Proceedings of the Royal Society of Edinburgh, page 1 of 22
DOI:10.1017/prm.2024.29

Eternal solutions to a porous medium equation with strong non-homogeneous absorption. Part I: radially non-increasing profiles

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(Received 9 October 2023; accepted 13 February 2024)

Existence of specific *eternal solutions* in exponential self-similar form to the following quasilinear diffusion equation with strong absorption

$$\partial_t u = \Delta u^m - |x|^\sigma u^q,$$

posed for $(t, x) \in (0, \infty) \times \mathbb{R}^N$, with $m > 1$, $q \in (0, 1)$ and $\sigma = \sigma_c := 2(1-q)/(m-1)$ is proved. Looking for radially symmetric solutions of the form

$$u(t, x) = \mathrm{e}^{-\alpha t} f(|x|\, \mathrm{e}^{\beta t}), \quad \alpha = \frac{2}{m-1}\beta,$$

we show that there exists a unique exponent $\beta^* \in (0, \infty)$ for which there exists a one-parameter family $(u_A)_{A>0}$ of solutions with compactly supported and non-increasing profiles $(f_A)_{A>0}$ satisfying $f_A(0) = A$ and $f_A'(0) = 0$. An important feature of these solutions is that they are bounded and do not vanish in finite time, a phenomenon which is known to take place for all non-negative bounded solutions when $\sigma \in (0, \sigma_c)$.

Keywords: porous medium equation; spatially inhomogeneous absorption; eternal solutions; exponential self-similarity; global solutions

2020 *Mathematics Subject Classification:* 35C06; 34D05; 35A24; 35B33; 35K65

1. Introduction and main results

The goal of the present paper (and also of its second part [22]) is to address the problem of existence and classification of some specific solutions to the following porous medium equation with strong absorption

$$\partial_t u - \Delta u^m + |x|^\sigma u^q = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N, \tag{1.1}$$

in the range of exponents

$$m > 1, \quad q \in (0,1), \quad \sigma = \sigma_c := \frac{2(1-q)}{m-1}. \tag{1.2}$$

On the one hand, equation (1.1) features, in the range of exponents given in (1.2), a competition between the degenerate diffusion term, which tends to conserve the total mass of the solutions while expanding their supports, and the absorption term which leads to a loss of mass. As it has been established and will be explained below, absorption becomes stronger as its exponent q decreases and dominant in the range we are dealing with, leading to specific, although sometimes surprising phenomena such as finite time extinction, instantaneous shrinking and localization of the supports of the solutions. On the other hand, the weight $|x|^\sigma$ with $\sigma > 0$ affects the absorption in the sense of enhancing its effect over regions far away from the origin, where $|x|$ is large, while reducing its strength near $x = 0$, where $|x|^\sigma$ is almost zero (and formally there is even no absorption at $x = 0$).

The balance between these two effects has been best understood in the spatially homogeneous case $\sigma = 0$ of equation (1.1). A lot of development has been done several decades ago in the range $q > m > 1$ where the diffusion is strong and the absorption is not leading the dynamics of the equations, see for example [**27**–**32**, **34**] and references therein. In this range, the previous knowledge of the porous medium equation and its self-similar behaviour had a strong influence in developing the theory. The intermediate range $1 < q \leqslant m$ is not yet totally understood in higher space dimensions. In dimension $N = 1$ it has been shown that solutions are global in time but their supports *are localized* if the initial condition is compactly supported; that is, there exists a radius $R > 0$ not depending on time such that $\operatorname{supp} u(t) \subseteq B(0, R)$ for any $t > 0$. Self-similar solutions might become unbounded [**12**, **33**] and thus a delicate analysis of the large time behaviour, involving the formation of boundary layers, is needed, see [**11**]. Such descriptions are still lacking in dimension $N \geqslant 2$.

More related to our study, still assuming that $\sigma = 0$, the range $q \in (0, 1)$ is the most striking one, where the absorption term dominates the diffusion and leads to two new mathematical phenomena. On the one hand, the *finite time extinction* stemming from the ordinary differential equation $\partial_t u = -u^q$ obtained by neglecting the diffusion has been established by Kalashnikov [**25**, **26**], emphasizing the dominance of the absorption term. On the other hand, *instantaneous shrinking of supports* of solutions to equation (1.1) (with $\sigma = 0$) emanating from a bounded initial condition u_0 such that $u_0(x) \to 0$ as $|x| \to \infty$ takes place; that is, for any non-negative initial condition $u_0 \in L^\infty(\mathbb{R}^N)$ such that $u_0(x) \to 0$ as $|x| \to \infty$ and $\tau > 0$, there is $R(\tau) > 0$ such that $\operatorname{supp} u(t) \subseteq B(0, R(\tau))$ for all $t \geqslant \tau$. This rather unexpected behaviour is once more due to the strength of the absorption, which involves a very quick loss of mass and has been proved in [**1**] after borrowing ideas from previous works [**14**, **26**] devoted to the semilinear case. Finer properties of the dynamics of equation (1.1) for $\sigma = 0$ in this range, such as the behaviour near

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the extinction time or even the extinction rates, are still lacking in a number of cases and seem (up to our knowledge) to be available only when $m + q = 2$ in [**16**], revealing a case of asymptotic simplification. Completing this picture with the cases when $m + q \neq 2$ appears to be a rather complicated open problem.

Drawing our attention now to the spatially inhomogeneous equation (1.1) when $\sigma > 0$, recent results have shown that the magnitude of σ has a very strong influence on the dynamics of equation (1.1) and, in some cases, the weight actually allows for a better understanding of the dynamics. More precisely, the analysis performed by Belaud and coworkers [**3**–**5**], along with the instantaneous shrinking of supports for bounded solutions to equation (1.1) proved in [**21**], shows that, for $0 < \sigma < \sigma_c$, *any non-negative solution* to equation (1.1) with bounded initial condition *vanishes in finite time*. A more direct proof of this result is given by the authors in the recent short note [**20**]. On the contrary, after developing the general theory of well-posedness for equation (1.1), we have focussed on the range $\sigma > \sigma_c$ in our previous work [**21**] and proved that, in the latter, finite time extinction *depends strongly on how concentrated* is the initial condition in a neighbourhood of the origin. More precisely, initial data which are positive in a ball $B(0,\,\delta)$ give rise to solutions with a non-empty positivity set for all times,

$$\{x \in \mathbb{R}^N \ : \ u(t,x) > 0\} \neq \emptyset \text{ for all } t > 0, \tag{1.3}$$

when $\sigma > \sigma_c$, while initial data which vanish in a suitable way as $x \to 0$ and with a sufficiently small L^∞ norm lead to solutions vanishing in finite time, as proved in [**20**] where optimal conditions are given. All these cases of different dynamics are consequences of the two types of competitions explained in the previous paragraphs.

The exponent $\sigma_c = 2(1-q)/(m-1)$ thus appears to separate the onset of extinction in finite time for arbitrary non-negative and bounded initial conditions which occurs for lower values of σ and the positivity property (1.3) which is known to take place for higher values of σ, in particular for initial conditions which are positive in a ball $B(0,\,\delta)$. According to [**20**], when $\sigma = \sigma_c$, there are non-negative solutions to equation (1.1) vanishing in finite time, their initial conditions having a sufficiently small L^∞-norm and decaying to zero in a suitable way as $x \to 0$, and the issue we address here is whether the positivity property (1.3) also holds true for some solutions to equation (1.1) when $\sigma = \sigma_c$. We actually construct specific solutions to equation (1.1) with $\sigma = \sigma_c$ featuring this property and these solutions turn out to have an exponential self-similar form as explained in detail below. In particular, they are defined for all $t \in \mathbb{R}$.

