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Realizations of free actions via their fixed point algebras

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Abstract. Let G be a compact group, let $\mathcal{B}$ be a unital C*-algebra, and let $(\mathcal{A}, G, \alpha)$ be a free C*-dynamical system, in the sense of Ellwood, with fixed point algebra $\mathcal{B}$. We prove that $(\mathcal{A}, G, \alpha)$ can be realized as the G-continuous part of the invariants of an equivariant coaction of G on a corner of $\mathcal{B} \otimes \mathcal{K}(\mathfrak{H})$ for a certain Hilbert space $\mathfrak{H}$ that arises from the freeness of the action. This extends a result by Wassermann for free and ergodic C*-dynamical systems. As an application, we show that any faithful *-representation of $\mathcal{B}$ on a Hilbert space $\mathfrak{H}_{\mathcal{B}}$ gives rise to a faithful covariant representation of $(\mathcal{A}, G, \alpha)$ on some truncation of $\mathfrak{H}_{\mathcal{B}} \otimes \mathfrak{H}$.

1 Introduction

Let $\mathcal{A}$ be a unital C*-algebra and let G be a compact group that acts strongly continuously on $\mathcal{A}$ by *-automorphisms $\alpha_g : \mathcal{A} \to \mathcal{A}$, $g \in G$. In this article, we call this data a *C*-dynamical system*, denote it briefly by $(\mathcal{A}, G, \alpha)$, and customarily write $\mathcal{B} := \mathcal{A}^G := \{x \in \mathcal{A} : (\forall g \in G)\, \alpha_g(x) = x\}$ for its fixed point algebra. Research into C*-dynamical systems is inherently interesting and has always been an area of active research both in operator algebras and noncommutative geometry. It is desirable to identify properties of C*-dynamical systems that are both commonly occurring and significant enough to obtain interesting results. To expedite matters, let us revisit the notion of *freeness*, which exemplifies one such property: A C*-dynamical system $(\mathcal{A}, G, \alpha)$ is called *free* if the so-called *Ellwood map*

$$\Phi : \mathcal{A} \otimes_{\text{alg}} \mathcal{A} \to C(G, \mathcal{A}), \qquad \Phi(x \otimes y)(g) := x\alpha_g(y)$$

has dense range with respect to the canonical C*-norm on $C(G, \mathcal{A})$. This condition, first introduced by Ellwood [10] for actions of quantum groups on C*-algebras, is known to be equivalent to Rieffel's saturatedness [19] and the Peter–Weyl–Galois condition [2]. By [16, Proposition 7.1.12 and Theorem 7.2.6], a continuous action $r : P \times G \to P$ of a compact group G on a compact space P is free in the classical sense, i. e., all stabilizer groups are trivial, if and only if the induced C*-dynamical system $(C(P), G, \alpha)$, where $\alpha_g(f)(p) := f(r(p, g))$ for all $g \in G$, $f \in C(P)$, and $p \in P$, is free in the sense of Ellwood. Free C*-dynamical systems thus provide a natural framework for noncommutative principal bundles. Because of this and their wide range of applications, these objects have garnered widespread interest and have been

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extensively studied by many researchers in recent years (see, e. g., [1, 2, 4–8, 11, 12, 17, 22, 24] and ref therein).

Free and ergodic C*-dynamical systems, also known as full multiplicity ergodic actions, have been the focal point of [26]. In particular, Wassermann proved the interesting result that each free and ergodic C*-dynamical system $(\mathcal{A}, G, \alpha)$ can be realized as the invariants of an equivariant coaction of G on $\mathcal{K}(L^2(G))$. Noteworthily, this constitutes an important step in the classification of such C*-dynamical systems by means of a generalized cocycle theory.

Now, let us consider a free C*-dynamical system $(\mathcal{A}, G, \alpha)$ with a general fixed point algebra $\mathcal{B}$. The overall purpose of this paper is to extend Wassermann's result by showing that $(\mathcal{A}, G, \alpha)$ can be realized as the G-continuous part of the invariants of an equivariant coaction of G on a corner of $\mathcal{B} \otimes \mathcal{K}(\mathfrak{H})$ for a certain Hilbert space $\mathfrak{H}$ that arises from the freeness property (Theorem 3.11). From this we derive the following representation theoretic result: Any faithful *-representation of $\mathcal{B}$ on a Hilbert space $\mathfrak{H}_{\mathcal{B}}$ gives rise to a faithful covariant representation of $(\mathcal{A}, G, \alpha)$ on some truncation of $\mathfrak{H}_{\mathcal{B}} \otimes \mathfrak{H}$ (Corollary 3.13). It is our hope that Theorem 3.11 can be further utilized to present conditions under which certain properties of the fixed point algebra $\mathcal{B}$ pass over to $(\mathcal{A}, G, \alpha)$. Note that we do not address the problem of classifying free C*-dynamical systems in this paper. In fact, this problem has been thoroughly treated in [22] using different methods.

