bar{F}_n = \bar{F}$, the system reliability reduces to

$$\bar{H}(t) = \rho(\bar{F}(t)) \quad \text{for all } t \geq 0,$$

where the univariate $\rho(u)$, called the *domination function* (DF), is such that $\rho(0) = 0$, $\rho(1) = 1$ and increasing on [0,1]. For more on DF and GDF we refer readers to [21] and [22]. It should be mentioned here that both GDF and DF are now called *distortion functions* in most of the references in this line of research.

Technically, the DF $\rho(u)$ and GDF $\varphi(u_1, \ldots, u_n)$ are utilized to adjust the probability distribution of random risk so as to put more emphasis on the tail area, and thus they are also called distortion functions in the theory of risk and actuarial science. For more on the DF and the generalized version, we refer readers to [9], [18], [19], [21], [22], and [23].

3. The models

Consider two components C and R with independent lifetimes X and Y having survival functions $\bar{F}$ and $\bar{G}$ and probability densities f and g, respectively. Assume that upon failure at

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time $x > 0$, the primary component C is instantly replaced by the secondary one R with exactly the same age x as that of the failed component. Let

$$Y_t = (Y - t \mid Y > t)$$

denote the residual lifetime of Y at time $t \geq 0$. Then such a system attains the lifetime

$$X + Y_X = (Y \mid Y > X),$$

which is known as the *relevation transform* of X by Y. For the residual lifetime at a random time, refer to [16]. It is easy to verify that

$$\mathbb{P}(X + Y_X > t) = \bar{F}(t) + \bar{G}(t) \int_0^t \frac{f(x)}{\bar{G}(x)} \, \mathrm{d}x \quad \text{for all } t > 0, \tag{3.1}$$

where, although the absolute continuity of $\bar{F}$ and $\bar{G}$ is not necessary, it is typically reasonable in real practice and simplifies the mathematical representation.

As was pointed out in [20], the parallel system of C and R has the lifetime

$$X \vee Y = \max\{X, Y\} = (X + Y_X)I(X < Y) + (Y + X_Y)I(Y < X),$$

where $I(A)$ is the indicator of the event A. The relevation transform can also be viewed as the generalization of the parallel structure in reliability. Thus from now on we denote $X \curlyvee Y \equiv X + Y_X$ for convenience. As an immediate consequence, the above equation is rewritten as

$$X \vee Y = (X \curlyvee Y)I(X < Y) + (Y \curlyvee X)I(Y < X) = (X \wedge Y) \curlyvee (X \vee Y),$$

where $X \wedge Y = \min\{X, Y\}$ denotes the series system lifetime of X and Y. This comes up with an interesting interpretation: a parallel system of two components with lifetimes X and Y is stochastically equivalent to the relevation transform of their minimum $X \wedge Y$ by the maximum $X \vee Y$, and the relevation transform $X \curlyvee Y$ is just one of the two possible realizations.

In this note we deal with a coherent system with components $C_1, \ldots, C_n$ having homogeneous lifetimes $X_1, \ldots, X_n$, and relevation redundancies $R_1, \ldots, R_n$ having homogeneous lifetimes $Y_1, \ldots, Y_n$. From now on, for system component lifetimes $X_1, \ldots, X_n$, we let F denote the common distribution function, $\bar{F}$ the survival function, and f the density function, and for relevation redundancy lifetimes $Y_1, \ldots, Y_n$, we let G denote the common distribution function, $\bar{G}$ the survival function, and g the density function.

In the context of allocating redundancy at component level, we apply the relevation redundancy R_i to the component C_i, $i = 1, \ldots, n$. See Figure 1(a) for an example. Thus the resulting coherent system attains the lifetime

$$T_c = \phi(X_1 \curlyvee Y_1, \ldots, X_n \curlyvee Y_n),$$

where $\phi(u_1, \ldots, u_n)$ is the structure function of the system. Assume that $(X_1 \curlyvee Y_1, \ldots, X_n \curlyvee Y_n)$ inherits the dependence structure of $(X_1, \ldots, X_n)$. Then, according to (3.1), T_c has the survival function

$$\bar{H}_c(t) = \rho\left(\bar{F}(t) + \bar{G}(t) \int_0^t \frac{f(x)}{\bar{G}(x)} \, \mathrm{d}x\right) \quad \text{for all } t \geq 0, \tag{3.2}$$

where $\rho(u)$ is the DF of the system.

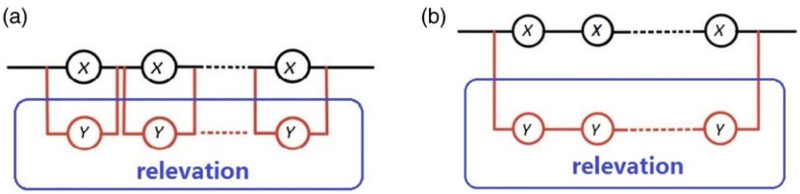

FIGURE 1. A series system with relevation redundancy. (a) T_c: redundancy at component level. (b) T_s: redundancy at system level.

