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THE SPECTRUM OF ORTHOGONAL STEINER TRIPLE SYSTEMS

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ABSTRACT. Two Steiner triple systems $(V, \mathcal{B})$ and $(V, \mathcal{D})$ are *orthogonal* if they have no triples in common, and if for every two distinct intersecting triples $\{x, y, z\}$ and $\{u, v, z\}$ of $\mathcal{B}$, the two triples $\{x, y, a\}$ and $\{u, v, b\}$ in $\mathcal{D}$ satisfy $a \neq b$. It is shown here that if $v \equiv 1, 3 \pmod 6$, $v \geq 7$ and $v \neq 9$, a pair of orthogonal Steiner triple systems of order v exist. This settles completely the question of their existence posed by O'Shaughnessy in 1968.

1. **Background.** We assume familiarity with standard terminology and results in combinatorial design theory [1]. A *Steiner triple system* of order v, or STS(v), is a pair $(V, \mathcal{B})$; V is a v-set of elements, and $\mathcal{B}$ is a set of 3-subsets of V called *triples* or *blocks*, with the property that each 2-subset of V occurs in exactly one triple of $\mathcal{B}$. Two STS(v) on the same set of elements, say $(V, \mathcal{A})$ and $(V, \mathcal{B})$, are *orthogonal* if $\mathcal{A} \cap \mathcal{B} = \emptyset$, and if $\big\{\{u, v, w\}, \{x, y, w\}\big\} \subset \mathcal{A}$ and $\big\{\{u, v, s\}, \{x, y, t\}\big\} \subset \mathcal{B}$ then $s \neq t$. We denote a pair of orthogonal Steiner triple systems of order v as OSTS(v).

Orthogonal Steiner triple systems were introduced in 1968 by O'Shaughnessy [9] as a means of constructing Room squares. O'Shaughnessy constructed OSTS(v) for orders $v \in \{7, 13, 19\}$. He conjectured that OSTS(v) exist whenever $v \equiv 1 \pmod 6$, and further conjectured that none exists when $v \equiv 3 \pmod 6$. It is trivial that no OSTS(3) can exist. Mullin and Nemeth [6, 7] established that no OSTS(9) exists. They further established that OSTS exist whenever the order v is a prime power congruent to 1 modulo 6. However, Rosa [10] disproved O'Shaughnessy's conjecture by establishing the existence of OSTS(27). Subsequently, Gibbons [2] found OSTS of order 15, and in fact enumerated all nonisomorphic OSTS(15).

A *pairwise balanced design* (or PBD) $(V, \mathcal{A})$ is a set V of elements, together with a set $\mathcal{A}$ of subsets of V each having size at least two, with the property that each 2-subset of elements occurs in exactly one of the sets in $\mathcal{A}$. The PBD is a (v, K)-PBD if $|V| = v$, and for every $A \in \mathcal{A}$, $|A| \in K$. Now define $B(K) = \{v : \exists (v, K)\text{-PBD}\}$. A set K is *PBD-closed* when $B(K) = K$. It is easy to see that the set $\mathcal{OSTS} = \{v : \exists\, \mathrm{OSTS}(v)\}$ is PBD-closed. Thus Wilson's techniques [12] ensure the existence of a finite v_0 so that if $v \geq v_0$ and $v \equiv 1, 3 \pmod 6$, an OSTS(v) exists. A major step in determining the spectrum for OSTS of orders congruent to 1 modulo 6 came when Mullin and Stinson [8] and Zhu and Chen (see [8]) examined the spectrum for pairwise balanced designs whose block

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sizes are prime powers congruent to 1 modulo 6. Their results ensure that an OSTS(v) exists for $v \equiv 1 \pmod 6$ for all $v \geq 1927$, and for all but 31 values less than 1927.

A similar closure result was obtained for the $v \equiv 3 \pmod 6$ class by examining PBD that permit blocks of sizes 15 and 27 in addition to prime powers congruent to 1 modulo 6. Stinson and Zhu [11] showed that an OSTS(v) exists for all $v \equiv 3 \pmod 6$, $v \geq 27369$, and for all but at most 917 orders in the range $15 \leq v \leq 27363$. In this class, the smallest unresolved case was that of OSTS(21).

Since that time, Greig [4] has improved the PBD result for the $1 \pmod 6$ class; he produces PBD whose block sizes are prime powers congruent to $1 \pmod 6$ for orders 295, 655, 1243, 1255, 1795, 1819 and 1921. Stinson and Zhu [11] eliminated two further cases for OSTS. Nevertheless, progress on the problem has been slow. In fact, the existence question for Room squares, originally a main motivation for defining OSTS, has been long settled [1].

In this paper, we settle the existence question for OSTS(v) completely, using a combination of results obtained by computational techniques [3], and a number of recursive techniques.