Following this introduction, the fundamental definitions and notations are presented in Section 2. The proofs are provided in Section 3, which is in essence divided into three parts: first, constructing an equivariant coaction from a given free C*-dynamical system; second, identifying the corresponding invariants as a free C*-dynamical system; and third, proving that the original and the derived free C*-dynamical systems are isomorphic.

2 Preliminaries and notations

2.1 About tensor products

In this article, tensor products of C*-algebras are taken with respect to the minimal tensor product, which is denoted by $\otimes$. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be unital C*-algebras. If there is no ambiguity, then we consider each one of them as a C*-subalgebra of $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}$ and extend maps on $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{C}$ canonically by tensoring with the respective identity map. For the sake of clarity, we also make use of the leg numbering notation, for instance, given $x \in \mathcal{A} \otimes \mathcal{C}$, we write x_{13} to denote the corresponding element in $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}$.

2.2 About multiplier algebras

Let $\mathcal{A}$ be a C*-algebra. A linear operator $m : \mathcal{A} \to \mathcal{A}$ is said to be a multiplier of $\mathcal{A}$ if for each $a \in \mathcal{A}$ there exists $c \in \mathcal{A}$ such that $a^* m(b) = c^* b$ for all $b \in \mathcal{A}$. The set of all multipliers of $\mathcal{A}$ is a unital C*-algebra which is denoted by $\mathcal{M}(\mathcal{A})$ and called the multiplier algebra of $\mathcal{A}$. For a faithful and nondegenerate *-representation $\pi : \mathcal{A} \to \mathcal{L}(\mathfrak{H})$ it may be identified with the following operator algebra:

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$$\mathcal{M}(\mathcal{A}) = \{m \in \mathcal{L}(\mathfrak{H}) : (\forall a \in \mathcal{A})\, ma, am \in \mathcal{A}\}.$$

Note that if $\mathcal{A}$ is a unital C*algebra, then $\mathcal{M}(\mathcal{A}) \cong \mathcal{A}$. For a thorough treatment of multipliers we refer to [15, Section 3.12].

Lemma 2.1 *Let $\pi_\mathcal{A} : \mathcal{A} \to \mathcal{L}(\mathfrak{H}_\mathcal{A})$ and $\pi_\mathcal{B} : \mathcal{B} \to \mathcal{L}(\mathfrak{H}_\mathcal{B})$ be faithful and nondegenerate *-homomorphisms of C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Then*

$$\mathcal{M}(\mathcal{A}) \otimes \mathcal{B} = (\mathcal{L}(\mathfrak{H}_\mathcal{A}) \otimes \mathcal{B}) \cap \mathcal{M}(\mathcal{A} \otimes \mathcal{B}).$$

Proof Clearly, $\mathcal{M}(\mathcal{A}) \otimes \mathcal{B} \subseteq (\mathcal{L}(\mathfrak{H}_\mathcal{A}) \otimes \mathcal{B}) \cap \mathcal{M}(\mathcal{A} \otimes \mathcal{B})$. Hence it suffices to establish the opposite inclusion. For this purpose, we first assume that $\mathcal{B}$ is unital. Let $x := \sum_{i=1}^n x_i \otimes b_i \in \mathcal{M}(\mathcal{A} \otimes \mathcal{B})$ for operators $x_1, \ldots, x_n \in \mathcal{L}(\mathfrak{H}_\mathcal{A})$ and linear independent elements $b_1, \ldots b_n \in \mathcal{B}$. Moreover, let ψ be a state on $\mathcal{B}$ such that $\psi(b_i^* b_i) > 0$ for all $1 \le i \le n$. Applying the Gram–Schmidt process, we may w. l. o. g. assume that $\psi(b_i^* b_j) = \delta_{i,j}$ for all $1 \le i, j \le n$, i. e., $b_1, \ldots, b_n$ is an orthonormal system. We write $\omega_i(b) := \psi(b_i^* b), b \in \mathcal{B}, 1 \le i \le n$, for the corresponding dual system of functionals on $\mathcal{B}$. Since $x \in \mathcal{M}(\mathcal{A} \otimes \mathcal{B})$, for each $a \in \mathcal{A}$ we have $x(a \otimes 1_\mathcal{B}) \in \mathcal{A} \otimes \mathcal{B}$. Consequently,

