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Isotopy in surface complexes from the computational viewpoint

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It is shown that the problem of deciding whether a curve in a finite surface complex is isotopic to a point is NP-complete. This contrasts with the recursively enumerable (RE)-completeness of the corresponding homotopy problem, and exhibits surface complexes as a common framework for NP-complete and RE-complete algorithmic problems.

For basic definitions and results on NP-complete problems the reader is referred to Cook [2] and Karp [4]. As is customary, algorithms will not be described in complete detail but anyone familiar with, say, Turing machines will be able to supply such details and verify the intuitively obvious claims about the lengths of computations. A general reference for computability, from the NP-level to recursive enumerability, is Machtey and Young [5]. For topological matters we refer to Seifert and Threlfall [6].

The *isotopy problem for surface complexes* is the problem of deciding, given a surface complex K and a closed curve c in K, whether c is isotopic to a point in K. The notion of isotopy assumed in the paper is stricter than the usual one and corresponds intuitively to a contraction of the curve during which no point is passed over more than once. This is equivalent to the property used in the proofs, namely that of bounding a topological disc in the complex.

In order to have a finite combinatorial description of (K, c), so that the problem can be considered algorithmic, we shall assume that K is

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a finite 2-dimensional simplicial complex, and that c is an edge path in K. Thus K is determined by finite sets $\{v_i\}$, $\{e_j\}$, and $\{f_k\}$, of "vertices", "edges", and "faces", respectively, each e_j being a pair of distinct v_i's and each f_k a triple of distinct v_i's. The curve is a finite sequence $(e_{j_1}, \ldots, e_{j_n})$ of edges such that e_{j_m} and $e_{j_{m+1}}$ have exactly one vertex in common, as do e_{j_n} and e_{j_1}, and each vertex of K appears at most once among these e_j's.

The *length* $|(K, c)|$ of a pair (K, c) is the total number of symbols in its description

$$\{v_1, v_2, \ldots, e_1, e_2, \ldots; f_1, f_2, \ldots; c = (e_{j_1}, \ldots, e_{j_n})\}$$

when the e_j's and f_k's are written as sets of e_i's.

Any other reasonable combinatorial description of a surface complex - for example a set of polygons with edge identifications - will be convertible to a simplicial one in time bounded by a polynomial function of the length of description chosen; so the notion of "polynomial time" computation relative to the length of description of (K, c) is invariant under reasonable notions of "description".

For definiteness we measure the length of computations relative to $|(K, c)|$.

LEMMA 1. *The isotopy problem for surface complexes is an NP problem.*

Proof. A closed curve c in K is isotopic to a point if and only if it bounds a subcomplex of K which is a topological disc. This is well known and follows, for example, by induction on the number of faces c is pulled across in contracting it to a point.

It then suffices to show how to decide in polynomial time whether a subcomplex $K_c = f_{k_1} \cup \ldots \cup f_{k_p}$ of K is a disc bounded by c, since a non-deterministic algorithm can correctly (but "unknowingly", as it were) select the disc bounded by c, if there is one, in a single sweep across the description of K.

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The nature of K_c can be established by checking the following conditions.

(i) The edges of $f_{k_1}, \ldots, f_{k_p}$ are identified in pairs, except for edges of c, each of which appears exactly once.

(ii) The f_{k_m} incident with a given vertex v_i in K_c form a closed cycle, unless v_i is on c, in which case they form an open cycle.

(Conditions (i) and (ii) say that K_c is a bounded 2-manifold, with boundary curve c.)

(iii) K_c has Euler characteristic 1.

(This condition establishes that the 2-manifold is a disc.)

It is easy to see how to check each of these conditions in polynomial time, in fact the time required is $O(|(K, c)|^2)$. □

It will now suffice to show that a known *NP*-complete problem is polynomial time reducible to the isotopy problem. A suitable problem for this purpose is the *hamiltonian path problem*, which we shall use in a form found in [5]: given a finite graph G and vertices v_1, v_n decide whether G contains a simple path from v_1 to v_n including all its vertices.

LEMMA 2. *The hamiltonian path problem is polynomial time reducible to the isotopy problem for surface complexes.*

Proof. Given a graph G with n vertices $v_1, \ldots, v_n$, construct a surface complex $K(G)$ as follows.

For each vertex v_i of G take a bouquet B_i of circles $c_i^1, \ldots, c_i^n$ with a single common point V_i and diameters $1/n, 2/n, \ldots, n-1/n, 1$ respectively. If $\{v_p, v_q\}$ is an edge of G we connect each circle c_p^m, $m \geq 2$, of B_p to the circle c_q^{m-1} of B_q by a "truncated cone" T_{pq}^m (and each circle c_q^m, $m \geq 2$, of B_q to the

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circle C_q^{m-1} by a truncated cone T_{qp}^m). It is assumed that the T_{pq}^m , $m, p, q \leq n$, meet *only* along shared boundary circles, and at points V_i common to different boundary circles. Finally we attach a cone T_n^1 to the circle C_n^1 at the target vertex V_n , tapering from diameter $1/n$ to 0 , and let c be the circle C_1^n at V_1 .

This construction, including a suitable triangulation of each truncated cone, can be completed in polynomial time.

If G contains a hamiltonian path $(v_1, v_{i_1}, \dots, v_{i_{n-2}}, v_n)$ then c will be the boundary of the cone (equals topological disc)

$$(*) \qquad T^n_{1i_1} \cup T^{n-1}_{i_1 i_2} \cup \dots \cup T^2_{i_{n-2}n} \cup T^1_n ,$$

and therefore isotopic to a point in $K(G)$.

Conversely, any disc bounded by c is of the form (*) and corresponds to a path in G . No disc D bounded by c can contain some, but not all, triangles from a T_{pq}^m , otherwise there will be free edges of D not in c . For the same reason, plus the fact that edges not in c must be identified in pairs, D has to contain exactly two T's incident with a given circle not equal to c . This determines a connected sequence of truncated cones, which must terminate with T_n^1 in order to close D to a disc.

Since the diameter changes by $1/n$ along each truncated cone, at least n T's are required to form D . If more than n are used then D will meet the same V_i twice and hence will not be a disc. The number is therefore exactly n , and to avoid meeting any V_i twice D must meet each of them exactly once, yielding a hamiltonian path from v_1 to v_n in G .

Thus the construction of $K(G)$ from G represents a polynomial time reduction of the hamiltonian path problem to the isotopy problem for

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surface complexes. (It should perhaps be stressed that the assignment of "diameters" to the c_i^m is made merely to assist visualization. It is of course irrelevant to the topological structure of $K(G)$ and need not be mentioned in its description.) □

The *NP*-completeness of the isotopy problem for surface complexes follows immediately from Lemmas 1 and 2.

In contrast, the *homotopy problem* for surface complexes, obtained by replacing the word "isotopic" by "homotopic", and consequently "disc" by "singular disc", is *RE*-complete. This follows from the classical results of Dehn [3] on the realizability of any finitely presented group as π_1 of a surface complex K and the equivalence between the homotopy problem and the word problem for $\pi_1(K)$, together with the results of Novikov and Boone which show that the word problem for groups is unsolvable and of the same degree as the complete recursively enumerable set (see for example [1]).

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