Main results. We are looking in this paper for some special solutions to equation (1.1) with m, q and $\sigma = \sigma_c$ as in (1.2) having an *exponential self-similar form*; that is,

$$u(t,x) = \mathrm{e}^{-\alpha t} f(|x|\mathrm{e}^{\beta t}), \quad (t,x) \in (0,\infty) \times \mathbb{R}^N. \tag{1.4}$$

Notice that solutions as in (1.4) are actually defined for all $t \in (-\infty,\,\infty)$; that is, they are not only global in time but *eternal*. Even if solutions of form (1.4) are rather unexpected for parabolic equations due to the irreversibility of time, several equations are known to have such solutions but usually in critical cases separating different behaviours. Parabolic equations featuring this property include the

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two-dimensional Ricci flow [13, 18], the fast diffusion equation with critical exponent $m_c = (N-2)/N$ in space dimension $N \geqslant 3$ [15], a viscous Hamilton–Jacobi equation featuring singular diffusion of p-Laplacian type, $p \in (2N/(N+1), 2)$ and critical gradient absorption [19], and the related reaction-diffusion equation $\partial_t u - \Delta u^m - |x|^\sigma u^q = 0$ [23, 24]. Concerning the latter, the critical value of σ is exactly the same as in (1.2), but the dynamic properties of the solutions strongly differ from the present work, since the spatially inhomogeneous part is a source term, introducing mass to the equation. Eternal solutions are also available for kinetic equations, such as the spatially homogeneous Boltzmann equation for Maxwell molecules [7, 9] or Smoluchowski's coagulation equation with coagulation kernel of homogeneity one [6, 8]. Let us finally mention that, besides solutions of the form (1.4), another important class of self-similar eternal solutions of evolution problems is that of travelling wave solutions of form $(t, x) \mapsto u(x - ct)$ in space dimension $N = 1$, which are available for scalar conservation laws and parabolic equations such as the celebrated Fisher-KPP equation, see [10, 17, 35] and the references therein.

Returning to ansatz (1.4), setting $\xi = |x|e^{\beta t}$ and performing some direct calculations, we readily find that the self-similar exponents must satisfy the condition

$$\alpha = \frac{2}{m-1}\beta, \tag{1.5}$$

where β becomes a free parameter for our analysis, while the profile f solves the differential equation

$$(f^m)''(\xi) + \frac{N-1}{\xi}(f^m)'(\xi) + \alpha f(\xi) - \beta\xi f'(\xi) - \xi^\sigma f^q(\xi) = 0, \quad \xi > 0. \tag{1.6}$$

The solutions to equation (1.6) we are looking for in this first part of a two-part work are solutions taking positive values at $\xi = 0$. To this end, let us observe that we can fix, without loss of generality, the initial condition as

$$f(0) = 1, \quad f'(0) = 0. \tag{1.7}$$

Indeed, given $a > 0$ and a solution f to (1.6)–(1.7), we can readily obtain by direct calculations that the rescaled function

$$g(\xi; a) = af(a^{-(m-1)/2}\xi) \tag{1.8}$$

solves (1.6) with initial conditions $g(0; a) = a$, $g'(0; a) = 0$. This leaves us with the task of solving the Cauchy problem (1.6)–(1.7), which is performed in the next result.

THEOREM 1.1. *Let m, q and $\sigma = \sigma_c$ as in (1.2). There exists a unique exponent $\beta^* > 0$ (and corresponding $\alpha^* = 2\beta^*/(m-1)$) such that, for $\alpha = \alpha^*$ and $\beta = \beta^*$, the Cauchy problem (1.6)–(1.7) has a compactly supported, non-negative and non-increasing solution $f^* \in C^1([0, \infty))$ with $(f^*)^m \in C^2([0, \infty))$. The function U^* defined by*

$$U^*(t, x) = e^{-\alpha^* t} f^*(|x|e^{\beta^* t}), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

is then a self-similar solution to equation (1.1) in exponential form (1.4).

Let us point out that, in strong contrast with the range $\sigma > 2(1-q)/(m-1)$ analysed in [**21**] and where the self-similarity exponents were uniquely determined, in the present case we have two free parameters for the shooting technique: both the initial value of the solution at $x = 0$ and the self-similar exponent β. Thus, in order to have uniqueness, we need to fix this initial value in view of rescaling (1.8), as explained above.

One of the interesting features of this work is the fact that the proof of theorem 1.1 is based on a *mix between various techniques*. We employ mainly a shooting technique with respect to the free parameter β, but in order to study the interface behaviour and establish the uniqueness in theorem 1.1, we transform (1.6) into a quadratic three-dimensional autonomous dynamical system and study a specific local behaviour and critical point in the associated phase space. Let us stress here that we have to go deeper than the analogous study of the interface behaviour in [**21**, § 4], since in some cases we need a second order local expansion near the interface point.

We end up this presentation by mentioning that the present work is the first part of a two-part analysis of eternal solutions to equation (1.1) and will be followed by a companion work [**22**] in which a second and rather surprising type of profiles, presenting a dead-core, is identified and classified, by employing a quite different bunch of techniques based on the complete analysis of an auxiliary dynamical system. Altogether, the existence of such a variety of self-similar solutions in exponential form shows that the dynamics of equation (1.1) in the critical case $\sigma = \sigma_c$ is expected to be rather complex and to depend on many features of the initial conditions (such as concentration near $x = 0$, magnitude of $\|u_0\|_\infty$ and location of the points where the maximum is attained, to name but a few) and is definitely a challenging problem.

2. Proof of theorem 1.1

The proof of theorem 1.1 is based on a shooting method with respect to the free exponent β and follows the same strategy as [**21**, § 4]. However, a number of preparatory results are proved in a different way and the analysis near the interface requires to be improved in some cases with the help of a phase space analysis. We divide this section into several subsections containing the main steps of the proof.

2.1. Existence of a compactly supported self-similar solution

Let $\beta > 0$ and $\alpha = 2\beta/(m-1)$. Recalling the differential equation (1.6) satisfied by the self-similar profiles f and setting for simplicity $F = f^m$, we study, as explained in the Introduction, the Cauchy problem

$$F''(\xi) + \frac{N-1}{\xi}F'(\xi) + \alpha f(\xi) - \beta\xi f'(\xi) - \xi^\sigma f^q(\xi) = 0, \tag{2.1a}$$

$$F(0) = 1, \quad F'(0) = 0. \tag{2.1b}$$

We obtain from the Cauchy–Lipschitz theorem that problem (2.1) has a unique positive solution $F(\cdot;\beta) \in C^2([0, \xi_{\max}(\beta)))$ defined on a maximal existence interval

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for which we have the following alternative: either $\xi_{\max}(\beta) = \infty$ or

$$\xi_{\max}(\beta) < \infty \quad \text{and} \quad \lim_{\xi \to \xi_{\max}(\beta)} \left[F(\xi;\beta) + \frac{1}{F(\xi;\beta)} \right] = \infty.$$

We next define

$$\xi_0(\beta) := \inf\{\xi \in (0, \xi_{\max}(\beta)) \ : \ f(\xi) = 0\} \in (0, \xi_{\max}(\beta)], \tag{2.2}$$

and

$$\xi_1(\beta) := \sup\{\xi \in (0, \xi_0(\beta)) \ : \ f' < 0 \text{ on } (0,\xi)\}. \tag{2.3}$$

We readily notice from (2.1a) and the C^2-regularity of F that

$$F''(0;\beta) = -\frac{2\beta}{(m-1)N} < 0, \tag{2.4}$$

so that $\xi_1(\beta) > 0$. Let us now study more precisely the behaviour of $F(\cdot;\beta)$ near $\xi_0(\beta)$ when $\xi_0(\beta)$ is finite.

Lemma 2.1. *Consider $\beta > 0$ such that $\xi_0(\beta) < \infty$. Then $\xi_{\max}(\beta) = \xi_0(\beta)$ and $F = F(\cdot;\beta) \in C^1([0,\, \xi_0(\beta)])$ satisfies $F(\xi_0(\beta)) = 0$ and*

$$F'(\xi_0(\beta)) = \xi_0(\beta)^{1-N} \int_0^{\xi_0(\beta)} \xi_*^{N-1} \left[\xi_*^{\sigma} f^q(\xi_*) - (\alpha + N\beta) f(\xi_*)\right] \mathrm{d}\xi_*,$$

recalling that $f = F^{1/m}$. Furthermore, if $\xi_0(\beta) = \xi_1(\beta)$ and $F'(\xi_0(\beta)) = 0$, then the extension of F by zero on $(\xi_0(\beta),\, \infty)$ belongs to $C^2([0,\, \infty))$ and is a solution to (2.1) on $[0,\, \infty)$ with

$$F(\xi_0(\beta)) = F'(\xi_0(\beta)) = F''(\xi_0(\beta)) = \left(F^{1/m}\right)'(\xi_0(\beta)) = 0.$$

Also, the extension of f by zero on $(\xi_0(\beta),\, \infty)$ belongs to $C^1([0,\, \infty))$.