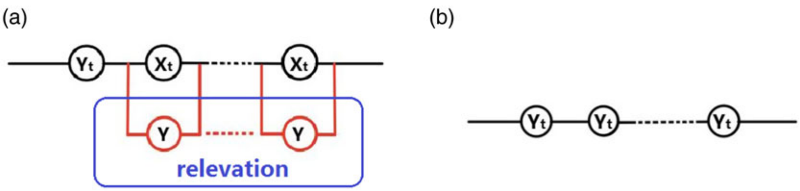

FIGURE 2. Structures of a series system with relevation redundancy. (a) T_c: redundancy at component level. (b) T_s: redundancy at system level.

In contrast, in the setting of redundancy at system level, we duplicate the coherent system of components $C_1, \ldots, C_n$ by $R_1, \ldots, R_n$ and then make it available as a relevation redundancy spare to the coherent system. See Figure 1(b) for an example. Thus the resulting coherent system gets the lifetime

$$T_s = \phi(X_1, \ldots, X_n) \curlyvee \phi(Y_1, \ldots, Y_n).$$

In accordance with (3.1), T_s attains the survival function

$$\bar{H}_s(t) = \rho(\bar{F}(t)) + \rho(\bar{G}(t)) \int_0^t \frac{\rho'(\bar{F}(x))f(x)}{\rho(\bar{G}(x))}\,\mathrm{d}x \quad \text{for all } t \geq 0. \tag{3.3}$$

As is mentioned in Section 1, in this note our interest lies in building the usual stochastic order between T_c and T_s, and this is to be accomplished through developing the uniform inequality between (3.2) and (3.3).

4. Series systems of components having independent lifetimes

In this section we pay specific attention to the series system with n independent component lifetimes $X_1, \ldots, X_n$ and n relevation redundancies with lifetimes $Y_1, \ldots, Y_n$. Assume that $X_1, \ldots, X_n$ has the same distribution as X, and $Y_1, \ldots, Y_n$ has the same distribution as Y. The system structures corresponding to relevation redundancy at component level and at system level are depicted in Figures 1(a) and 1(b), respectively.

Let $t > 0$ denote the time of the occurrence of the first component failure. The structure of the system in Figure 1(a) is further depicted in Figure 2(a). According to [17], the reliability function of the used series system with age $t > 0$ is exactly equal to that of the series system composed of same number of used components with common age t. Therefore the structure of the system in Figure 1(b) is stochastically equivalent to the one in Figure 2(b).

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Based on the structures of Figures 2(a) and 2(b), one naturally comes up with the idea that $X_t \geq_{\rm st} Y_t$ for any $t \geq 0$ implies $T_c \geq_{\rm st} T_s$. Since $X \geq_{\rm hr} Y$ if and only if $X_t \geq_{\rm st} Y_t$ for any $t \geq 0$, motivated by this intuition we present our first main result in Theorem 4.1.

Before developing the proof of Theorem 4.1, it is worth pointing out that n independent component lifetimes have independence copula $C_I(u_1, \ldots, u_n)$ and thus the series system with these components has the domination function $\rho(u) = u^n$. As a consequence, based on (3.2) and (3.3), the survival function of system lifetime T_c is further represented as

$$\bar{H}_c(t) = \left[\bar{F}(t) + \bar{G}(t) \int_0^t \frac{f(x)}{\bar{G}(x)} \, \mathrm{d}x\right]^n \quad \text{for all } t \geq 0, \tag{4.1}$$

and the survival function of T_s is also rephrased as

$$\bar{H}_s(t) = \bar{F}^n(t) + \bar{G}^n(t) \int_0^t \frac{n\bar{F}^{n-1}(x) f(x)}{\bar{G}^n(x)} \, \mathrm{d}x \quad \text{for all } t \geq 0. \tag{4.2}$$

Now let us present the first main result.

Theorem 4.1. *For a series system with components having mutually independent lifetimes, if $X \geq_{\rm hr} Y$ then $T_c \geq_{\rm st} T_s$.*

Proof. Since $X \geq_{\rm hr} Y$ implies

$$\frac{\bar{F}(t)}{\bar{G}(t)} \geq \frac{\bar{F}(x)}{\bar{G}(x)} \quad \text{for } t \geq x, \tag{4.3}$$

it follows from (4.1) and (4.2) that

$$\begin{aligned}
&\bar{H}_c(t) - \bar{H}_s(t) \\
&= \left[\bar{F}(t) + \bar{G}(t) \int_0^t \frac{f(x)}{\bar{G}(x)} \, \mathrm{d}x\right]^n - \left[\bar{F}^n(t) + \bar{G}^n(t) \int_0^t \frac{n\bar{F}^{n-1}(x) f(x)}{\bar{G}^n(x)} \, \mathrm{d}x\right] \\
&\geq \bar{F}^n(t) + n\bar{F}^{n-1}(t)\bar{G}(t) \int_0^t \frac{f(x)}{\bar{G}(x)} \, \mathrm{d}x - \bar{F}^n(t) - \bar{G}^n(t) \int_0^t \frac{n\bar{F}^{n-1}(x) f(x)}{\bar{G}^n(x)} \, \mathrm{d}x \\
&= n\bar{F}^{n-1}(t)\bar{G}(t) \int_0^t \frac{f(x)}{\bar{G}(x)} \, \mathrm{d}x - \bar{G}^n(t) \int_0^t \frac{n\bar{F}^{n-1}(x) f(x)}{\bar{G}^n(x)} \, \mathrm{d}x \\
&\overset{\rm sgn}{=} \int_0^t \frac{f(x)}{\bar{G}(x)} \left[\left(\frac{\bar{F}(t)}{\bar{G}(t)}\right)^{n-1} - \left(\frac{\bar{F}(x)}{\bar{G}(x)}\right)^{n-1}\right] \mathrm{d}x \\
&\geq 0
\end{aligned}$$

for all $t \geq 0$, where the last inequality follows from (4.3). Owing to the arbitrary $t \geq 0$, this yields $T_c \geq_{\rm st} T_s$, the desired ordering. □