We employ definitions and results from Stinson and Zhu [11] without further comment, but state some definitions and their results upon which we rely most heavily here.

The key ingredient that we use repeatedly is a generalization of OSTS(v) suggested by Stinson and Zhu [11]. A *group-divisible design*, or GDD, is a triple $(V, \mathcal{G}, \mathcal{A})$ where V is a set of elements, $\mathcal{G}$ is a partition of V whose classes are called *groups*, and $\mathcal{A}$ is a set of subsets of V called blocks, with the property that every 2-subset appears either in a group of $\mathcal{G}$ or a block of $\mathcal{A}$, but not both. We consider here only GDD in which every block has size three. An *orthogonal group-divisible design* (OGDD) $(X, \mathcal{G}, \mathcal{B}_1, \mathcal{B}_2)$ is a set X and a partition $\mathcal{G}$ of X into classes (again called *groups*), and two disjoint sets $\mathcal{B}_1$ and $\mathcal{B}_2$ of 3-subsets of X, so that each pair $\{x, y\}$ of elements of X appears once in a 3-subset of $\mathcal{B}_1$ and once in a 3-subset of $\mathcal{B}_2$ if x and y are from different groups, and does not appear in a 3-subset of either if x and y are from the same group. Moreover, if $\{x, y, a\} \in \mathcal{B}_1$ and $\{x, y, b\} \in \mathcal{B}_2$, then a and b are in different groups; and for two distinct intersecting triples $\{x, y, z\}$ and $\{u, v, z\}$ of $\mathcal{B}_1$, the triples $\{x, y, a\}$ and $\{u, v, b\}$ of $\mathcal{B}_2$ satisfy $a \neq b$. We abuse the notation somewhat by referring to the pair of GDD $(X, \mathcal{G}, \mathcal{B}_1)$ and $(X, \mathcal{G}, \mathcal{B}_2)$ as an OGDD, and also saying that one of the GDD is *orthogonal* to the other. It is easy to see that an OSTS(v) is precisely the same as an OGDD in which there are v groups, each having a single element. Adopting usual notation, we say that an OGDD has *type* $(g_1)^{u_1} \cdots (g_s)^{u_s}$ if the OGDD has u_i groups of size g_i for $1 \leq i \leq s$, and no other groups. Thus an OSTS(v) is the same as an OGDD of type 1^v.

We shall in addition use conjugate orthogonal quasigroups (COQ), but use them in a very standard way. Two quasigroups are *conjugate orthogonal* if each conjugate of one is orthogonal to each conjugate of the other. We refer the reader to [11] for more details concerning COQ.

The following lemmas summarize basic constructions from [11].

LEMMA 1.1. *Suppose that there exists an* OGDD *of type* g^u. *Further suppose that* $v = 1$ *or that a* COQ(v) *exists. Then there exists an* OGDD *of type* $(gv)^u$. *If in addition there exists an* OSTS(gv) (OSTS($gv+1$)) *then there exists an* OSTS(guv) (OSTS($guv+1$), *respectively).*

LEMMA 1.2 (DIRECT PRODUCT). *If there exists an* OSTS(u) *and a* COQ(v), *then there exists an* OGDD *of type* v^u. *If, in addition, there exists an* OSTS(v), *there exists an* OSTS(uv).

LEMMA 1.3 (SINGULAR DIRECT PRODUCT). *If there exists an* OSTS(u), *a* COQ($v-1$) *and an* OSTS(v), *then there exists an* OSTS$\big(u(v-1)+1\big)$.

To apply these results, COQ are needed. These are produced by the following two lemmas without further comment:

LEMMA 1.4. *If v is a prime power and* $v \notin \{2,3\}$, *then there exists a* COQ(v).

LEMMA 1.5. *If there exists a* COQ(u) *and a* COQ(v), *then there exists a* COQ(uv).

We require a small extension of the results of [11] for use with the Main construction that we introduce later.

LEMMA 1.6 (FILLING IN GROUPS). *If there exists an* OGDD *of type* $u^g v^h$ *and there exists an* OSTS($u+1$) *and an* OSTS($v+1$), *then there exists an* OSTS($gu+hv+1$).

PROOF. This is implicit in Section 6 of [11]. ∎

2. **Constructions and uses of** OGDD. In this section, we first develop some constructions for OGDD, and then describe applications of Wilson's Fundamental Construction [12] to produce OSTS from the OGDD found.

Let G be a finite abelian group of order v and let H be a subgroup of G. Then a (v,k,λ)-*relative difference set* based on G and H is a triple (G,H,F) where F is a family of k-subsets of G which has the property that every element of $G \setminus H$ occurs exactly λ times as a difference of elements in the subsets of F, and no member of H appears as such a difference. Further, let d be a fixed element of $G \setminus H$. Then a (v,k,λ)-*near relative difference set* based on G, H and d is a quadruple (G,H,F,d) where F is a family of k-subsets which has the property that every element of $G \setminus (H \cup \{\pm d\})$ occurs exactly λ times as a difference of elements in the subsets of F, and no member of $H \cup \{\pm d\}$ occurs as such a difference.