Proof. As $\xi_0(\beta) < \infty$, then the above alternative implies that $\xi_{\max}(\beta) = \xi_0(\beta)$ and

$$\lim_{\xi \to \xi_0(\beta)} F(\xi) = 0. \tag{2.5}$$

Moreover, it follows from (2.1a) that

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \left[\xi^{N-1} F'(\xi) - \beta \xi^N f(\xi)\right] = \xi^{N-1} \left[\xi^{\sigma} f^q(\xi) - (\alpha + N\beta) f(\xi)\right] \tag{2.6}$$

for $\xi \in [0,\, \xi_0(\beta))$; hence, after integration over $(0,\, \xi)$,

$$\xi^{N-1} F'(\xi) - \beta \xi^N f(\xi) = \int_0^{\xi} \xi_*^{N-1} \left[\xi_*^{\sigma} f^q(\xi_*) - (\alpha + N\beta) f(\xi_*)\right] \mathrm{d}\xi_*.$$

Since we have already established in (2.5) that F and f have a continuous extension on $[0,\, \xi_0(\beta)]$, we may take the limit $\xi \to \xi_0(\beta)$ in the above identity and complete the proof of the first statement of lemma 2.1.

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Now, assuming that $\xi_0(\beta) = \xi_1(\beta)$ and $F'(\xi_0(\beta)) = 0$, we integrate identity (2.6) over $(\xi,\, \xi_0(\beta))$ and find

$$-\xi^{N-1}F'(\xi) + \beta\xi^N f(\xi) = \int_{\xi}^{\xi_0(\beta)} \xi_*^{N-1}\left[\xi_*^{\sigma} f^q(\xi_*) - (\alpha + N\beta) f(\xi_*)\right] \mathrm{d}\xi_*$$

for $\xi \in (0,\, \xi_0(\beta))$. Owing to the non-negativity of f and $-F'$ on $(0,\, \xi_0(\beta))$, we further obtain

$$0 \leqslant -\xi^{N-1}\frac{F'(\xi)}{\xi_0(\beta) - \xi} \leqslant \frac{1}{\xi_0(\beta) - \xi}\int_{\xi}^{\xi_0(\beta)} \xi_*^{N-1}\left[\xi_*^{\sigma} f^q(\xi_*) - (\alpha + N\beta) f(\xi_*)\right] \mathrm{d}\xi_*$$

for $\xi \in (0,\, \xi_0(\beta))$. Since $f(\xi_0(\beta)) = 0$, the right-hand side of the above inequality converges to zero as $\xi \nearrow \xi_0(\beta)$ and we conclude that $F''(\xi_0(\beta))$ is well-defined and equal to zero. Therefore, the extension of F by zero on $(\xi_0(\beta),\, \infty)$ is a C^2-smooth function on $[0,\, \infty)$, as claimed. Similarly, for $\xi \in (0,\, \xi_0(\beta))$,

$$0 \leqslant \beta\xi^{N}\frac{f(\xi)}{\xi_0(\beta) - \xi} \leqslant \frac{1}{\xi_0(\beta) - \xi}\int_{\xi}^{\xi_0(\beta)} \xi_*^{N-1}\left[\xi_*^{\sigma} f^q(\xi_*) - (\alpha + N\beta) f(\xi_*)\right] \mathrm{d}\xi_*$$

from which we deduce that $f'(\xi_0(\beta))$ is well-defined and equal to zero. Hence, the extension of f by zero on $(\xi_0(\beta),\, \infty)$ belongs to $C^1([0,\, \infty))$. □

We now introduce the following three sets:

$$\mathcal{A} := \{\beta > 0 : \xi_0(\beta) < \infty \text{ and } F'(\xi;\beta) < 0 \text{ for } \xi \in (0, \xi_0(\beta)]\},$$
$$\mathcal{C} := \{\beta > 0 : \xi_1(\beta) < \xi_0(\beta)\},$$
$$\mathcal{B} := (0, \infty) \setminus (\mathcal{A} \cup \mathcal{C}),$$

and observe that $\mathcal{A} \cap \mathrm{C} = \emptyset$. Let us first show that the sets $\mathcal{A}$ and $\mathcal{C}$ are non-empty and open.

LEMMA 2.2. *The set $\mathcal{A}$ is non-empty and open and there exists $\beta_u > 0$ such that $(\beta_u,\, \infty) \subseteq \mathcal{A}$.*

Proof. Set $g(\xi;\beta) = f(\xi/\sqrt{\beta};\beta)$ for $\xi \in [0,\, \sqrt{\beta}\xi_0(\beta)]$, or equivalently $f(\xi;\beta) = g(\xi\sqrt{\beta};\beta)$ for $\xi \in [0,\, \xi_0(\beta)]$. Setting also $G := g^m$, we obtain by straightforward calculations that g (and thus G) solves the Cauchy problem

$$G''(\zeta) + \frac{N-1}{\zeta}G'(\zeta) + \frac{2}{m-1}g(\zeta) - \zeta g'(\zeta) - \beta^{-(\sigma+2)/2}\zeta^{\sigma}g^q(\zeta) = 0, \tag{2.7a}$$

$$G(0) = 1, \quad G'(0) = 0, \tag{2.7b}$$

where $\zeta = \xi\sqrt{\beta}$. Noticing that in the limit $\beta \to \infty$ the last term in (2.7a) vanishes, we proceed exactly as in the proof of [**21**, lemma 4.4] (see also the proof of [**36**, theorem 2] from where the idea comes) to conclude that there exists $\beta_u > 0$ such that $(\beta_u,\, \infty) \subseteq \mathcal{A}$. We omit here the details as they are totally similar to the ones in the quoted references. That $\mathcal{A}$ is open is an immediate consequence of the continuous dependence of $f(\cdot;\beta)$ on β. □

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As for the set $\mathcal{C}$, we do not need a rescaling in order to prove that it is non-empty.

Lemma 2.3. *The set $\mathcal{C}$ is non-empty and open and there exists $\beta_l > 0$ such that $(0, \beta_l) \subseteq \mathcal{C}$.*

Proof. We obtain by letting $\beta \to 0$ in (2.1) that the limit equation is

$$H''(\xi) + \frac{N-1}{\xi} H'(\xi) - \xi^\sigma H^{q/m}(\xi) = 0, \tag{2.8a}$$

with initial conditions

$$H(0) = 1, \quad H'(0) = 0. \tag{2.8b}$$

By the Cauchy–Lipschitz theorem, problem (2.8) has a unique positive solution $H \in C^2([0, \xi_H))$ defined on a maximal existence interval for which we have the following alternative: either $\xi_H = \infty$ or

$$\xi_H < \infty \quad \text{and} \quad \lim_{\xi \to \xi_H} \left[H(\xi) + \frac{1}{H(\xi)} \right] = \infty.$$

It follows from (2.8) that

$$\frac{\mathrm{d}}{\mathrm{d}\xi}(\xi^{N-1} H'(\xi)) = \xi^{N-1} \left[H''(\xi) + \frac{N-1}{\xi} H'(\xi) \right] = \xi^{N+\sigma-1} H^{q/m}(\xi) > 0.$$

Hence $\xi^{N-1} H'(\xi) > 0$ and thus $H'(\xi) > 0$ for any $\xi \in (0, \xi_H)$. Given $\delta \in (0, \xi_H)$ fixed, we have $H'(\delta) > 0$ and $H(\xi) > 1$ for any $\xi \in (0, \delta)$. The continuous dependence with respect to the parameter β in (2.1) ensures that there exists $\beta_l > 0$ such that

$$F(\xi; \beta) > \frac{1}{2}, \quad \xi \in [0, \delta], \quad F'(\delta; \beta) > \frac{H'(\delta)}{2} > 0$$

for any $\beta \in (0, \beta_l)$. Recalling (2.2) and (2.3), we conclude that $\xi_1(\beta) \in (0, \delta)$ and $\xi_0(\beta) > \delta$ for any $\beta \in (0, \beta_l)$; that is, $\xi_1(\beta) < \xi_0(\beta)$ for $\beta \in (0, \beta_l)$ and $(0, \beta_l) \subseteq \mathcal{C}$. We use once more the continuous dependence with respect to the parameter β of $F(\cdot; \beta)$ to conclude that $\mathcal{C}$ is open. □

We infer from lemmas 2.2 and 2.3 that the set $\mathcal{B}$ is non-empty and closed. The instantaneous shrinking of supports of bounded solutions to equation (1.1) proved in [**21**, theorem 1.1], together with the definition of the set $\mathcal{A}$, readily gives the following characterization of the elements in the set $\mathcal{B}$.