With Theorem 4.1, we confirm that relevation redundancy at component level results in a more reliable series system than relevation at system level. This is consistent with the seminal conclusion on active redundancy in [2]. Naturally, one may conjecture that the fact that X is larger than Y in some stochastic sense is necessary to some extent. However, as per Figures

2(a) and 2(b), even if $X \geq_{\rm hr} Y$ is violated, T_c may still outperform T_s due to the relevation. The next result just confirms this intuition.

Theorem 4.2. *For a series system with components having mutually independent lifetimes, if $X \leq_{\rm hr} Y$, then $T_c \geq_{\rm lr} T_s$.*

Proof. Based on (4.1) and (4.2), T_c has the probability density function

$$h_c(t) = ng(t)\left[\bar{F}(t) + \bar{G}(t)\int_0^t \frac{f(x)}{\bar{G}(x)}\,\mathrm{d}x\right]^{n-1}\int_0^t \frac{f(x)}{\bar{G}(x)}\,\mathrm{d}x \quad \text{for all } t \geq 0,$$

and the probability density function of T_s is

$$h_s(t) = ng(t)\bar{G}^{n-1}(t)\int_0^t \frac{n\bar{F}^{n-1}(x)f(x)}{\bar{G}^n(x)}\,\mathrm{d}x \quad \text{for all } t \geq 0.$$

Thus, for all $t \geq 0$,

$$\begin{aligned}
\frac{h_c(t)}{h_s(t)} &= \frac{ng(t)\left[\bar{F}(t) + \bar{G}(t)\int_0^t \frac{f(x)}{\bar{G}(x)}\,\mathrm{d}x\right]^{n-1}\int_0^t \frac{f(x)}{\bar{G}(x)}\,\mathrm{d}x}{ng(t)\bar{G}^{n-1}(t)\int_0^t \frac{n\bar{F}^{n-1}(x)f(x)}{\bar{G}^n(x)}\,\mathrm{d}x} \\
&= \left[\frac{\bar{F}(t)}{\bar{G}(t)} + \int_0^t \frac{f(x)}{\bar{G}(x)}\,\mathrm{d}x\right]^{n-1} \cdot \int_0^t \frac{f(x)}{\bar{G}(x)}\,\mathrm{d}x \bigg/ \int_0^t \frac{n\bar{F}^{n-1}(x)f(x)}{\bar{G}^n(x)}\,\mathrm{d}x \\
&= [\ell_1(t)]^{n-1} \cdot \ell_2(t).
\end{aligned}$$

On one hand,

$$\begin{aligned}
[\ell_1(t)]' &= \left[\frac{\bar{F}(t)}{\bar{G}(t)} + \int_0^t \frac{f(x)}{\bar{G}(x)}\,\mathrm{d}x\right]' \\
&= \frac{-f(t)\bar{G}(t) + g(t)\bar{F}(t)}{\bar{G}^2(t)} + \frac{f(t)}{\bar{G}(t)} \\
&= \frac{g(t)\bar{F}(t)}{\bar{G}^2(t)} \\
&\geq 0 \quad \text{for all } t \geq 0.
\end{aligned}$$

This implies that $\ell_1(t)$ is non-negative and increasing in $t \in [0, \infty)$. On the other hand, the hazard rate order $X \leq_{\rm hr} Y$ implies that $\bar{F}(t)/\bar{G}(t)$ is decreasing. Thus

$$\begin{aligned}
[\ell_2(t)]' &= \left[\int_0^t \frac{f(x)}{\bar{G}(x)}\,\mathrm{d}x \bigg/ \int_0^t \frac{n\bar{F}^{n-1}(x)f(x)}{\bar{G}^n(x)}\,\mathrm{d}x\right]' \\
&\overset{\rm sgn}{=} \frac{f(t)}{\bar{G}(t)}\int_0^t \frac{n\bar{F}^{n-1}(x)f(x)}{\bar{G}^n(x)}\,\mathrm{d}x - \frac{n\bar{F}^{n-1}(t)f(t)}{\bar{G}^n(t)}\int_0^t \frac{f(x)}{\bar{G}(x)}\,\mathrm{d}x \\
&\geq \frac{nf(t)}{\bar{G}(t)}\left(\frac{\bar{F}(t)}{\bar{G}(t)}\right)^{n-1}\int_0^t \frac{f(x)}{\bar{G}(x)}\,\mathrm{d}x - \frac{n\bar{F}^{n-1}(t)f(t)}{\bar{G}^n(t)}\int_0^t \frac{f(x)}{\bar{G}(x)}\,\mathrm{d}x \\
&= 0 \quad \text{for all } t \geq 0,
\end{aligned}$$

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and then

$$\ell_2(t) = \int_0^t \frac{f(x)}{\bar{G}(x)}\,\mathrm{d}x \bigg/ \int_0^t \frac{n\bar{F}^{n-1}(x)f(x)}{\bar{G}^n(x)}\,\mathrm{d}x$$

is non-negative and increasing in $t \ge 0$.