Let $T = \{x,y,z\}$ be a triple of elements of an abelian group G. Then the three pairs $\{x-z, y-z\}$, $\{x-y, z-y\}$ and $\{y-x, z-x\}$ are the *fundamental pairs* of T. Let F be the set of triples of a $(v,3,1)$-relative difference set or near relative difference set. The multiset of $3|F|$ fundamental pairs corresponding to the triples of F is the *set of fundamental pairs* of F.

Now let $S_1 = (G,H,F_1)$ and $S_2 = (G,H,F_2)$ be a pair of $(v,3,1)$-relative difference sets based on G and H. Then for each fundamental pair $P = \{x,y\}$ of S_1 there is a unique triple $T(P)$ in S_2 having a translate $T'(P)$ containing P, say $T'(P) = \{x,y,z(P)\}$. The multiset $\mathcal{L} = \{z(P) : P$ is a fundamental pair of $S_1\}$ is an *orthogonality certificate* for the (ordered) pair (S_1, S_2) of relative difference sets if all members of $\mathcal{L}$ are distinct and no member of $\mathcal{L}$ lies in H.

Similarly, let $S_1 = (G, H, F_1, d_1)$ and $S_2 = (G, H, F_2, d_2)$ be $(v, 3, 1)$-near relative difference sets. The multiset $\mathcal{L} = \{z(P) : P$ is a fundamental pair of $S_1\} \cup \{\pm d_1 \pm d_2\}$ is an *orthogonality certificate* for the ordered pair (S_1, S_2) if all members of $\mathcal{L}$ are distinct and no member of $\mathcal{L}$ lies in H.

The significance of orthogonality certificates for pairs of relative difference sets is given in the following two lemmas.

LEMMA 2.1. *Let $S_1 = (G, H, F_1)$ and $S_2 = (G, H, F_2)$ be a pair of $(v, 3, 1)$-relative difference sets. Let $h = |H|$, $v = |G|$, and $n = v/h$. If there exists an orthogonality certificate for the pair (S_1, S_2) then there exists an* OGDD *of type h^n.*

PROOF. We consider the members of G to be the elements of the OGDD, and the cosets of H in G to be the groups of the OGDD. The $v|F_1|$ triples $\{T+g : T \in F_1, g \in G\}$ forms a $\{3\}$ − GDD on G, and similarly the $v|F_2|$ triples $\{T+g : T \in F_2, g \in G\}$ forms a $\{3\}$ − GDD on G. The second GDD formed is orthogonal to the first, as is easily verified using the definition of the orthogonality certificate. ∎

The situation for near relative difference sets is somewhat more complicated. Let $S = (G, H, F, d)$ be a $(v, 3, 1)$-near relative difference set. Then S is *partitionable* if d and $-d$ are distinct in G, and there exists a subset P of G containing $|G|/2$ elements and having the property that $\big\{\{0, d\} + g : g \in G\big\} = \big\{\{0, d\} + p : p \in P\big\} \cup \big\{\{0, -d\} + p : p \in P\big\}$. The set P is a *partitioning set*. We write $S = (G, H, F, d, P)$ if (G, H, F, d) is a near relative difference set with partitioning set P.

LEMMA 2.2. *Let $S_1 = (G, H, F_1, d_1, P_1)$ and $S_2 = (G, H, F_2, d_2, P_2)$ be a pair of partitionable $(v, 3, 1)$-near relative difference sets. Let $h = |H|$, $v = |G|$ and $n = v/h$. If there exists a orthogonality certificate for (S_1, S_2) then there exists an* OGDD *of type $h^n 2^1$.*

PROOF. We take the elements of G, together with two new elements ∞_1 and ∞_2, to be the points of the OGDD. The cosets of H in G form the n groups each of size h, and $\{\infty_1, \infty_2\}$ forms a group of size 2. The set of $v|F_1|$ triples $\{T + g : T \in F_1, g \in G\}$ together with the triples $\big\{\{\infty_1, 0, d_1\} + p : p \in P_1\big\} \cup \big\{\{\infty_2, 0, -d_1\} + p : p \in P_1\big\}$ form the triples of a GDD. (Here we adopt the usual convention that $\infty_i + g = \infty_i$.)