Lemma 2.4. *Let $\beta \in \mathcal{B}$. Then $\xi_0(\beta) = \xi_1(\beta) < \infty$ and $(f^m)'(\xi_0(\beta); \beta) = 0$.*

The proof is immediate and is given with details in [**21**, lemma 4.6]. We thus conclude that, for any element $\beta \in \mathcal{B}$, we have an eternal self-similar solution to equation (1.1) in form (1.4) with profile $f(\cdot; \beta)$ as in lemma 2.4.

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2.2. Monotonicity

In this section, we prove the following general monotonicity property of the profiles $f(\cdot;\beta)$ solving (2.1) with respect to the parameter β.

LEMMA 2.5. *Let* $0 < \beta_1 < \beta_2 < \infty$*. Then*

$$f(\xi;\beta_1) > f(\xi;\beta_2) \quad \text{for any} \quad \xi \in (0, \min\{\xi_1(\beta_1), \xi_1(\beta_2)\}) .$$

Proof. Consider $\beta_2 > \beta_1 > 0$ and pick $X \in (0, \min\{\xi_1(\beta_1), \xi_1(\beta_2)\})$. Then

$$F_i := F(\cdot;\beta_i) > 0, \quad F_i' < 0, \quad \text{in } (0, X).$$

Since $\beta_2 > \beta_1$ and $F_1(0) = F_2(0) = 1$, $F_1'(0) = F_2'(0) = 0$, we infer from (2.4) that $F_2 < F_1$ in a right-neighbourhood of $\xi = 0$. We may thus define

$$\xi_* := \inf\{\xi \in (0, X) : F_1(\xi) = F_2(\xi)\} > 0,$$

and notice that $F_2(\xi) < F_1(\xi)$ for any $\xi \in (0, \xi_*)$. Assume for contradiction that $\xi_* < X$. Then $F_2(\xi_*) = F_1(\xi_*)$. We introduce for any $\lambda \geqslant 1$ the following family of rescaled functions

$$G_\lambda(\xi) := \lambda^m F_2(\lambda^{-(m-1)/2}\xi), \quad \xi \in [0, \xi_*], \tag{2.9}$$

which are also solutions to (2.1a) with $\beta = \beta_2$, and adapt an optimal barrier argument from [**37**] (see also [**21**, lemma 4.12]). Owing to the monotonicity of F_1 and F_2 on $[0, X]$, we first note that

$$\min_{\xi\in[0,\xi_*]} G_\lambda(\xi) = G_\lambda(\xi_*) = \lambda^m F_2(\lambda^{-(m-1)/2}\xi_*) \geqslant \lambda^m F_2(\xi_*),$$

whence

$$\lim_{\lambda\to\infty} \min_{\xi\in[0,\xi_*]} G_\lambda(\xi) = \infty,$$

while $F_1(\xi) \leqslant 1$ for $\xi \in [0, \xi_*]$. Consequently, the optimal parameter

$$\lambda_0 := \inf\{\lambda \geqslant 1 : G_\lambda(\xi) > F_1(\xi),\ \xi \in [0, \xi_*]\} \tag{2.10}$$

is well defined and finite. Since $F_2 < F_1$ on $(0, \xi_*)$, we also deduce that $\lambda_0 > 1$. The definition of λ_0 guarantees that there exists $\eta \in [0, \xi_*]$ such that

$$G_{\lambda_0}(\eta) = F_1(\eta), \quad G_{\lambda_0} \geqslant F_1 \text{ in } [0, \xi_*]. \tag{2.11}$$

On the one hand, we infer from the monotonicity of F_2 and the property $\lambda_0 > 1$ that

$$F_1(\xi_*) = F_2(\xi_*) < \lambda_0^m F_2(\xi_*) < \lambda_0^m F_2(\lambda_0^{-(m-1)/2}\xi_*) = G_{\lambda_0}(\xi_*),$$

which rules out the possibility that $\eta = \xi_*$. On the other hand,

$$G_{\lambda_0}(0) = \lambda_0^m F_2(0) = \lambda_0^m > 1 = F_1(0),$$

so that $\eta > 0$. Consequently, $\eta \in (0, \xi_*)$ and we derive from (2.11) that $G_{\lambda_0} - F_1$ attains a strict minimum at $\xi = \eta$, which, together with the definition of η, implies that

$$G_{\lambda_0}(\eta) = F_1(\eta), \quad G'_{\lambda_0}(\eta) = F'_1(\eta), \quad G''_{\lambda_0}(\eta) \geqslant F''_1(\eta). \tag{2.12}$$

Since both G_{λ_0} and F_1 are solutions to (2.1a) with parameters β_2 and β_1, respectively, we infer from (2.12) that

$$\begin{aligned}
0 &= G''_{\lambda_0}(\eta) + \frac{N-1}{\eta}G'_{\lambda_0}(\eta) + \frac{2\beta_2}{m-1}G_{\lambda_0}^{1/m}(\eta) - \beta_2\eta\left(G_{\lambda_0}^{1/m}\right)'(\eta) - \eta^\sigma G_{\lambda_0}^{q/m}(\eta)\\
&\geqslant F''_1(\eta) + \frac{N-1}{\eta}F'_1(\eta) + \frac{2\beta_2}{m-1}F_1^{1/m}(\eta) - \beta_2\eta\left(F_1^{1/m}\right)'(\eta) - \eta^\sigma F_1^{q/m}(\eta)\\
&= -\frac{2\beta_1}{m-1}F_1^{1/m}(\eta) + \beta_1\frac{\eta}{m}F_1^{(1-m)/m}(\eta)F'_1(\eta) + \frac{2\beta_2}{m-1}F_1^{1/m}(\eta)\\
&\quad - \beta_2\frac{\eta}{m}F_1^{(1-m)/m}(\eta)F'_1(\eta)\\
&= (\beta_2-\beta_1)F_1^{(1-m)/m}(\eta)\left[\frac{2}{m-1}F_1(\eta) - \frac{\eta}{m}F'_1(\eta)\right] > 0,
\end{aligned}$$

which leads to a contradiction. We have thus established that $F_2 < F_1$ on $(0, X)$ and the proof is complete due to the arbitrary choice of $X \in (0, \xi_1(\beta_2)) \cap (0, \xi_1(\beta_1))$. □

Let us remark that, in contrast to the range $\sigma > \sigma_c$ studied in [**21**, § 3], in our case the profiles $f(\cdot;\beta)$ are ordered in a decreasing way with respect to the shooting parameter β.

2.3. Interface behaviour

The goal of this section is deriving the local behaviour near the interface point $\xi_0(\beta)$ for profiles $f(\cdot;\beta)$ with $\beta \in \mathcal{B}$. We begin with a formal calculation. Let us drop for simplicity β from the notation and assume that, at the interface, we have

$$f(\xi) \sim A(\xi_0-\xi)^\theta, \quad f'(\xi) \sim -A\theta(\xi_0-\xi)^{\theta-1}, \quad \text{as } \xi \to \xi_0 = \xi_0(\beta),$$

for some $A > 0$ and $\theta > 0$ to be determined. We also obtain formally that

$$(f^m)'(\xi) \sim -m\theta A^m(\xi_0-\xi)^{m\theta-1}, \quad (f^m)''(\xi) \sim m\theta(m\theta-1)A^m(\xi_0-\xi)^{m\theta-2},$$

both equivalences holding true as $\xi \to \xi_0$. Inserting this ansatz in (1.6) gives, as $\xi \to \xi_0$,

$$\begin{aligned}
&m\theta(m\theta-1)A^m(\xi_0-\xi)^{m\theta-2} - \frac{N-1}{\xi_0}m\theta A^m(\xi_0-\xi)^{m\theta-1}\\
&\quad + \beta\xi_0 A\theta(\xi_0-\xi)^{\theta-1} + \frac{2\beta}{m-1}A(\xi_0-\xi)^\theta - A^q\xi_0^\sigma(\xi_0-\xi)^{q\theta} = 0.
\end{aligned}$$

We thus have four possibilities of balancing the dominating powers.

- $m\theta - 2 = \theta - 1 < q\theta$. This implies $\theta = 1/(m-1)$, but in this case $m\theta - 1 = \theta > 0$ and thus this choice leads to $A = 0$.

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- $\theta - 1 = q\theta < m\theta - 2$. This implies $\theta = 1/(1-q)$ and $m\theta - 2 > q\theta$ leads straightforwardly to $m + q > 2$.
- $m\theta - 2 = q\theta < \theta - 1$. This implies $\theta = 2/(m-q)$ and the inequality $\theta - 1 > q\theta$ easily gives $m + q < 2$.
- $m\theta - 2 = q\theta = \theta - 1$. This implies that $\theta = 1/(m-1) = 1/(1-q)$ and $m + q = 2$.