$$x_i a = (\mathrm{id} \otimes \omega_i(x))a = \mathrm{id} \otimes \omega_i(x(a \otimes 1_\mathcal{B})) \in \mathcal{A}$$

for all $1 \le i \le n$ and $a \in \mathcal{A}$. Likewise, we find $ax_i \in \mathcal{A}$. It follows that $x_i \in \mathcal{M}(\mathcal{A})$ for all $1 \le i \le n$, and hence $x \in \mathcal{M}(\mathcal{A}) \otimes \mathcal{B}$. Taking limits, we thus get

$$\big(\mathcal{L}(\mathfrak{H}_A) \otimes \mathcal{B}\big) \cap \mathcal{M}(\mathcal{A} \otimes \mathcal{B}) \subseteq \mathcal{M}(\mathcal{A}) \otimes \mathcal{B}.$$

For non-unital $\mathcal{B}$ we may replace $1_\mathcal{B}$ by an approximate unit of $\mathcal{B}$ and use similar arguments, the detailed verification being left to the reader. ∎

2.3 About coactions of compact groups

Let G be a locally compact Hausdorff group. We denote by $\lambda : G \to \mathcal{U}(L^2(G))$ the left regular representation given by $(\lambda_g f)(h) := f(g^{-1}h)$ and by $r : G \to \mathcal{U}(L^2(G))$ the right regular representation given by $(r_g f)(h) := f(hg)$. For $f \in C_0(G)$ we write $r(f)$ for the integrated form with respect to the right regular representation and consider $C_r^*(G)$ as the norm closure of the *-algebra $r(C(G)) \subseteq \mathcal{L}(L^2(G))$ or, equivalently, as the fixed point algebra of $\mathcal{K}(L^2(G))$ under the adjoint action $\mathrm{Ad}[\lambda_g]$, $g \in G$.

Usually, we consider $C_r^*(G)$ as a quantum group with respect to the faithful and nondegenerate *-homomorphism $\delta_G : C_r^*(G) \to \mathcal{M}(C_r^*(G) \otimes C_r^*(G))$ defined by the integrated form of the diagonal representation $G \ni g \mapsto r_g \otimes r_g \in \mathcal{U}(L^2(G) \otimes L^2(G))$. Due to [14, p. 255] (see also [25, p. 48]), the unitary W_G on $L^2(G \times G) \cong L^2(G) \otimes L^2(G)$ defined by $(W_G f)(g, h) := f(g, hg)$ implements the map δ_G in the sense that $\delta_G(x) = W_G^*\big(x \otimes 1_G\big)W_G$ for all $x \in C_r^*(G)$. We shall also utilize the fact that $W_G \in \mathcal{M}\big(C_0(G) \otimes C_r^*(G)\big)$ as well as the identities

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$$(W_G)_{23}(W_G)_{12}(W_G)_{13} = (W_G)_{12}(W_G)_{23}, \tag{2.1}$$

$$(r_g \otimes 1_G)W_G = W_G(r_g \otimes r_g) \qquad \forall g \in G, \tag{2.2}$$

$$(1_G \otimes \lambda_g)W_G = W_G(1_G \otimes \lambda_g) \qquad \forall g \in G, \tag{2.3}$$

$$(\lambda_g \otimes r_g)W_G = W_G(\lambda_g \otimes 1_G) \qquad \forall g \in G. \tag{2.4}$$

More generally, we are concerned with coactions of compact groups on C*-algebras and refer to [9, Appendix A] for a detailed discussion on the subject. However, for expediency we now repeat the basic definition.

Definition 2.1 A *coaction* of a locally compact Hausdorff group G on a C*-algebra $\mathcal{A}$ is a faithful and nondegenerate *-homomorphism $\delta : \mathcal{A} \to \mathcal{M}(\mathcal{A} \otimes C_r^*(G))$ satisfying the *coaction identity*

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta_G) \circ \delta. \tag{2.5}$$

As coactions take values in multiplier algebras, the coaction identity involves the extensions of the maps $\delta \otimes \mathrm{id}$ and $\mathrm{id} \otimes \delta_G$ to the multiplier algebras $\mathcal{M}(\mathcal{A} \otimes C_r^*(G))$ and $\mathcal{M}(\mathcal{A} \otimes C_r^*(G) \otimes C_r^*(G))$ in domain and codomain, respectively.

Remark 2.2 Let $\mathbb{G}$ be a quantum group and let $\mathcal{A}$ be a C*-algebra. We wish to mention that in the quantum group literature, a map $\delta : \mathcal{A} \to \mathcal{M}(\mathcal{A} \otimes C_r^*(\mathbb{G}))$ satisfying (2.5) is called "a coaction of $C_r^*(\mathbb{G})$ on $\mathcal{A}$" or "a coaction of the dual quantum group $\widehat{\mathbb{G}}$ on $\mathcal{A}$".