A second GDD is defined similarly using S_2. The second is orthogonal to the first, as is easily verified using the definition of an orthogonality certificate. ∎

We employ these two lemmas to construct a number of OGDD, in each case taking G to be the cyclic group of integers modulo v, and the subgroup H is uniquely determined by its order. In order that a near relative difference set be partitionable, it is sufficient that $v/\gcd(v, d)$ be even and that $d \neq v/2$. The following arrays are to be read as follows. The first line gives the set of starter blocks for the first system, and the second line those of the orthogonal system. The third line gives the orthogonality certificate, specifically what occurs in the second system with the fundamental pairs of the starter blocks of the first system. In the cases of near relative difference sets, the triple $\{\infty, 0, d\}$ stands for both $\{\infty_1, 0, d\}$ and $\{\infty_2, 0, -d\}$, and the entries of the orthogonality certificate are $\pm d_1 \pm d_2$.

$2^{11}2^1 = 2^{12}$ in Z_{22} with two infinite points

0,1,3	0,4,10	0,5,13	∞,0,7
0,1,9	0,2,18	0,3,15	∞,0,5
18,13,16	14,4,3	19,∞,8	12,2,20,10

2^{13} in Z_{26}

0,1,3	0,4,10	0,5,14	0,7,15
0,1,7	0,2,10	0,3,15	0,4,9
23,14,22	21,2,15	20,4,12	18,17,1

$2^{14}2^1 = 2^{15}$ in Z_{28} with two infinite points

0,1,3	0,4,9	0,6,18	0,7,20	∞,0,11
0,1,10	0,2,8	0,3,7	0,5,17	∞,0,13
5,15,21	20,13,19	12,17,22	27,∞,7	24,26,2,4

2^{16} in Z_{32}

0,1,3	0,4,9	0,6,17	0,7,19	0,8,18
0,1,4	0,2,21	0,5,25	0,6,23	0,8,22
2,8,31	9,13,10	18,7,3	12,21,26	30,6,28

3^{11} in Z_{33}

0,1,3	0,4,10	0,5,18	0,7,19	0,8,24
0,1,8	0,2,20	0,3,9	0,4,14	0,5,17
20,23,28	7,13,12	32,14,25	8,17,29	31,4,1

3^{17} in Z_{51}

0,1,3	0,4,9	0,6,16	0,7,26	0,8,28	0,11,24
0,1,4	0,2,13	0,5,12	0,6,26	0,8,36	0,9,27
2,8,50	26,38,37	31,25,35	49,1,41	16,14,43	48,29,28
	0,12,30	0,14,29			
	0,10,32	0,14,35			
	46,15,13	47,27,45			

$4^9 2^1$ in Z_{36} with two infinite points

0,1,3	0,4,10	0,5,16	0,7,19	0,8,21	∞,0,14
0,1,17	0,2,24	0,3,7	0,5,11	0,8,23	∞,0,10
6,23,22	∞,15,11	13,16,30	1,26,5	31,7,34	24,4,32,12

4^{10} in Z_{40}

0,1,3	0,4,9	0,6,18	0,7,21	0,8,23	0,11,24
0,1,4	0,2,14	0,5,27	0,6,31	0,7,23	0,8,19
2,8,39	36,31,25	33,11,9	23,29,27	34,1,38	32,18,4

6^8 in Z_{48}

0,1,3	0,4,9	0,6,18	0,7,26	0,10,25	0,11,28	0,13,27
0,1,20	0,2,41	0,3,34	0,4,22	0,5,42	0,10,35	0,12,33
42,30,31	45,17,6	23,18,15	10,25,35	22,7,2	20,4,38	46,11,9

6^9 in Z_{54}

0,1,3	0,4,10	0,5,16	0,7,28	0,8,30	0,12,25
0,1,5	0,2,12	0,3,23	0,6,19	0,7,33	0,8,38
11,15,52	6,23,41	38,34,32	48,33,31	40,19,49	16,35,46
	0,14,31	0,15,34			
	0,11,25	0,15,37			
	42,1,24	7,8,51			

In addition to the OGDD produced using relative difference sets, we produce three more OGDD by employing smaller groups of automorphisms. On $Z_7 \times \{0, 1\}$, define a GDD of type 2^7 by taking $\Big\{\{i \times \{0, 1\}\} : i \in Z_7\Big\}$ to form the seven groups, and develop the starter blocks

$$\Big\{\{0_0, 1_0, 4_1\}, \{0_0, 2_0, 1_1\}, \{0_0, 3_0, 5_1\}, \{0_1, 1_1, 3_1\}\Big\}$$

modulo $(7, -)$ to form the triples. Form a second GDD by developing

$$\Big\{\{0_0, 3_1, 4_1\}, \{0_0, 1_1, 6_1\}, \{0_0, 2_1, 5_1\}, \{0_0, 1_0, 3_0\}\Big\}.$$

It is easily verified that the two GDD are orthogonal.