Looking now at the constant A in front of the previous ansatz, we find the following three cases:

Case 1. $m + q > 2$. According to the formal calculation, we expect $\theta = 1/(1-q)$ and then $\beta\xi_0 A\theta = A^q\xi_0^\sigma$, which leads to

$$A^{1-q} = \frac{(1-q)\xi_0^{\sigma-1}}{\beta}. \tag{2.13}$$

Case 2. $m + q = 2$. We expect $\theta = 1/(1-q) = 2/(m-q)$ and

$$m\theta(m\theta - 1)A^m + \beta\xi_0 A\theta - A^q\xi_0^\sigma = 0;$$

that is, $A = A_*$ with A_* being the unique positive solution to

$$\frac{m(m+q-1)}{(1-q)^2}A_*^{m-q} + \frac{\beta\xi_0}{1-q}A_*^{1-q} - \xi_0^\sigma = 0.$$

Since $m + q = 2$ and $\sigma = 2$ in that case, the above equation simplifies to

$$\frac{m}{(1-q)^2}A_*^{m-q} + \frac{\beta\xi_0}{1-q}A_*^{(m-q)/2} - \xi_0^2 = 0. \tag{2.14}$$

Case 3. $m + q < 2$. We expect $\theta = 2/(m-q)$ and $m\theta(m\theta - 1)A^m = A^q\xi_0^\sigma$, hence

$$A^{m-q} = \frac{(m-q)^2}{2m(m+q)}\xi_0^\sigma. \tag{2.15}$$

In order to prove in a rigorous way all these estimates near the interface, we proceed as in [**21**]. We start with some general upper bounds at the interface, but omit the proof, as it is totally similar to that of [**21**, lemma 4.7].

LEMMA 2.6. *Assume that* $\beta \in \mathcal{B}$ *and set* $f = f(\cdot;\beta)$ *and* $\xi_0 = \xi_0(\beta)$. *Then*

$$|(f^{m-q})'(\xi)| \leqslant 2^{N-1}\xi_0^\sigma(\xi_0 - \xi), \quad \xi \in \left(\frac{\xi_0}{2}, \xi_0\right), \tag{2.16}$$

and

$$f(\xi) \leqslant \beta^{q-1}\xi_0^{(\sigma-1)/(1-q)}(\xi_0 - \xi)^{1/(1-q)}, \quad \xi \in \left(\frac{\xi_0}{2}, \xi_0\right). \tag{2.17}$$

Moreover, there exists $C_1 > 0$ *depending only on* N, m *and* q *such that*

$$f(\xi) \leqslant C_1\xi_0^{\sigma/(m-q)}(\xi_0 - \xi)^{2/(m-q)}, \quad \xi \in \left(\frac{\xi_0}{2}, \xi_0\right). \tag{2.18}$$

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The following consequences of lemma 2.6 are drawn in the same way as in [21, lemmas 4.8 and 4.9].

Corollary 2.7. *Let $\beta \in \mathcal{B}$ and set $f = f(\cdot;\beta)$ and $\xi_0 = \xi_0(\beta)$. Then*

$$\limsup_{\xi \to \xi_0} \left(f^{(m-q)/2} \right)' (\xi) > -\infty.$$

In addition, if $m+q > 2$ then

$$\limsup_{\xi \to \xi_0} \left(f^{m-1} \right)' (\xi) = 0.$$

The estimates given in corollary 2.7 allow us to proceed as in [21, propositions 4.10 and 4.11] in order to identify the precise algebraic rate at which $f(\cdot;\beta)$ vanishes at the interface, which depends on the sign of $m+q-2$ as follows.

Proposition 2.8. *Let $\beta \in \mathcal{B}$ and set $f = f(\cdot;\beta)$ and $\xi_0 = \xi_0(\beta)$.*

(a) If $m+q<2$, then, as $\xi \to \xi_0$,

$$f(\xi) = K_1 \xi_0^{\sigma/(m-q)} (\xi_0 - \xi)^{2/(m-q)} + o((\xi_0-\xi)^{2/(m-q)}), \tag{2.19}$$

where

$$K_1 := \left[\frac{m-q}{\sqrt{2m(m+q)}} \right]^{2/(m-q)}.$$

(b) If $m+q=2$, then $\sigma = 2$ and, as $\xi \to \xi_0$,

$$f(\xi) = K_1 \xi_0^{2/(m-q)} K_2(\beta) (\xi_0 - \xi)^{2/(m-q)} + o((\xi_0-\xi)^{2/(m-q)}), \tag{2.20}$$

where K_1 is defined in part (a) and

$$K_2(\beta) := \left[\sqrt{1+\frac{\beta^2}{4m}} - \frac{\beta}{2\sqrt{m}} \right]^{2/(m-q)}.$$

(c) If $m+q>2$, then, as $\xi \to \xi_0$,

$$f(\xi) = K_3(\beta) \xi_0^{(\sigma-1)/(1-q)} (\xi_0 - \xi)^{1/(1-q)} + o((\xi_0-\xi)^{1/(1-q)}), \tag{2.21}$$

where

$$K_3(\beta) := \left[\frac{1-q}{\beta} \right]^{1/(1-q)}.$$

Let us notice here that the values of K_1, $K_2(\beta)$ and $K_3(\beta)$ in (2.19), (2.20) and (2.21) correspond to the values of A obtained through the formal deduction in (2.15), (2.14) and (2.13), respectively. It is now worth pointing out that there is no explicit dependence on β in the behaviour (2.19) when $m+q<2$. This is why we need to perform some rather serious extra work in order to identify the second

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order of the expansion at the interface when $m+q \in (1, 2)$, as formal computations (which are rather tedious and we do not give here) reveal that β shows up in an explicit way in this next order, a feature that will be very helpful in the proof of the uniqueness issue. More precisely, we have the following asymptotic expansions.

PROPOSITION 2.9. *Let $m+q<2$, $\beta \in \mathcal{B}$ and set $f = f(\cdot;\beta)$ and $\xi_0 = \xi_0(\beta)$. Then, as $\xi \to \xi_0$,*

$$\begin{aligned} f(\xi) = {} & K_1 \xi_0^{\sigma/(m-q)} (\xi_0 - \xi)^{2/(m-q)} \\ & - K_0(\beta) \xi_0^{(\sigma+m+q-2)/(m-q)} (\xi_0 - \xi)^{(4-m-q)/(m-q)} \\ & + o((\xi_0 - \xi)^{(4-m-q)/(m-q)}), \end{aligned} \tag{2.22}$$

where K_1 is defined in (2.19) and

$$K_0(\beta) := \frac{(m-q)\beta K_1^{2-m}}{m(1-q)(m+q+2)}. \tag{2.23}$$

Proof. As in the proof of [**21**, proposition 4.10], we introduce the new dependent variables

$$\begin{aligned} \mathcal{X}(\xi) &:= \sqrt{m} \xi^{-(\sigma+2)/2} f^{(m-q)/2}(\xi), \\ \mathcal{Y}(\xi) &:= \sqrt{m} \xi^{-\sigma/2} f^{(m-q-2)/2}(\xi) f'(\xi), \\ \mathcal{Z}(\xi) &:= \frac{\alpha}{\sqrt{m}} \xi^{(2-\sigma)/2} f^{(2-m-q)/2}(\xi), \end{aligned} \tag{2.24}$$

as well as a new independent variable η via the integral representation

$$\eta(\xi) := \frac{1}{\sqrt{m}} \int_0^\xi f^{(q-m)/2}(\xi_*) \xi_*^{\sigma/2} \, d\xi_*, \quad \xi \in [0, \xi_0). \tag{2.25}$$

Introducing (X, Y, Z) defined by $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = (X \circ \eta, Y \circ \eta, Z \circ \eta)$, we see that (X, Y, Z) solves the quadratic autonomous dynamical system

$$\begin{cases} \dot{X} = X \left[\dfrac{m-q}{2} Y - \dfrac{\sigma+2}{2} X \right] \\ \dot{Y} = -\dfrac{m+q}{2} Y^2 - \left(N - 1 + \dfrac{\sigma}{2} \right) XY - XZ + \dfrac{m-1}{2} YZ + 1 \\ \dot{Z} = Z \left[\dfrac{2-m-q}{2} Y + \dfrac{2-\sigma}{2} X \right]. \end{cases} \tag{2.26}$$