2.4 About free C*-dynamical systems

One of the key tools utilized in this article is a characterization of freeness that we previously introduced in [22, Lemma 3.2], namely that a C*-dynamical system $(\mathcal{A}, G, \alpha)$ is free if and only if for each irreducible representation (σ, V_σ) of G there exists a finite-dimensional Hilbert space $\mathfrak{H}_\sigma$ and an isometry $s(\sigma) \in \mathcal{A} \otimes \mathcal{L}(V_\sigma, \mathfrak{H}_\sigma)$ satisfying $\alpha_g(s(\sigma)) = s(\sigma)(1_{\mathcal{A}} \otimes \sigma_g)$ for all $g \in G$. However, to simplify notation we patch this family of isometries together and use the following characterization instead.

Lemma 2.3 (see [24, Lemma 3.1]) *For a C*-dynamical system $(\mathcal{A}, G, \alpha)$ the following statements are equivalent:*

(a) *$(\mathcal{A}, G, \alpha)$ is free.*
(b) *There is a unitary representation $\mu : G \to \mathcal{U}(\mathfrak{H})$ with finite-dimensional multiplicity spaces and, given any faithful covariant representation (π, u) of $(\mathcal{A}, G, \alpha)$ on some Hilbert space $\mathfrak{H}_{\mathcal{A}}$, an isometry $S \in \mathcal{L}(\mathfrak{H}_{\mathcal{A}} \otimes L^2(G), \mathfrak{H}_{\mathcal{A}} \otimes \mathfrak{H})$ satisfying*

$$S\mathcal{A} \otimes \mathcal{K}(L^2(G)) \subseteq \mathcal{A} \otimes \mathcal{K}(L^2(G), \mathfrak{H}), \tag{2.6}$$

$$(u_g \otimes 1_{\mathfrak{H}})S = S(u_g \otimes r_g) \qquad \forall g \in G, \tag{2.7}$$

$$(1_{\mathcal{A}} \otimes \mu_g)S = S(1_{\mathcal{A}} \otimes \lambda_g) \qquad \forall g \in G. \tag{2.8}$$

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Here, we do not distinguish between $\mathcal{A}$ and $\pi(\mathcal{A}) \subseteq \mathcal{L}(\mathfrak{H}_{\mathcal{A}})$ for sake of brevity. Furthermore, the tensor product $\mathcal{A} \otimes \mathcal{K}(L^2(G), \mathfrak{H})$ is closed with respect to the operator norm, where $\mathcal{K}(L^2(G), \mathfrak{H})$ is regarded as the respective corner of $\mathcal{K}(L^2(G) \oplus \mathfrak{H})$.

The adjoint $S^* \in \mathcal{L}(\mathfrak{H}_{\mathcal{A}} \otimes \mathfrak{H}, \mathfrak{H}_{\mathcal{A}} \otimes L^2(G))$ of the isometry S in Lemma 2.3 satisfies $S^* \mathcal{A} \otimes \mathcal{K}(\mathfrak{H}) \subseteq \mathcal{A} \otimes \mathcal{K}(\mathfrak{H}, L^2(G))$ or, equivalently,

$$\mathcal{A} \otimes \mathcal{K}(\mathfrak{H}) S \subseteq \mathcal{A} \otimes \mathcal{K}(L^2(G), \mathfrak{H}). \tag{2.9}$$

For this reason, we can assert that S is, in fact, a multiplier for $\mathcal{A} \otimes \mathcal{K}(L^2(G) \oplus \mathfrak{H})$, i. e., $S \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K}(L^2(G) \oplus \mathfrak{H}))$, with $(1_{\mathcal{A}} \otimes p_{L^2(G)}) S = 0 = S(1_{\mathcal{A}} \otimes p_{\mathfrak{H}})$, where $p_{\mathfrak{H}}$ and $p_{L^2(G)}$ denote the canonical projections onto $\mathfrak{H}$ and $L^2(G)$, respectively.