Similarly, on $Z_9 \times \{0, 1\}$, form two GDD by developing the following starter blocks modulo $(9, -)$:

GDD # 1	GDD # 2
$0_0, 2_0, 8_1$	$0_1, 2_1, 8_0$
$0_0, 4_0, 7_1$	$0_1, 4_1, 7_0$
$0_0, 1_0, 5_1$	$0_1, 1_1, 5_0$
$0_0, 1_1, 2_1$	$0_1, 1_0, 2_0$
$0_1, 2_1, 5_1$	$0_0, 2_0, 5_0$
$0_0, 3_0, 6_0$	$0_1, 3_1, 6_1$

Each of the first five starter blocks develops into nine blocks, while the sixth generates only three distinct blocks. It is an easy exercise to verify that these two GDD form OGDD of type 2^9.

Finally, on $Z_5 \times \{0, 1, 2, 3\}$, we present OGDD of type 2^{10} having groups $\Big\{\{i \times \{0, 1\}\} : i \in Z_5\Big\} \cup \Big\{\{i \times \{2, 3\}\} : i \in Z_5\Big\}$.

The starter blocks for the two GDD to be developed modulo $(5, -)$ are:

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GDD # 1	GDD # 2
$0_0, 3_0, 4_1$	$0_0, 3_0, 2_2$
$0_0, 4_0, 2_1$	$0_0, 1_0, 1_3$
$0_0, 1_2, 4_2$	$0_0, 3_1, 2_1$
$0_0, 2_2, 3_2$	$0_0, 1_1, 3_2$
$0_0, 0_2, 2_3$	$0_0, 4_1, 3_3$
$0_0, 1_3, 3_3$	$0_0, 0_2, 1_2$
$0_0, 4_3, 0_3$	$0_0, 2_3, 4_3$
$0_1, 2_1, 4_2$	$0_1, 3_1, 0_3$
$0_1, 1_1, 2_3$	$0_1, 3_2, 1_2$
$0_1, 3_2, 4_3$	$0_1, 4_2, 3_3$
$0_1, 1_2, 0_3$	$0_1, 0_2, 1_3$
$0_1, 0_2, 3_3$	$0_2, 2_3, 3_3$

We summarize the constructions given thus far:

LEMMA 2.3. *There exist* OGDD *of type*
1. 2^n *for* $n \in \{7, 9, 10, 12, 13, 15, 16\}$;
2. 3^n *for* $n \in \{11, 17\}$;
3. 4^{10} *and* $4^9 2^1$;
4. 6^n *for* $n \in \{8, 9\}$.

We have in fact found many more OGDD than those presented here, but have chosen to include here just those that assist us in settling the existence problem for OSTS.

Now we turn to constructions that use OGDD to produce OSTS. Our main construction is an application of Wilson's Fundamental Construction [12]:

LEMMA 2.4 (MAIN CONSTRUCTION). *If there exist* OGDD *of type* $g^n u^1$, OGDD *of type* $g^n v^1$, *and a* TD$(n+1, m)$, *then there exists an* OGDD *of type* $(mg)^n\big((m-t)u + tv\big)^1$ *for all* $0 \le t \le m$.

PROOF. Let $G_1, \ldots, G_{n+1}$ be the groups of the TD$(n+1, m)$. Apply Wilson's Fundamental Construction, giving each point of groups $G_1, \ldots, G_n$ weight g, $m - t$ points of G_{n+1} weight u and the remaining t points of G_{n+1} weight v. The result is a pair of GDD of type $(mg)^n\big((m-t)u+tv\big)^1$ whose orthogonality can be verified easily using the definition of OGDD. ∎

Next we consider some applications of the Main construction.

LEMMA 2.5. *Let* $X = \{18, 24, 30\}$. *Let* $x \in X$ *and suppose that there exists a* TD$\big((x/2)+1, m\big)$, *an* OSTS$(2m+1)$ *and an* OSTS$(2t+1)$, *where* $0 \le t \le m$. *Then there exists an* OSTS$(xm + 2t + 1)$.

PROOF. For each $x \in X$, there exist OGDD of type $2^{x/2}$ and of type $2^{(x/2)+1}$, by Lemma 2.3. Apply Lemma 2.4 with $u = 0$ and $v = 2$ to obtain OGDD of type $(2m)^{x/2}(2t)^1$. Apply Lemma 1.6 to complete the proof. ∎

LEMMA 2.6. *Suppose that there exists a* TD$(10, m)$, *an* OSTS$(4m+1)$ *and an* OSTS$(2m+2t+1)$ *where* $0 \le t \le m$. *Then there exists an* OSTS$(38m + 2t + 1)$.

PROOF. There exists an OGDD of type $4^9 2^1$ and of type 4^{10}. Apply Lemma 2.4 with $u = 2$ and $v = 4$, and then apply Lemma 1.6. ■

In order to apply Lemma 2.5, we require that $m \equiv 0, 1 \pmod 3$ and that $t \equiv 0, 1 \pmod 3$, $t \notin \{1, 4\}$. In order to apply Lemma 2.6, we require that $m \equiv 0, 2 \pmod 3$ and that $t \equiv 0, 1 \pmod 3$; moreover, when $m \equiv 2 \pmod 3$, we require that $t \equiv 1 \pmod 3$ for an OSTS$(2m + 2t + 1)$ to exist. In particular, the smallest order obtained when $m \equiv 2 \pmod 3$ is $38m + 3$.