As a consequence, we now conclude that $h_c(t)/h_s(t)$ is increasing, and this invokes $T_c \ge_{\rm lr} T_s$. □

It is worth mentioning that the relevation $X \curlyvee Y$ not only preserves the IFR (increasing failure rate) property of the redundant Y in the context of Theorem 4.2 (see Theorem 3.5 of [29]) but also upgrades the $T_c \ge_{\rm st} T_s$ of Theorem 4.1 to $T_c \ge_{\rm lr} T_s$. Technically, it is of natural interest to check whether the hazard rate ordering $X \le_{\rm hr} Y$ can be relaxed to $X \le_{\rm st} Y$ in Theorem 4.2. In what follows, we attempt to answer this question for series systems with only two components.

Theorem 4.3. *For a series system with two components having independent lifetimes, $T_c \ge_{\rm st} T_s$ whenever*

$$\bar{G}(t) \ge \bar{F}(t) \ge \frac{1}{2}[\bar{G}(t) + \bar{G}^2(t)] \quad \textit{for all } t \ge 0. \tag{4.4}$$

Proof. Denote

$$a(t) \equiv \int_0^t \frac{f(x)}{\bar{G}(x)}\,\mathrm{d}x \quad \text{for all } t \ge 0.$$

By (4.1) we have

$$\bar{H}_c(t) = \bar{F}^2(t) + 2\bar{F}(t)\bar{G}(t)a(t) + \bar{G}^2(t)a^2(t) \quad \text{for all } t \ge 0.$$

Based on (4.2) and the first inequality of (4.4), we have

$$\bar{H}_s(t) = \bar{F}^2(t) + \bar{G}^2(t)\int_0^t \frac{2\bar{F}(x)f(x)}{\bar{G}^2(x)}\,\mathrm{d}x \le \bar{F}^2(t) + 2\bar{G}^2(t)a(t).$$

In view of

$$a(t) \longrightarrow \int_0^\infty \frac{f(x)}{\bar{G}(x)}\,\mathrm{d}x > 1 \quad \text{as } t \to \infty,$$

we consider the following two cases individually.

For $t \ge 0$ such that $a(t) > 1$, it follows from the second inequality of (4.4) that

$$\begin{aligned}\bar{H}_c(t) - \bar{H}_s(t) &\ge 2\bar{F}(t)\bar{G}(t)a(t) + \bar{G}^2(t)a^2(t) - 2\bar{G}^2(t)a(t)\\ &\ge [2\bar{F}(t) - \bar{G}(t)]\bar{G}(t)a(t)\\ &\ge 0,\end{aligned}$$

which gives rise to $\bar{H}_c(t) \ge \bar{H}_s(t)$.

For $t \ge 0$ such that $a(t) \le 1$, owing to the first inequality of (4.4), we have

$$2\bar{F}(t) \ge \bar{G}(t) + \bar{G}^2(t) \ge \bar{G}(t) + \bar{F}(t)\bar{G}(t).$$

Equivalently,

$$F(t) \ge 2[1 - \bar{F}(t)/\bar{G}(t)].$$

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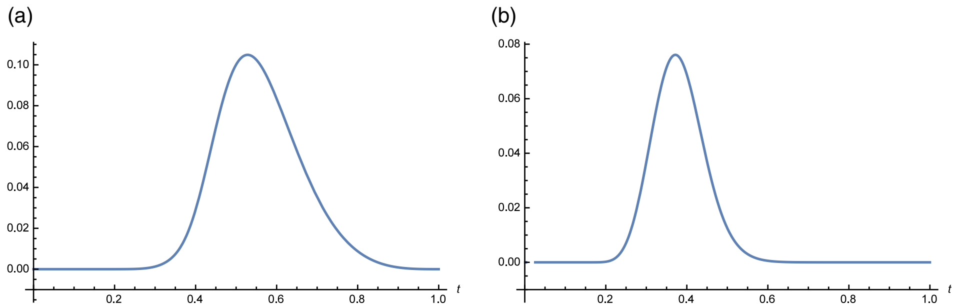

FIGURE 3. Difference curves of $\bar{H}_c(1/t-1)-\bar{H}_s(1/t-1)$. (a) Series system. (b) Parallel system.

Thus

$$a(t)=\int_0^t \frac{f(x)}{\bar{G}(x)}\,\mathrm{d}x \geq F(t) \geq 2[1-\bar{F}(t)/\bar{G}(t)].$$

Equivalently,

$$2\bar{F}(t)-2\bar{G}(t)+\bar{G}(t)a(t)\geq 0.$$

As a result,

$$\bar{H}_c(t)-\bar{H}_s(t)\geq [2\bar{F}(t)-2\bar{G}(t)+\bar{G}(t)a(t)]\bar{G}(t)a(t) \quad \text{for all } t\geq 0.$$

This implies $\bar{H}_c(t)-\bar{H}_s(t)\geq 0$ and hence $\bar{H}_c(t)\geq \bar{H}_s(t)$ again.