Observe that, owing to (2.19),

$$\lim_{\xi \to \xi_0} \eta(\xi) = \infty,$$

so that studying the behaviour of $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})(\xi)$ as $\xi \to \xi_0$ amounts to that of $(X, Y, Z)(\eta)$ as $\eta \to \infty$. Furthermore, we argue as in [**21**, proposition 4.10] to deduce from (2.19) and corollary 2.7 that

$$(X, Y, Z)(\eta) \in (0, \infty) \times (-\infty, 0) \times (0, \infty), \quad \eta > 0,$$

and

$$\lim_{\eta \to \infty} (X, Y, Z)(\eta) = \left(0, -\sqrt{\frac{2}{m+q}}, 0\right).$$

We are thus interested in the behaviour near the critical point $(0, -\sqrt{2/(m+q)}, 0)$. We translate this point to the origin of coordinates by setting

$$W = Y + \sqrt{\frac{2}{m+q}}. \tag{2.27}$$

We then find by direct calculation that system (2.26) becomes

$$\begin{cases} \dot{X} &= -\dfrac{m-q}{\sqrt{2(m+q)}} X + \dfrac{m-q}{2} XW - \dfrac{\sigma+2}{2} X^2 \\ \dot{W} &= \left(N - 1 + \dfrac{\sigma}{2}\right) \sqrt{\dfrac{2}{m+q}} X + \sqrt{2(m+q)} W - \dfrac{m-1}{\sqrt{2(m+q)}} Z \\ & \quad - \left(N - 1 + \dfrac{\sigma}{2}\right) XW - XZ - \dfrac{m+q}{2} W^2 + \dfrac{m-1}{2} WZ \\ \dot{Z} &= -\dfrac{2-m-q}{\sqrt{2(m+q)}} Z + \dfrac{2-m-q}{2} WZ + \dfrac{2-\sigma}{2} XZ. \end{cases} \tag{2.28}$$

Introducing $\mathbf{F}(\mathbf{V}) = (F_1, F_2, F_3)(\mathbf{V})$ defined for $\mathbf{V} = (V_1, V_2, V_3) \in \mathbb{R}^3$ by

$$\begin{aligned} F_1(\mathbf{V}) &:= -\frac{m-q}{\sqrt{2(m+q)}} V_1 + \frac{m-q}{2} V_1 V_2 - \frac{\sigma+2}{2} V_1^2 \\ F_2(\mathbf{V}) &:= \left(N - 1 + \frac{\sigma}{2}\right) \sqrt{\frac{2}{m+q}} V_1 + \sqrt{2(m+q)} V_2 - \frac{m-1}{\sqrt{2(m+q)}} V_3 \\ & \qquad - \left(N - 1 + \frac{\sigma}{2}\right) V_1 V_2 - V_1 V_3 - \frac{m+q}{2} V_2^2 + \frac{m-1}{2} V_2 V_3 \\ F_3(\mathbf{V}) &:= -\frac{2-m-q}{\sqrt{2(m+q)}} V_3 + \frac{2-m-q}{2} V_2 V_3 + \frac{2-\sigma}{2} V_1 V_3, \end{aligned}$$

and denoting the semiflow associated with the dynamical system

$$\dot{\mathbf{V}}(\eta) = \mathbf{F}(\mathbf{V}(\eta)), \quad \eta > 0, \quad \mathbf{V}(0) = \mathbf{V}_0 \in \mathbb{R}^3, \tag{2.29}$$

by $\varphi(\cdot; \mathbf{V}_0)$, we deduce from (2.28) that $\mathbf{V}_* := (X, W, Z) = \varphi(\cdot; \mathbf{V}_*(0))$ is defined on $[0, \infty)$ with

$$\lim_{\eta \to \infty} \mathbf{V}_*(\eta) = 0. \tag{2.30}$$

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The matrix associated with the linearization of system (2.29) at the origin is

$$\mathcal{M} = \sqrt{\frac{2}{m+q}} \begin{pmatrix} -\frac{m-q}{2} & 0 & 0 \\ N-1+\frac{\sigma}{2} & m+q & -\frac{m-1}{2} \\ 0 & 0 & -\frac{2-m-q}{2} \end{pmatrix}$$

having three distinct eigenvalues

$$\lambda_1 = -\frac{m-q}{\sqrt{2(m+q)}}, \quad \lambda_2 = \sqrt{2(m+q)}, \quad \lambda_3 = -\frac{2-m-q}{\sqrt{2(m+q)}},$$

with corresponding eigenvectors (not normalized)

$$E_1 = \left(1, -\frac{2(N-1)+\sigma}{3m+q}, 0\right), \quad E_2 = (0,1,0), \quad E_3 = \left(0, \frac{m-1}{2+m+q}, 1\right).$$

Then $\mathbf{0}$ is a hyperbolic point of φ and has a two-dimensional stable manifold $\mathcal{W}_s(\mathbf{0})$. According to the proof of the stable manifold theorem (see e.g. [**2**, theorem 19.11]), there is an open neighbourhood $\mathcal{V}$ of zero in $\mathbb{R}^3$, an open neighbourhood $\mathcal{V}_0$ of zero in $\mathbb{R}^2$ and a C^2-smooth function $h : \mathcal{V}_0 \to \mathbb{R}$ such that $h(0, 0) = \partial_x h(0, 0) = \partial_z h(0, 0) = 0$ and the local stable manifold

$$\mathcal{W}_s^{\mathcal{V}}(\mathbf{0}) := \{\mathbf{V}_0 \in \mathcal{W}_s(\mathbf{0}) \ : \ \varphi(\eta; \mathbf{V}_0) \in \mathcal{V} \text{ for all } \eta \geqslant 0\}$$

satisfies

$$\mathcal{W}_s^{\mathcal{V}}(\mathbf{0}) \subseteq \{xE_1 + h(x,z)E_2 + zE_3 : (x,z) \in \mathcal{V}_0\},$$

its tangent space at $\mathbf{0}$ being $\mathbb{R}E_1 \oplus \mathbb{R}E_3$. Since $\{\varphi(\eta; \mathbf{V}_*(0)) \ : \ \eta \geqslant \eta_0\}$ is included in $\mathcal{W}_s(\mathbf{0}) \cap \mathcal{V}$ for η_0 large enough by (2.30), we conclude that $\varphi(\eta; \mathbf{V}_*(0))$ belongs to $\mathcal{W}_s^{\mathcal{V}}(\mathbf{0})$ for $\eta \geqslant \eta_0$. Consequently, there are functions $(\overline{x}, \overline{z}) : [\eta_0, \infty) \to \mathcal{V}_0$ such that

$$(X, W, Z)(\eta) = \varphi(\eta; \mathbf{V}_*(0)) = \overline{x}(\eta)E_1 + h(\overline{x}(\eta), \overline{z}(\eta))E_2 + \overline{z}(\eta)E_3$$

for $\eta \geqslant \eta_0$. In fact, $\overline{x}(\eta) = X(\eta)$, $\overline{z}(\eta) = Z(\eta)$ and

$$W(\eta) = -\frac{2(N-1)+\sigma}{3m+q} X(\eta) + \frac{m-1}{2+m+q} Z(\eta) + h(X(\eta), Z(\eta)). \tag{2.31}$$

Let us notice from (2.24) that

$$\mathcal{Z}(\xi) = \alpha m^{(q-1)/(m-q)} \mathcal{X}^{(2-m-q)/(m-q)}(\xi),$$

which implies that $X(\eta) = o(Z(\eta))$ as $\eta \to \infty$, since $(2-m-q)/(m-q) < 1$. Recalling also that h is C^2-smooth with $h(0, 0) = \partial_x h(0, 0) = \partial_z h(0, 0) = 0$, we

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infer from (2.31) that

$$W(\eta) = \frac{m-1}{2+m+q} Z(\eta) + o(Z(\eta)) \quad \text{as } \eta \to \infty,$$

or equivalently, undoing the change of variable (2.25) and the translation (2.27), we get as $\xi \to \xi_0$,

$$\mathcal{Y}(\xi) = -\sqrt{\frac{2}{m+q}} + \frac{m-1}{2+m+q} \mathcal{Z}(\xi) + o(\mathcal{Z}(\xi)). \tag{2.32}$$

Moreover, we readily infer from the already obtained local behaviour (2.19) and the definition of $\mathcal{Z}$ in (2.24) that, as $\xi \to \xi_0$,

$$\mathcal{Z}(\xi) \sim \frac{\alpha}{\sqrt{m}} K_1^{(2-m-q)/2} \xi_0^{(2-\sigma)/2+\sigma(2-m-q)/2(m-q)} (\xi_0 - \xi)^{(2-m-q)/(m-q)}.$$