For dealing with the case when $v \equiv 1 \pmod 6$, one further construction of this type is useful:

LEMMA 2.7. *Suppose that there exists a* TD$(9, m)$, *an* OSTS$(6m + 1)$ *and an* OSTS$(6t + 1)$ *where* $0 \leq t \leq m$. *Then there exists an* OSTS$(48m + 6t + 1)$.

PROOF. There exist OGDD of types 6^8 and 6^9 by Lemma 2.3. Apply Lemma 2.4 with $u = 0$ and $v = 6$, and then apply Lemma 1.6. ■

Each of Lemmas 2.5, 2.6 and 2.7 require transversal designs. Fortunately, all of the transversal designs that we require are produced by a single classical construction due to MacNeish [5]:

LEMMA 2.8. *Let* $n \geq 2$ *be an integer. Suppose that* $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ *where* $p_1, \ldots, p_s$ *are distinct primes. Then there exists a* TD(m, n) *for all* $2 \leq m \leq \min_i(p_i^{\alpha_i}) + 1$.

In the following, Lemma 2.8 is used to produce all of the needed transversal designs.

3. OSTS with $v \equiv 1 \pmod 6$.

In this section, we show that if $v \equiv 1 \pmod 6$, there is an OSTS(v). In the process, we consider a number of smaller cases for the class $v \equiv 3 \pmod 6$, but for the most part we treat the two classes separately.

We require two direct constructions of Gibbons and Mathon [3]:

LEMMA 3.1. *There exists an* OSTS(v) *for* $v \in \{115, 145\}$.

In [8], it is shown that if $v \equiv 1 \pmod 6$ and $v \geq 1927$, there exists an OSTS(v). Some of the remaining cases were handled by Stinson and Zhu [11], who show:

LEMMA 3.2. *For* $v \equiv 1 \pmod 6$, *there exists an* OSTS(v) *with the possible exception of* $v \in \mathcal{A}$, *where*

$$\mathcal{A} = \{55, 115, 145, 205, 235, 265, 295, 319, 355, 391, 415, 445, 451, 493, 649, 655, 679, 697, 745, 781, 799, 805, 1243, 1255, 1315, 1585, 1795, 1819, 1921\}.$$

Recently, Greig [4] has shown the existence of OSTS(v) for eight values in $\mathcal{A}$.

LEMMA 3.3. *For* $v \equiv 1 \pmod 6$, $v \leq 307$, *there exists an* OSTS(v).

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PROOF. By Lemma 3.2 and Lemma 3.1, we need only consider $v \in \{55, 205, 235, 265, 295\}$. For these values, apply Lemmas 1.2 and 1.1 using the following ingredients:

Order	OGDD	COQ	OSTS
55	6^9	-	7
205	3^{17}	4	13
235	2^{13}	9	19
265	3^{11}	8	25

Greig [4] gives a PBD on 295 points having a unique block of size 49 and all other blocks of size 7, producing an OSTS(295). This establishes the lemma. ∎

LEMMA 3.4. *For* $v \equiv 3 \pmod 6$, $15 \leq v \leq 201$, *there exists an* OSTS(v).

PROOF. Rosa [10] constructed an OSTS(27), and Gibbons [2] constructed an OSTS(15). Stinson and Zhu [11] constructed OSTS(v) for $v \in \{105, 189, 195\}$. Gibbons and Mathon [3] have constructed OSTS(v) for $v \in \{21, 33, 39, 45, 51, 57, 63, 69, 75, 81, 87, 93, 99, 111, 117, 123, 129, 135, 153, 159, 171\}$. This leaves the values $v \in \{141, 147, 165, 177, 183, 201\}$. The values $v \in \{141, 147\}$ are handled by Lemmas 1.2 and 1.3 using the following ingredients:

Order	OSTS	COQ	Construction
141	7	20	Lemma 1.3
147	7	21	Lemma 1.2

The values $v \in \{165, 183, 201\}$ are handled by Lemma 1.1 using the following ingredients:

Order	OGDD	COQ	OSTS
165	3^{11}	5	15
183	2^{13}	7	15
201	4^{10}	5	21

For $v = 177$, apply Lemma 2.5 with $x = 18$, $m = 9$ and $t = 7$ to form an OGDD of type $(18)^9(14)^1$, and apply Lemma 1.6 using OSTS(19) and OSTS(15).

