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A characterization of left semiartinian rings

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In defining the torsion-theoretic Krull dimension of an associative ring R we make use of a function δ from the complete lattice of all subsets of the torsion-theoretic spectrum of R to the complete lattice of all hereditary torsion theories on R-mod. In this note we give necessary and sufficient conditions for δ to be injective, surjective, and bijective. In particular, δ is bijective if and only if R is a left semiartinian ring.

Throughout the following R will always designate an associative (but not necessarily commutative) ring with unit element and R-tors will denote the complete lattice of all hereditary torsion theories on the category R-mod of unitary left R-modules. If $\tau \in R$-tors, we denote by T_τ the class of all τ-torsion left R-modules, by F_τ the class of all τ-torsionfree left R-modules, and by $T_\tau(-)$ the τ-torsion radical. The smallest element ξ of R-tors is characterized by $T_\xi = \{0\}$; the largest element χ of R-tors is characterized by $F_\chi = \{0\}$. If M is a left R-module, we denote by $\chi(M)$ the largest element of R-tors relative to which M is torsionfree.

If $\tau \in R$-tors, a nonzero left R-module M is said to be τ-*cocritical* if and only if M is τ-torsionfree while M/N is τ-torsion for every nonzero submodule N of M. A nonzero left R-module M is said to be *cocritical* if and only if it is $\chi(M)$-cocritical. The elements

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of R-tors of the form $\pi = \chi(M)$ for M a cocritical left R-module are called *prime* torsion theories [2]. The set of all prime elements of R-tors is called the *left spectrum* of R and is denoted by R-sp . If M is a left R-module, we define the *assassin* of M by

$$\text{ass}(M) = \{\pi \in R\text{-sp} \mid M \text{ has a } \pi\text{-cocritical submodule}\} .$$

A ring R is said to be *left semiartinian* if and only if every nonzero left R-module has a simple submodule.

In [1] we defined the function

$$\delta : \text{subsets of } R\text{-sp} \to R\text{-tors}$$

as follows: if $U \subseteq R$-sp then

$$T_{\delta(U)} = \{{}_RM \mid \emptyset \neq \text{ass}(M/N) \subseteq U \text{ for every proper submodule } N \text{ of } M\} .$$

This function is used in defining a dimension for rings analogous to the classical Krull dimension for commutative rings. In this note we wish to point out some properties of the function δ itself. It is easy and straightforward to check that δ is a morphism of complete lattices and that $\delta(\emptyset) = \xi$.

PROPOSITION 1. *The following conditions are equivalent:*

(1) δ *is injective;*

(2) *if* $\pi \in R$-sp *then there exists a* π*-cocritical simple left* R*-module.*

Proof. (1) ⇒ (2). If $\pi \in R$-sp then $\delta(\{\pi\}) \neq \delta(\emptyset) = \xi$ by the injectiveness of δ and so there exists a nonzero left R-module M which is $\delta(\{\pi\})$-torsion. In particular M has a π-cocritical submodule M' . If $0 \neq N$ is a proper submodule of M' then $\text{ass}(M'/N) = \{\pi\}$ and so M'/N has a π-cocritical submodule. But M'/N is π-torsion and so we have a contradiction. Thus M' can have no proper submodules other than 0 and so M' is simple.

(2) ⇒ (1). Assume that $\delta(U) = \delta(U')$ for $U \neq U' \subseteq R$-sp . Without loss of generality we can assume that there exists a $\pi \in U \backslash U'$. If M is a simple π-cocritical left R-module then $M \in T_{\delta(U)} = T_{\delta(U')}$ and so $\pi \in U'$ – a contradiction. Thus δ is injective. □

PROPOSITION 2. *The following conditions are equivalent:*

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(1) δ *is surjective;*

(2) $\emptyset \neq \operatorname{ass}(M)$ *for every nonzero left* R*-module* M .

Proof. (1) ⇒ (2). If δ is surjective then there exists a $U \subseteq R\text{-sp}$ for which $\delta(U) = \chi$. Therefore $T_{\delta(U)} = R\text{-mod}$ which implies (2).

(2) ⇒ (1). Let $\tau \in R\text{-tors}$ and let $U = \bigcup\{\operatorname{ass}(M) \mid 0 \neq M \in T_\tau\}$. Then $\emptyset \neq \operatorname{ass}(M/N) \subseteq U$ for every proper submodule N of $M \in T_\tau$ and so $\tau \leq \delta(U)$. Assume that $\tau \neq \delta(U)$ and let $0 \neq M \in T_{\delta(U)} \backslash T_\tau$. Then we have $0 \neq \overline{M} = M/T_\tau(M) \in T_{\delta(U)} \cap F_\tau$. Let $\pi \in \operatorname{ass}(\overline{M})$ and let N be a π-cocritical submodule of $\overline{M}$. Then $\pi = \chi(N) \geq \tau$. But $\overline{M} \in T_{\delta(U)}$ and so $\pi \in U$. Therefore there exists a π-cocritical left R-module $N' \in T_\tau \subseteq T_\pi$, a contradiction. Therefore we must have $\tau = \delta(U)$. □

PROPOSITION 3. *The following conditions are equivalent:*

(1) δ *is bijective;*

(2) R *is a left semiartinian ring.*

Proof. (1) ⇒ (2). Let M be a nonzero left R-module. By Proposition 2, $\emptyset \neq \operatorname{ass}(M)$. If $\pi \in \operatorname{ass}(M)$ then by Proposition 1 there exists a simple left R-module N' which is π-cocritical. Moreover, M has a π-cocritical submodule N . Since N' is π-torsionfree, $\hom_R(N', E(N)) \neq 0$. Since N' is π-cocritical and $E(N)$ is π-torsion-free, any nonzero homomorphism $\alpha : N' \to E(N)$ is a monomorphism. Since N' is simple, $N'\alpha \subseteq N$. Therefore M has a simple submodule. This proves that R is left semiartinian.

(2) ⇒ (1). If M is a nonzero left R-module then by (2), M has a simple submodule. Since all simple left R-modules are cocritical, this implies that $\operatorname{ass}(M) \neq \emptyset$. If $\pi \in R\text{-sp}$ and N is a π-cocritical left R-module then N has a simple submodule N' which is also π-cocritical and so $\pi = \chi(N')$. By Propositions 1 and 2, δ is then bijective. □

PROPOSITION 4. *If* R *is a left semiartinian ring then* δ^{-1} *is defined by* $\delta^{-1} : \tau \mapsto \{\chi(M) \mid M$ *is simple and* τ*-torsion*$\}$.

Proof. Let $\tau \in R\text{-tors}$ and let

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$$U = \{\chi(M) \mid M \text{ is simple and } \tau\text{-torsion}\} .$$

If M is a τ-torsion left R-module then so is M/N for every proper submodule N of M. If $\pi \in \text{ass}(M/N)$ then there exists a π-cocritical submodule M'/N of M/N. Since R is left semiartinian, M'/N in turn has a simple submodule M''/N which is also τ-torsion and τ-cocritical. Then $\pi = \chi(M''/N) \in U$. Hence $T_\tau \subseteq T_{\delta(U)}$.

Conversely assume that M is a left R-module which is not τ-torsion. Then $0 \neq \overline{M} = M/T_\tau(M)$ and so $\overline{M}$ has a simple submodule N' which is τ-torsionfree. This implies that $\chi(N') \in \text{ass}(\overline{M})\setminus U$ and so $M \notin T_{\delta(U)}$. □

References

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