Inserting the previous expansion into (2.32) and recalling the definition of $\mathcal{Y}$ in (2.24), we find

$$\begin{aligned}\frac{2\sqrt{m}}{m-q} \xi^{-\sigma/2} \left(f^{(m-q)/2}\right)'(\xi) = & -\sqrt{\frac{2}{m+q}} \\ & + \frac{\alpha(m-1)K_1^{(2-m-q)/2}}{(2+m+q)\sqrt{m}} \xi_0^{(2-\sigma)/2+\sigma(2-m-q)/2(m-q)} (\xi_0 - \xi)^{(2-m-q)/(m-q)} \\ & + o\left((\xi_0-\xi)^{(2-m-q)/(m-q)}\right),\end{aligned}$$

which leads to, since $\alpha = 2\beta/(m-1)$,

$$\begin{aligned}& \left(f^{(m-q)/2}\right)'(\xi) \\ & = -K_1^{(m-q)/2} \xi_0^{\sigma/2} \left(1 - \frac{\xi_0-\xi}{\xi_0}\right)^{\sigma/2} \\ & \quad + (1-q)K_0(\beta) K_1^{(m-q-2)/2} \xi_0^{[2(m-q)+\sigma(2-m-q)]/2(m-q)} (\xi_0-\xi)^{(2-m-q)/(m-q)} \\ & \quad \times \left(1 - \frac{\xi_0-\xi}{\xi_0}\right)^{\sigma/2} + o\left((\xi_0-\xi)^{(2-m-q)/(m-q)}\right) \\ & = -K_1^{(m-q)/2} \xi_0^{\sigma/2} \left(1 - \frac{\sigma(\xi_0-\xi)}{2\xi_0}\right) + o(\xi_0-\xi) \\ & \quad + (1-q)K_0(\beta) K_1^{(m-q-2)/2} \xi_0^{[2(m-q)+\sigma(2-m-q)]/2(m-q)} (\xi_0-\xi)^{(2-m-q)/(m-q)} \\ & \quad + o\left((\xi_0-\xi)^{(2-m-q)/(m-q)}\right).\end{aligned}$$

Recalling that $(2-m-q)/(m-q) < 1$, we end up with

$$\begin{aligned}\left(f^{(m-q)/2}\right)'(\xi) = -K_1^{(m-q)/2}\xi_0^{\sigma/2} \\ &+ (1-q)K_0(\beta)K_1^{(m-q-2)/2}\xi_0^{[2(m-q)+\sigma(2-m-q)]/2(m-q)} \\ &\times (\xi_0-\xi)^{(2-m-q)/(m-q)} \\ &+ o\left((\xi_0-\xi)^{(2-m-q)/(m-q)}\right).\end{aligned} \tag{2.33}$$

Integrating (2.33) over $(\xi,\, \xi_0)$ and then taking powers $2/(m-q)$ give

$$\begin{aligned} f(\xi) &= K_1\xi_0^{\sigma/(m-q)}(\xi_0-\xi)^{2/(m-q)} \\ &\quad\times \left[1 - \frac{(m-q)K_0(\beta)}{2K_1}\xi_0^{(m+q-2)/(m-q)}(\xi_0-\xi)^{(2-m-q)/(m-q)}\right. \\ &\quad\left.+ o\left((\xi_0-\xi)^{(2-m-q)/(m-q)}\right)\right]^{2/(m-q)} \\ &= K_1\xi_0^{\sigma/(m-q)}(\xi_0-\xi)^{2/(m-q)} - K_0(\beta)\xi_0^{(\sigma+m+q-2)/(m-q)}(\xi_0-\xi)^{(4-m-q)/(m-q)} \\ &\quad + o\left((\xi_0-\xi)^{(4-m-q)/(m-q)}\right),\end{aligned}$$

as stated. □

2.4. Uniqueness

We are now ready to complete the proof of theorem 1.1 by showing that the set $\mathcal{B}$ contains at most one element. Taking into account the previous preparations, this proof borrows ideas from the analogous one in [**21**, § 4.4].

Proof of theorem 1.1: uniqueness. Assume for contradiction that there are $\beta_1 \in \mathcal{B}$ and $\beta_2 \in \mathcal{B}$ such that $0 < \beta_1 < \beta_2 < \infty$. By lemma 2.4, we have $\xi_0(\beta_1) = \xi_1(\beta_1)$ and $\xi_0(\beta_2) = \xi_1(\beta_2)$, so that lemma 2.5 implies that $f_1(\xi) > f_2(\xi)$ and $F_1(\xi) > F_2(\xi)$ for any $\xi \in (0,\, \min\{\xi_0(\beta_1),\, \xi_0(\beta_2)\})$, with $f_i := f(\cdot;\beta_i)$ and $F_i := f_i^m$ for $i = 1,\, 2$. In particular, $\xi_0(\beta_2) < \xi_0(\beta_1)$.

As in the proof of lemma 2.5, see (2.9)–(2.10), we introduce the rescaled version G_λ of F_2 defined by

$$G_\lambda(\xi) := \lambda^m F_2\left(\lambda^{-(m-1)/2}\xi\right), \quad \xi \in [0,\infty), \quad \lambda \geqslant 1, \tag{2.34}$$

recalling that F_2 is well-defined on $[0,\, \infty)$ by lemma 2.1, and define the optimal parameter

$$\lambda_0 := \inf\{\lambda \geqslant 1 \;:\; G_\lambda(\xi) > F_1(\xi),\ \xi \in [0, \xi_0(\beta_1)]\} \in (1,\infty), \tag{2.35}$$

its existence being ensured by the fact that

$$\begin{aligned}\lim_{\lambda\to\infty}\min_{\xi\in[0,\xi_0(\beta_1)]} G_\lambda(\xi) &= \lim_{\lambda\to\infty} G_\lambda(\xi_0(\beta_1)) = \lim_{\lambda\to\infty}\lambda^m F_2(\lambda^{-(m-1)/2}\xi_0(\beta_1)) \\ &\geqslant \lim_{\lambda\to\infty}\lambda^m F_2\left(\frac{\xi_0(\beta_2)}{2}\right) = \infty.\end{aligned}$$

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According to the definition of λ_0 in (2.35) and the compactness of the interval $[0,\,\xi_0(\beta_1)]$, we deduce that there is $\eta\in[0,\,\xi_0(\beta_1)]$ such that $F_1(\eta)=G_{\lambda_0}(\eta)$ and $F_1\leqslant G_{\lambda_0}$ on $[0,\,\xi_0(\beta_1)]$. Arguments very similar to the ones employed in the proof of lemma 2.5, along with lemma 2.1, then discard the possibility that either $\eta=0$ or $\eta\in(0,\,\xi_0(\beta_1))$, thus showing that $\eta=\xi_0(\beta_1)$. Consequently,

$$F_1(\xi_0(\beta_1))=0=G_{\lambda_0}(\xi_0(\beta_1)),\quad 0<F_1(\xi)<G_{\lambda_0}(\xi),\quad \xi\in[0,\xi_0(\beta_1)),\tag{2.36}$$

and we also obtain the following equality implied by the equality of the supports in (2.36) and rescaling (2.34)

$$\xi_0(\beta_1)=\lambda_0^{(m-1)/2}\xi_0(\beta_2).\tag{2.37}$$

We now split the analysis into the three cases already set apart at the beginning of § 2.3, according to the sign of $m+q-2$.