This completes the proof. ∎

LEMMA 3.5. *Suppose that* $v \equiv 1, 3 \pmod 6$ *and that* v *satisfies one of*

1. $301 \leq v \leq 381$;
2. $397 \leq v \leq 441$;
3. $463 \leq v \leq 541$;
4. $571 \leq v \leq 741$;
5. $757 \leq v \leq 865$;
6. $877 \leq v \leq 993$;
7. $1027 \leq v \leq 1743$; *or*
8. $1759 \leq v \leq 1941$.

Then there exists an OSTS(v).

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PROOF. Two tables are given below for applications of Lemma 2.5 and Lemma 2.6, respectively. In employing the first table, the OSTS($2m+1$) exists as a consequence of Lemmas 3.4 and Lemma 3.3. Similarly, for the second table, an OSTS($4m+1$) exists by Lemmas 3.4 and Lemma 3.3.

Applications of Lemma 2.5

m	x	$mx+13$	$m(x+2)+1$
16	18	301	321
13	24	325	339
19	18	355	381
16	24	397	417
25	18	463	501
27	18	499	541
31	18	571	621
25	24	613	651
37	18	679	741
31	24	757	807
43	18	787	861
27	30	823	865
49	18	895	981
31	30	943	993
43	24	1045	1119
61	18	1113	1221
67	18	1219	1341
73	18	1327	1461
79	18	1435	1581
64	24	1549	1665
67	24	1621	1743
97	18	1759	1941

Applications of Lemma 2.6

m	$38m+1$	$38m+3$	$40m+1$
9	343		361
11		421	441
17		649	681
23		877	921
27	1027		1081

This completes the proof. ■

Now we are in a position to state the definitive result for $v \equiv 1 \pmod 6$.

THEOREM 3.6. *If* $v \equiv 1 \pmod 6$, *there exists an* OSTS(v).

PROOF. In view of Lemma 3.3, we need only consider $v \geq 313$, and in view of Lemma 3.2, we need only consider $v \leq 1921$. By Lemma 3.5, if $301 \leq v \leq 1941$ and $v \equiv 1, 3 \pmod 6$, then there exists an OSTS(v) except possibly when

1. $385 \leq v \leq 393$;
2. $445 \leq v \leq 459$;

3. $543 \leq v \leq 567$;
4. $867 \leq v \leq 873$;
5. $997 \leq v \leq 1023$; or
6. $1747 \leq v \leq 1755$.

Upon considering the set $\mathcal{A}$ of Lemma 3.2, it suffices to consider values in the first two of these intervals. We cover these intervals using Lemma 2.7 as follows:

Applications of Lemma 2.7

m	$48m+1$	$54m+1$
8	385	433
9	433	487

This completes the proof of the theorem. ∎

4. OSTS **with** $v \equiv 3 \pmod 6$**.** In this section, we complete the proof of the Main Theorem, showing that if $v \equiv 3 \pmod 6$ and $v \geq 15$, there is an OSTS(v). We shall require a few more direct constructions due to Gibbons and Mathon [3]:

LEMMA 4.1. *If* $v \in \{207, 213, 219, 237, 243, 279, 291, 387, 447, 453, 543, 549, 1011, 1017\}$ *then there exists an* OSTS(v).

LEMMA 4.2. *If* $v \equiv 3 \pmod 6$ *and* $15 \leq v \leq 297$, *there exists an* OSTS(v).

PROOF. In view of Lemma 3.4, we need only consider $207 \leq v \leq 297$. By Lemma 4.1, we need only consider

$$v \in \{225, 231, 249, 255, 261, 267, 273, 285, 297\}.$$

Applying Lemma 2.5 with $x = 18$, $m = 13$ and $t \in \{7, 10, 13\}$ yields OSTS(v) for $v \in \{249, 255, 261\}$. Then applying Lemmas 1.2 and 1.3 gives OSTS(v) for $v \in \{225, 231, 273, 285\}$, using ingredients as follows:

Order	OSTS	COQ	Construction
225	7	32	Lemma 1.3
231	7	33	Lemma 1.2
273	21	13	Lemma 1.2
285	15	19	Lemma 1.2

The final value, $v = 297$, is handled by Lemma 1.1 using the following ingredients:

Order	OGDD	COQ	OSTS
297	3^{11}	9	27

This completes the proof. ∎

Now we extend the interval of orders for which OSTS are known:

LEMMA 4.3. *If* $v \equiv 3 \pmod 6$ *and* $15 \leq v \leq 1941$, *there exists an* OSTS(v).

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PROOF. By Lemma 4.2, we need only consider $v \geq 303$. The following list of intervals is what remains to consider after applying Lemma 3.5 (as summarized in the proof of Theorem 3.6), restricting the intervals to those values congruent to 3 (mod 6):

1. $387 \leq v \leq 393$;
2. $447 \leq v \leq 459$;
3. $543 \leq v \leq 567$;
4. $867 \leq v \leq 873$;
5. $999 \leq v \leq 1023$; or
6. $1749 \leq v \leq 1755$.