Now, based on the above, we conclude with $\bar{H}_c(t)\geq \bar{H}_s(t)$ for all $t\geq 0$, i.e. $T_c \geq_{\text{st}} T_s$. □

As for Theorem 4.1, one referee suggested the possibility of extending such a conclusion to the case of the series system with independent but heterogeneous components. Here we present an example showing that relevation redundancy at component level also results in a more reliable series system than relevation at system level. However, it remains an open problem in a general context.

Example 4.1. Consider the system of two components in series. Assume that the component lifetimes $X_1 \sim \mathcal{E}(1)$, the exponential distribution with hazard rate 1, and $X_2 \sim \mathcal{E}(2)$ are independent, and the relevation redundancy lifetimes $Y_1 \sim \mathcal{W}(0.5, 2)$, Weibull distribution with survival function $\mathrm{e}^{-0.5t^2}$, and $Y_2 \sim \mathcal{W}(1, 3)$ are independent. It is not difficult to check that for all $t\geq 0$,

$$\bar{H}_c(t)=\left(\mathrm{e}^{-t}+\mathrm{e}^{-0.5t^2}\int_0^t \mathrm{e}^{-x+0.5x^2}\,\mathrm{d}x\right)\cdot\left(\mathrm{e}^{-2t}+2\mathrm{e}^{-t^3}\int_0^t \mathrm{e}^{-2x+x^3}\,\mathrm{d}x\right)$$

and

$$\bar{H}_s(t)=\mathrm{e}^{-3t}+3\mathrm{e}^{-0.5t^2-t^3}\int_0^t \mathrm{e}^{-3x+0.5x^2+x^3}\,\mathrm{d}x.$$

As is seen in Figure 3(a), $\bar{H}_c(1/t-1)\geq \bar{H}_s(1/t-1)$ for all $t\in[0, 1]$, where we use the time transform $1/t-1$ to map the positive half of the real line to the unit interval. Markedly, this leads to $T_c \geq_{\text{st}} T_s$.

Also, for the parallel system with independent and homogeneous component lifetimes and relevation redundancy lifetimes, Example 4.2 shows that relevation redundancy at component

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level also results in a more reliable system than relevation at system level. Likewise, we have no idea for any general conclusion of the parallel structure at this moment.

Example 4.2. Consider the parallel system of two components. Assume that the component lifetimes $X_i \sim \mathcal{W}(1, 3)$, $i = 1, 2$, are independent, and the relevation redundancy lifetimes $Y_i \sim \mathcal{W}(0.5, 2)$, $i = 1, 2$, are independent. It is routine to check that for all $t \geq 0$

$$\bar{H}_c(t) = 1 - \left(1 - \mathrm{e}^{-t^3} - 2\mathrm{e}^{-0.5t^2} \int_0^t x\mathrm{e}^{-x^3+0.5x^2}\,\mathrm{d}x\right)^2$$

and

$$\bar{H}_s(t) = 1 - \big(1 - \mathrm{e}^{-t^3}\big)^2 + 6\big[1 - \big(1 - \mathrm{e}^{-0.5t^2}\big)^2\big] \int_0^t \frac{x\mathrm{e}^{-x^3}(1 - \mathrm{e}^{-x^3})}{1 - (1 - \mathrm{e}^{-0.5x^2})^2}\,\mathrm{d}x.$$

Figure 3(b) shows that $\bar{H}_c(1/t - 1) \geq \bar{H}_s(1/t - 1)$ holds for all $t \in [0, 1]$. Thus $T_c \geq_{\mathrm{st}} T_s$.

5. Coherent systems of components having dependent lifetimes

In traditional reliability theory, system components are usually assumed to be of mutually independent lifetimes for mathematical tractability. However, this is far from the truth in engineering practice because components of a complex system usually suffer common stresses and share load and thus their lifetimes are statistically dependent in the final analysis. As a study in depth, in this section we pay attention to coherent systems with components having statistically dependent lifetimes, and we attempt to gain some insight on relevation redundancy at both system and component level.

As is seen in Section 4, there is no guarantee that relevation redundancy at component level outperforms that at system level even in the context of the series system with independent and identically distributed component lifetimes. As the second main result of this note, Theorem 5.1 presents a sufficient condition for the stochastic order between the lifetime of system with relevation redundancy at component level and that at system level.