Case 1. $m+q<2$. We recall that, in this case, proposition 2.9 gives

$$\begin{aligned} f_i(\xi)=&\,K_1\xi_0(\beta_i)^{\sigma/(m-q)}(\xi_0(\beta_i)-\xi)^{2/(m-q)}\\ &-K_0(\beta_i)\xi_0(\beta_i)^{(\sigma+m+q-2)/(m-q)}(\xi_0(\beta_i)-\xi)^{(4-m-q)/(m-q)}\\ &+o((\xi_0(\beta_i)-\xi)^{(4-m-q)/(m-q)}),\end{aligned}\tag{2.38}$$

as $\xi\to\xi_0(\beta_i)$, $i=1,\,2$. In order to simplify the calculations, we can work at the level of f_i by noticing that rescaling (2.34) reduces to

$$g_{\lambda_0}(\xi):=G_{\lambda_0}^{1/m}(\xi)=\lambda_0 f_2\left(\lambda_0^{-(m-1)/2}\xi\right).\tag{2.39}$$

We thus infer from (2.38) and (2.39) that

$$\begin{aligned} g_{\lambda_0}(\xi)=&\,\lambda_0K_1\xi_0(\beta_2)^{\sigma/(m-q)}\left(\xi_0(\beta_2)-\lambda_0^{-(m-1)/2}\xi\right)^{2/(m-q)}\\ &-K_0(\beta_2)\lambda_0\xi_0(\beta_2)^{(\sigma+m+q-2)/(m-q)}(\xi_0(\beta_2)-\lambda_0^{-(m-1)/2}\xi)^{(4-m-q)/(m-q)}\\ &+o\left(\left(\xi_0(\beta_2)-\lambda_0^{-(m-1)/2}\xi\right)^{(4-m-q)/(m-q)}\right)\\ =&\,\lambda_0K_1\left(\lambda_0^{-(m-1)/2}\xi_0(\beta_1)\right)^{\sigma/(m-q)}\lambda_0^{-(m-1)/(m-q)}(\xi_0(\beta_1)-\xi)^{2/(m-q)}\\ &-K_0(\beta_2)\lambda_0\left(\lambda_0^{-(m-1)/2}\xi_0(\beta_1)\right)^{(\sigma+m+q-2)/(m-q)}\lambda_0^{-(m-1)(4-m-q)/2(m-q)}\\ &\times(\xi_0(\beta_1)-\xi)^{(4-m-q)/(m-q)}+o\left((\xi_0(\beta_1)-\xi)^{(4-m-q)/(m-q)}\right).\end{aligned}$$

Noticing that the powers of λ_0 appearing in the (rather tedious) previous calculations cancel out due to the precise value of σ given in (1.2), we further

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obtain

$$\begin{aligned} g_{\lambda_0}(\xi) &= K_1\xi_0(\beta_1)^{\sigma/(m-q)}(\xi_0(\beta_1)-\xi)^{2/(m-q)} \\ &\quad - K_0(\beta_2)\xi_0(\beta_1)^{(\sigma+m+q-2)/(m-q)}(\xi_0(\beta_1)-\xi)^{(4-m-q)/(m-q)} \\ &\quad + o\left((\xi_0(\beta_1)-\xi)^{(4-m-q)/(m-q)}\right) \\ &= f_1(\xi) + (K_0(\beta_1)-K_0(\beta_2))\xi_0(\beta_1)^{(\sigma+m+q-2)/(m-q)} \\ &\quad \times (\xi_0(\beta_1)-\xi)^{(4-m-q)/(m-q)} \\ &\quad + o\left((\xi_0(\beta_1)-\xi)^{(4-m-q)/(m-q)}\right). \end{aligned}$$

Since $\beta_1 < \beta_2$, we deduce from (2.23) that $K_0(\beta_1) < K_0(\beta_2)$. Thus, $g_{\lambda_0}(\xi) < f_1(\xi)$ in a left neighbourhood of $\xi_0(\beta_1)$, whence (by raising to power m) $G_{\lambda_0}(\xi) < F_1(\xi)$ in the same left neighbourhood of $\xi_0(\beta_1)$, and we have reached a contradiction to (2.36).

Case 2. $m+q=2$. In this case, proposition 2.8 (b) gives

$$F_i(\xi) = K_1^m\xi_0(\beta_i)^{2m/(m-q)}K_2^m(\beta_i)(\xi_0(\beta_i)-\xi)^{2m/(m-q)} + o\left((\xi_0(\beta_i)-\xi)^{2m/(m-q)}\right)$$

as $\xi \to \xi_0(\beta_i)$, $i=1$, 2. We thus have

$$\begin{aligned} G_{\lambda_0}(\xi) &= \lambda_0^m K_1^m\xi_0(\beta_2)^{2m/(m-q)}K_2^m(\beta_2)\left(\xi_0(\beta_2)-\lambda_0^{-(m-1)/2}\xi\right)^{2m/(m-q)} \\ &\quad + o\left((\xi_0(\beta_2)-\lambda_0^{-(m-1)/2}\xi)^{2m/(m-q)}\right) \\ &= \lambda_0^m K_1^m(\lambda_0^{-(m-1)/2}\xi_0(\beta_1))^{2m/(m-q)}K_2^m(\beta_2)\lambda_0^{-m(m-1)/(m-q)} \\ &\quad \times (\xi_0(\beta_1)-\xi)^{2m/(m-q)} \\ &\quad + o\left((\xi_0(\beta_1)-\xi)^{2m/(m-q)}\right) \\ &= K_1^m\xi_0(\beta_1)^{2m/(m-q)}K_2^m(\beta_2)(\xi_0(\beta_1)-\xi)^{2m/(m-q)} \\ &\quad + o\left((\xi_0(\beta_1)-\xi)^{2m/(m-q)}\right) \\ &= \left[\frac{K_2(\beta_2)}{K_2(\beta_1)}\right]^m F_1(\xi) + o\left((\xi_0(\beta_1)-\xi)^{2m/(m-q)}\right), \end{aligned}$$

the powers of λ_0 cancelling out due to $m+q=2$. Noticing that we can write

$$K_2(\beta) = \left[\sqrt{1+\frac{\beta^2}{4m}} + \frac{\beta}{2\sqrt{m}}\right]^{-2/(m-q)},$$

we easily observe that K_2 is a decreasing function of β, thus $K_2(\beta_2) < K_2(\beta_1)$ since $\beta_2 > \beta_1$. Therefore, $G_{\lambda_0}(\xi) < F_1(\xi)$ in a left neighbourhood of $\xi_0(\beta_1)$, which contradicts (2.36).

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Case 3. $m+q>2$. We recall that, in this case, proposition 2.8 (c) gives

$$F_i(\xi)=K_3^m(\beta_i)\xi_0(\beta_i)^{m(\sigma-1)/(1-q)}(\xi_0(\beta_i)-\xi)^{m/(1-q)}+o((\xi_0(\beta_i)-\xi)^{m/(1-q)})$$

as $\xi\to\xi_0(\beta_i)$, $i=1$, 2. Using then rescaling (2.34) and identity (2.37), we readily infer that

$$\begin{aligned}
G_{\lambda_0}(\xi) &= \lambda_0^m K_3^m(\beta_2)\xi_0(\beta_2)^{m(\sigma-1)/(1-q)}\left(\xi_0(\beta_2)-\lambda_0^{-(m-1)/2}\xi\right)^{m/(1-q)} \\
&\quad + o\left(\left(\xi_0(\beta_2)-\lambda_0^{-(m-1)/2}\xi\right)^{m/(1-q)}\right) \\
&= \lambda_0^m K_3^m(\beta_2)\left(\lambda_0^{-(m-1)/2}\xi_0(\beta_1)\right)^{m(\sigma-1)/(1-q)}\lambda_0^{-(m-1)m/2(1-q)} \\
&\quad \times(\xi_0(\beta_1)-\xi)^{m/(1-q)} \\
&\quad + o\left((\xi_0(\beta_1)-\xi)^{m/(1-q)}\right) \\
&= K_3^m(\beta_2)\xi_0(\beta_1)^{m(\sigma-1)/(1-q)}(\xi_0(\beta_1)-\xi)^{m/(1-q)}+o((\xi_0(\beta_1)-\xi)^{m/(1-q)}) \\
&= \left[\frac{K_3(\beta_2)}{K_3(\beta_1)}\right]^m F_1(\xi)+o((\xi_0(\beta_1)-\xi)^{m/(1-q)}).
\end{aligned}$$

Since $K_3(\beta_2)<K_3(\beta_1)$ for $\beta_2>\beta_1$, we find that $G_{\lambda_0}(\xi)<F_1(\xi)$ in a left neighbourhood of $\xi_0(\beta_1)$, which is again a contradiction to (2.36).

The previous contradictions imply that there cannot be two different values of the exponent β in the set $\mathcal{B}$, completing the proof. □

Acknowledgements

This work is partially supported by the Spanish project PID2020-115273GB-I00 and by the Grant RED2022-134301-T (Spain). Part of this work has been developed during visits of R. G. I. to Institut de Mathématiques de Toulouse and to Laboratoire de Mathématiques LAMA, Université de Savoie, and of P. L. to Universidad Rey Juan Carlos, and both authors thank these institutions for hospitality and support. The authors wish to thank Ariel Sánchez (Universidad Rey Juan Carlos) for interesting comments and suggestions. We also thank the referees for carefully reading the manuscript.

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