Employing Lemma 4.1, what remains is then only

$$v \in \{393, 459, 555, 561, 567, 867, 999, 1005, 1011, 1023, 1749, 1755\}.$$

For $v \in \{459, 867\}$, we apply Lemma 1.1 using the following ingredients:

Order	OGDD	COQ	OSTS
459	3^{17}	9	27
867	3^{17}	17	51

For the remaining cases, we apply Lemmas 1.2 and 1.3, using the following ingredients:

Order	OSTS	COQ	Construction
393	7	56	Lemma 1.3
555	15	37	Lemma 1.2
561	7	80	Lemma 1.3
567	7	81	Lemma 1.2
999	37	27	Lemma 1.2
1005	15	67	Lemma 1.2
1023	33	31	Lemma 1.2
1749	135	13	Lemma 1.2
1755	19	92	Lemma 1.3

This completes the proof. ∎

Although we have already settled all cases when $v \equiv 1 \pmod 6$, in the remainder we treat both classes $v \equiv 1, 3 \pmod 6$ because it is convenient to do so.

LEMMA 4.4. *Let* $(m_1, m_2, \ldots, m_u)$ *be a sequence of positive integers, and let* s *be a positive integer, satisfying*

1. $m_i \equiv 1 \pmod 6$ *for* $1 \leq i \leq u$;
2. *there exists a* $\mathrm{TD}(10, m_i)$ *for* $1 \leq i \leq u$;
3. $0 < m_{i+1} - m_i \leq 6s$ *for* $1 \leq i < u$; *and*
4. $m_i \geq 54s + 6$.

Suppose further that if $v \equiv 1, 3 \pmod 6$ *and* $13 \leq v \leq 18m_1 + 9$, *there exists an* $\mathrm{OSTS}(v)$.

Then there exists an $\mathrm{OSTS}(v)$ *for* $v \equiv 1, 3 \pmod 6$ *and* $13 \leq v \leq 20m_u + 1$.

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PROOF. Apply a straightforward induction based on Lemma 2.5 with $x = 18$. The fact that $m \geq 54s + 6$ implies $20m + 1 \geq 18(m + 6s) + 13$ ensures that all values are produced. ■

COROLLARY 4.5. *Let* $(m_1, m_2, \ldots)$ *be an infinite sequence of positive integers, and let* s *be a positive integer, satisfying*

1. $m_i \equiv 1 \pmod 6$ *for* $1 \leq i \leq u$;
2. *there exists a* TD$(10, m_i)$ *for* $1 \leq i \leq u$;
3. $0 < m_{i+1} - m_i \leq 6s$ *for* $1 \leq i < u$; *and*
4. $m_i \geq 54s + 6$.

Suppose further that if $v \equiv 1, 3 \pmod 6$ *and* $13 \leq v \leq 18m_1 + 9$, *there exists an* OSTS(v).

Then there exists an OSTS(v) *for* $v \equiv 1, 3 \pmod 6$, $v \geq 13$.

Corollary 4.5 provides the vehicle to complete the solution.

MAIN THEOREM 4.6. *If* $v \equiv 1, 3 \pmod 6$, $v \geq 7$, *and* $v \neq 9$, *there exists an* OSTS(v).

PROOF. First, if $m \equiv 1 \pmod 6$ and $(m, 35) = 1$, there exists a TD$(10, m)$ by Lemma 2.8. Since at least one of $6m$, $6m + 6$ and $6m + 12$ is relatively prime to 35, we can apply Corollary 4.5 with $s = 3$ to the sequence whose entries are elements of $M = \{m : m \geq 169, m \equiv 1 \pmod 6), (m, 35) = 1\}$, provided that an OSTS$(v)$ exists for all $v \equiv 1, 3 \pmod 6$, $13 \leq v \leq 3051$. We begin by applying Lemma 4.4 with $s = 1$ to the sequence (103,109), noting that $18 \cdot 103 + 9 = 1863$ and that $20 \cdot 109 + 1 = 2181$. Thus, together with Lemma 4.3 and Theorem 3.6, we have the result for $13 \leq v \leq 2181$. Extend this interval to include 2185 and 2187 by noting that $2185 \equiv 1 \pmod 6$, and $2187 = 3^7$, so we can apply Lemma 1.2 to an OSTS(27) and a COQ(81) to obtain an OGDD of type 81^{27}, and thus an OSTS(2187).

Now apply Lemma 4.4 with $s = 2$ to the sequence $(121, 127, 139, 151, 157)$, noting that $121 \cdot 18 + 9 = 2187$ and $157 \cdot 20 + 1 = 3141$. Since $3141 > 3041$, we can now apply the corollary as stated to complete the proof. ■

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