Theorem 5.1. *Suppose that the DF $\rho(u)$ is convex and $u\rho'(u)/\rho(u)$ is decreasing in $u \in [0, 1]$. If $X \geq_{\mathrm{hr}} Y$, then $T_c \geq_{\mathrm{st}} T_s$.*

Proof. Since the domination function ρ is convex, it follows from (3.2) and (3.3) that for all $t \geq 0$,

$$\begin{aligned}
&\bar{H}_c(t) - \bar{H}_s(t)\\
&\quad= \rho\left(\bar{F}(t) + \bar{G}(t)\int_0^t \frac{f(x)}{\bar{G}(x)}\,\mathrm{d}x\right) - \rho(\bar{F}(t)) - \rho(\bar{G}(t))\int_0^t \frac{\rho'(\bar{F}(x))f(x)}{\rho(\bar{G}(x))}\,\mathrm{d}x\\
&\quad\geq \rho'(\bar{F}(t))\bar{G}(t)\int_0^t \frac{f(x)}{\bar{G}(x)}\,\mathrm{d}x - \rho(\bar{G}(t))\int_0^t \frac{\rho'(\bar{F}(x))f(x)}{\rho(\bar{G}(x))}\,\mathrm{d}x\\
&\quad= \rho(\bar{G}(t))\int_0^t \frac{f(x)}{\bar{G}(x)}\left[\frac{\bar{G}(t)\rho'(\bar{F}(t))}{\rho(\bar{G}(t))} - \frac{\bar{G}(x)\rho'(\bar{F}(x))}{\rho(\bar{G}(x))}\right]\mathrm{d}x.
\end{aligned} \tag{5.1}$$

The convexity of $\rho(u)$ implies that $\rho'(u)$ is non-negative and increasing. Since $u\rho'(u)/\rho(u)$ is decreasing in $u \in [0, 1]$, the $u/\rho(u)$ is decreasing, and thus

$$\frac{u\rho'(u)}{\rho(u)} - 1 \geq 0 \quad \text{for all } u \in [0, 1]. \tag{5.2}$$

Also, $X \geq_{\mathrm{hr}} Y$ implies that

$$\frac{f(x)}{\bar{F}(x)} \leq \frac{g(x)}{\bar{G}(x)}, \quad \bar{F}(x) \geq \bar{G}(x) \quad \text{for all } x \geq 0. \tag{5.3}$$

Owing to the decreasing property of $u\rho'(u)/\rho(u)$, we have

$$u[\rho'(u)]^2 - \rho(u)\rho'(u) - u\rho''(u)\rho(u) \geq 0 \quad \text{for all } u \in [0, 1] \tag{5.4}$$

and

$$\frac{\bar{F}(x)\rho'(\bar{F}(x))}{\rho(\bar{F}(x))} - \frac{\bar{G}(x)\rho'(\bar{G}(x))}{\rho(\bar{G}(x))} \leq 0 \quad \text{for all } x \geq 0. \tag{5.5}$$

Note that

$$\begin{aligned}
\left[\frac{\bar{G}(x)\rho'(\bar{F}(x))}{\rho(\bar{G}(x))}\right]' &\overset{\text{sgn}}{=} -\frac{g(x)}{\bar{G}(x)} - \frac{f(x)\rho''(\bar{F}(x))}{\rho'(\bar{F}(x))} + \frac{g(x)\rho'(\bar{G}(x))}{\rho(\bar{G}(x))} \\
&= \frac{g(x)}{\bar{G}(x)}\left[\frac{\bar{G}(x)\rho'(\bar{G}(x))}{\rho(\bar{G}(x))} - 1\right] - \frac{f(x)\rho''(\bar{F}(x))}{\rho'(\bar{F}(x))} \\
&\geq \frac{f(x)}{\bar{F}(x)}\left[\frac{\bar{G}(x)\rho'(\bar{G}(x))}{\rho(\bar{G}(x))} - 1\right] - \frac{f(x)\rho''(\bar{F}(x))}{\rho'(\bar{F}(x))} \\
&\geq \frac{f(x)}{\bar{F}(x)}\left[\frac{\bar{F}(x)\rho'(\bar{F}(x))}{\rho(\bar{F}(x))} - 1\right] - \frac{f(x)\rho''(\bar{F}(x))}{\rho'(\bar{F}(x))} \\
&\overset{\text{sgn}}{=} \bar{F}(x)[\rho'(\bar{F}(x))]^2 - \rho(\bar{F}(x))\rho'(\bar{F}(x)) - \bar{F}(x)\rho''(\bar{F}(x))\rho(\bar{F}(x)) \\
&\geq 0 \quad \text{for } x \geq 0,
\end{aligned}$$

where '$\overset{\text{sgn}}{=}$' means both sides have the same sign, the first inequality follows from (5.2) and (5.3), the second one stems from (5.5), and the last is due to (5.4). As a consequence, we conclude that $\bar{G}(x)\rho'(\bar{F}(x))/\rho(\bar{G}(x))$ is increasing in x, and hence

$$\frac{\bar{G}(t)\rho'(\bar{F}(t))}{\rho(\bar{G}(t))} \geq \frac{\bar{G}(x)\rho'(\bar{F}(x))}{\rho(\bar{G}(x))} \quad \text{for all } t \geq x \geq 0.$$

Now, from (5.1) it follows immediately that $\bar{H}_c(t) \geq \bar{H}_s(t)$ for all $t \geq 0$. That is, $T_c \geq_{\mathrm{st}} T_s$, the desired ordering. □

It is postulated in Theorem 5.1 that the DF $\rho(u)$ is convex and $u\rho'(u)/\rho(u)$ is decreasing in $u \in [0, 1]$. Evidently, this assumption is fulfilled for series systems with independent component lifetimes, and thus Theorem 4.1 follows immediately as a special case from Theorem 5.1. In addition, such an assumption integrates both the system structure and the statistical dependence among component lifetimes. It is worth pointing out the following conditions due to [13], which are sufficient to this assumption:

- a series system with system component lifetimes linked by Archimedean survival copula C_ψ such that $-\psi'(\mathrm{e}^x)$ is log-concave and $-\psi'(\mathrm{e}^x)/\psi(\mathrm{e}^x)$ is log-concave, e.g. $\psi(y) = 1 - y^{1/\theta}$ with $\theta \geq 1$;

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- a parallel system with component lifetimes linked by Archimedean copula C_ψ such that $-\psi'(\mathrm{e}^x)$ is log-convex and $-\psi'(\mathrm{e}^x)/[1-\psi(\mathrm{e}^x)]$ is log-convex, e.g. $\psi(y)=\mathrm{e}^{-y^{1/\theta}}$ with $\theta \ge 1$;
- the series-parallel system $\max\{\min\{X_1, X_2, X_3\}, \min\{X_2, X_3, X_4\}\}$ composed of identically distributed component lifetimes $X_1, \ldots, X_4$, which are linked by the Ali–Mikhail–Haq (AMH) survival copula generated by $\psi(y)=(1-\theta)/(\mathrm{e}^y-\theta)$ with $\theta \in [0, 0.15]$.

Naturally, one may speculate that the usual stochastic order between T_c and T_s can be upgraded to either the hazard rate or the reversed hazard rate order in the context of Theorem 5.1. On the other hand, due to Theorems 5.1 and 4.2, one may also conjecture that for a series system $X_{1:n}$ with independent component lifetimes $X_1, \ldots, X_n$, the likelihood ratio order between T_c and T_s holds whenever $X \ge_{\mathrm{hr}} Y$. However, Example 5.1 below serves as a negative answer to both conjectures.

Example 5.1.

(i) For the series system $X_{1:3}$ with independent component lifetimes X_1, X_2, X_3, the DF $\rho(u)=u^3$ for $u \in [0, 1]$. Since

$$\rho''(u)=6u \ge 0, \qquad \frac{u\rho'(u)}{\rho(u)}=3,$$

the $\rho(u)$ is convex and $u\rho'(u)/\rho(u)$ is a constant for $u \in [0, 1]$.

Assume that $X \sim \mathcal{E}(1)$, the standard exponential distribution, and Y has survival function

$$\bar{G}(t)=\begin{cases}\mathrm{e}^{-t}, & 0<t<1,\\ \mathrm{e}^{-t^2}, & t>1.\end{cases}$$

It is routine to verify that $X \ge_{\mathrm{hr}} Y$. Thus all assumptions in Theorem 5.1 are fulfilled.

Even now, as is seen in Figure 4(a), $\bar{H}_c(t)/\bar{H}_s(t)$ is not monotone at all, and this negates the hazard rate order between T_c and T_s in the setting of Theorem 5.1.

(ii) Consider the parallel-series system $\max\{\min\{X_1, X_2, X_3\}, \min\{X_2, X_3, X_4\}\}$ with independent component lifetimes X_1, X_2, X_3, X_4. It is easy to check that $\rho(u)=2u^3-u^4$ for $u \in [0, 1]$. Then

$$\rho''(u)=12u(1-u) \ge 0, \qquad \frac{u\rho'(u)}{\rho(u)}=\frac{6-4u}{2-u}, \qquad \frac{\partial}{\partial u}\left[\frac{u\rho'(u)}{\rho(u)}\right]=-2.$$

That is, $\rho(u)$ is convex and $u\rho'(u)/\rho(u)$ is decreasing in $u \in [0, 1]$.

For $Y \sim \mathcal{E}(1)$ and X with survival function

$$\bar{F}(t)=\begin{cases}\mathrm{e}^{-0.5t^2}, & 0<t<1,\\ \mathrm{e}^{0.5-t}, & t>1,\end{cases}$$

it is routine to verify that $X \ge_{\mathrm{hr}} Y$. Thus all assumptions of Theorem 5.1 are verified. Again, as depicted in Figure 4(b), the ratio $H_c(t)/H_s(t)$ is not monotone either. Consequently, the reversed hazard rate order between T_c and T_s in the setting of Theorem 5.1 is invalidated.

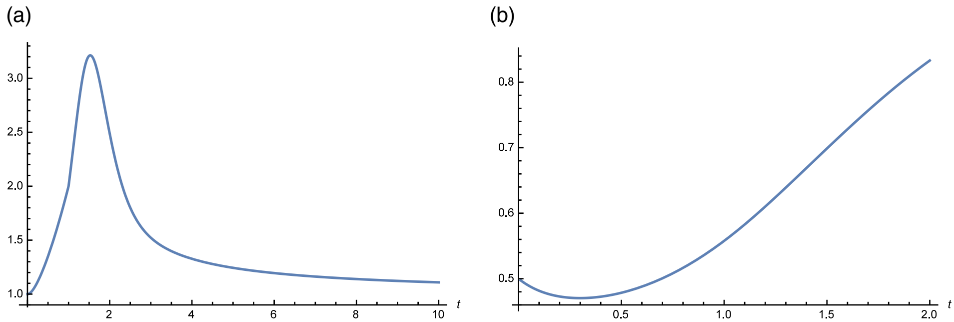

FIGURE 4. Two curves of ratios concerned with system lifetimes T_c and T_s. (a) Ratio of survival functions. (b) Ratio of distribution functions.

6. Some remarks

For coherent systems, due to the complexity incurred by the statistical dependence and heterogeneity of component lifetimes, it is unfeasible to build the usual consistent stochastic order between the system lifetime with relevation redundancy at system level and that with relevation redundancy at component level. Apart from the results developed in previous sections, further in-depth research is needed in order to provide more insight into such a fundamental problem in reliability. Finally, we close this note by presenting two propositions, illuminating future research on series systems of independent components.

As an immediate consequence of Theorems 4.1 and 4.2, we present the first proposition.

Proposition 6.1. *For a series system with components having independent and homogeneous lifetimes $X_1, \ldots, X_n$ and relevation redundancies having independent and homogeneous lifetimes $Y_1, \ldots, Y_n$, if X_1 and Y_1 are of proportional hazard rates, then $T_c \geq_{\text{st}} T_s$.*

It is of natural interest to further check whether $T_c \geq_{\text{st}} T_s$ still holds whenever X_i and Y_i are of proportional hazard rates, $i = 1, \ldots, n$, for mutually independent but heterogeneous $X_1, \ldots, X_n$ in further research.

Recall that a non-negative random variable X is said to be smaller than another one Y in the *stochastic tail order* (denoted as $X \leq_{\text{sto}} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all $x > x_0$ and some threshold $x_0 > 0$. For more on this order we refer readers to [7] and [14]. As for Theorem 4.3, we speculate that the $X \leq_{\text{st}} Y$ of (4.4) itself is not sufficient. Although we come up with the following stochastic tail order, further discussions are expected to clarify the mechanism in future.

Proposition 6.2. *For a series system with components having mutually independent lifetimes, if $X \leq_{\text{st}} Y$ then $T_c \geq_{\text{sto}} T_s$.*

Proof. Recall that

$$a(t) = \int_0^t \frac{f(x)}{\bar{G}(x)}\,\mathrm{d}x \quad \text{for } t \geq 0.$$

For $t \geq 0$ such that $a(t) \geq 1$, it stems from (4.1) that

$$\begin{aligned}\bar{H}_c(t) &= [\bar{F}(t) + \bar{G}(t)a(t)]^n \\ &= \bar{F}^n(t) + \sum_{k=1}^{n} \binom{n}{k} \bar{F}^{n-k}(t)\bar{G}^k(t)a^k(t)\end{aligned}$$

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$$\geq \bar{F}^n(t) + \sum_{k=1}^{n} \binom{n}{k} \bar{F}^{n-k}(t)\bar{G}^k(t)a(t).$$

On the other hand, $X \leq_{\mathrm{st}} Y$ implies $\bar{F}(x) \leq \bar{G}(x)$ for all $x \geq 0$, and thus it follows from (4.2) that

$$\begin{aligned}\bar{H}_s(t) &= \bar{F}^n(t) + \bar{G}^n(t) \int_0^t \frac{n\bar{F}^{n-1}(x)f(x)}{\bar{G}^n(x)}\,\mathrm{d}x \\ &\leq \bar{F}^n(t) + n\bar{G}^n(t)a(t).\end{aligned}$$

Then we have

$$\begin{aligned}\bar{H}_c(t) - \bar{H}_s(t) &\geq \left[n\bar{F}^{n-1}(t) + \sum_{k=2}^{n} \binom{n}{k} \bar{F}^{n-k}(t)\bar{G}^{k-1}(t) - n\bar{G}^{n-1}(t)\right]\bar{G}(t)a(t) \\ &\overset{\mathrm{sgn}}{=} \sum_{k=1}^{n} \binom{n}{k} \bar{F}^{n-k}(t)\bar{G}^{k-1}(t) - n\bar{G}^{n-1}(t) \\ &= \sum_{k=1}^{n} \frac{n}{k}\binom{n-1}{k-1} \bar{F}^{n-k}(t)\bar{G}^{k-1}(t) - n\bar{G}^{n-1}(t) \\ &= n\bar{G}^{n-1}(t)\left[\sum_{l=0}^{n-1} \frac{1}{l+1}\binom{n-1}{l}\left(\frac{\bar{F}(t)}{\bar{G}(t)}\right)^{n-l-1} - 1\right] \\ &\geq n\bar{G}^{n-1}(t)\left[\sum_{l=0}^{n-1} \left(\frac{\bar{F}(t)}{\bar{G}(t)}\right)^{n-l-1} - 1\right] \\ &\geq 0 \quad \text{for all } t \geq 0,\end{aligned}$$

where the second to the last inequality is due to

$$\frac{1}{l+1}\binom{n-1}{l} = \frac{1}{l+1} \cdot \frac{n-1}{n-l-1} \cdot \cdots \cdot \frac{l+2}{2} \cdot \frac{l+1}{1} > 1 \quad \text{for } l = 0, 1, \ldots, n-1.$$

This gives rise to $\bar{H}_c(t) \geq \bar{H}_s(t)$ for all $t \geq 0$ such that $a(t) \geq 1$. That is, $T_c \geq_{\mathrm{sto}} T_s$. □

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Competing interests

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