A particular simple class of free actions is given by so-called *cleft actions* (see [21]). Regarded as noncommutative principal bundles, these actions are essentially characterized by the fact that all associated noncommutative vector bundles are trivial. For convenience of the reader we now recall the definition. Indeed, we call a C*-dynamical system $(\mathcal{A}, G, \alpha)$ *cleft* if there is a unitary $U \in \mathcal{M}(\mathcal{A} \otimes C_r^*(G))$ such that $\bar{\alpha}_g(U) = U(1_{\mathcal{A}} \otimes r_g)$ for all $g \in G$. Here, $\bar{\alpha}_g$, $g \in G$, denotes the strictly continuous extension of $\alpha_g \otimes 1_G$, $g \in G$, to $\mathcal{M}(\mathcal{A} \otimes C_r^*(G))$ (see, e. g., [15, Proposition 3.12.10]), which is continuous for the strict topology. It is clear that each cleft C*-dynamical system is free with a possible choice for μ and $\mathfrak{H}$ given by λ and $L^2(G)$, respectively.

3 The realization

To commence, we fix a free C*-dynamical system $(\mathcal{A}, G, \alpha)$ with fixed point algebra $\mathcal{B}$ along with a faithful covariant representation (π, u) thereof on some Hilbert space $\mathfrak{H}_{\mathcal{A}}$. By Lemma 2.3, there is a unitary representation $\mu : G \to \mathcal{U}(\mathfrak{H})$ with finite-dimensional multiplicity spaces and an isometry $S \in \mathcal{L}(\mathfrak{H}_{\mathcal{A}} \otimes L^2(G), \mathfrak{H}_{\mathcal{A}} \otimes \mathfrak{H})$ satisfying (2.6), (2.7), (2.8), and (2.9).

We see at once that $P := SS^* \in \mathcal{L}(\mathfrak{H}_{\mathcal{A}} \otimes \mathfrak{H})$ is a projection satisfying $(u_g \otimes \mu_h) P = P(u_g \otimes \mu_h)$ for all $g, h \in G$. Moreover, we have $P \in \mathcal{M}(\mathcal{B} \otimes \mathcal{K}(\mathfrak{H}))$, which is easily checked. In what follows, the central object of interest is the corresponding corner

$$\mathcal{D} := P(\mathcal{B} \otimes \mathcal{K}(\mathfrak{H}))P \subseteq \mathcal{B} \otimes \mathcal{K}(\mathfrak{H}) \cap P\mathcal{L}(\mathfrak{H}_{\mathcal{A}} \otimes \mathfrak{H})P. \tag{3.1}$$

Clearly, $\mathcal{D}$ is pointwise fixed by $\mathrm{Ad}[u_g \otimes 1_{\mathfrak{H}}]$, $g \in G$. Furthermore, $\mathrm{Ad}[1_{\mathcal{A}} \otimes \mu_g]$, $g \in G$, is a strongly continuous action on $\mathcal{D}$, i. e., $(\mathcal{D}, G, \mathrm{Ad}[1_{\mathcal{A}} \otimes \mu])$ is a C*-dynamical system. Our first task is to construct a coaction of G on $\mathcal{D}$ in terms of S and W_G (see Section 2.3). We begin with a series of lemmas.

Lemma 3.1 *For the element $W_S := S_{12}(W_G)_{23}S_{12}^*$ in $\mathcal{L}(\mathfrak{H}_{\mathcal{A}} \otimes \mathfrak{H} \otimes L^2(G))$ the following assertions hold:*

1. *W_S is a partial isometry with initial and final projection $P \otimes 1_G$.*
2. *$(u_g \otimes \mu_h \otimes r_h) W_S = W_S(u_g \otimes \mu_h \otimes r_g)$ for all $g, h \in G$.*
3. *$W_S \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K}(\mathfrak{H}) \otimes C_r^*(G))$.*[1]

[1] Note that $\mathcal{A} \otimes \mathcal{K}(\mathfrak{H}) \otimes C_r^*(G) \subseteq \mathcal{L}(\mathfrak{H}_{\mathcal{A}} \otimes \mathfrak{H} \otimes L^2(G))$ is nondegenerate.

Proof

1. Since S is an isometry, we see at once that the initial and final projections are given by $W_S^* W_S = P \otimes 1_G$ and $W_S W_S^* = P \otimes 1_G$, respectively.
2. Let $g, h \in G$. Then an easy verification yields

$$\begin{aligned}(u_g \otimes \mu_h \otimes r_h) W_S &= (u_g \otimes \mu_h \otimes r_h) S_{12} (W_G)_{23} S_{12}^* \\ &\overset{(2.7),(2.8)}{=} S_{12} (u_g \otimes r_g \lambda_h \otimes r_h)(W_G)_{23} S_{12}^* \\ &\overset{(2.2),(2.4)}{=} S_{12} (W_G)_{23} (u_g \otimes r_g \lambda_h \otimes r_g) S_{12}^* \\ &\overset{(2.7),(2.8)}{=} S_{12} (W_G)_{23} S_{12}^* (u_g \otimes \mu_h \otimes r_g) = W_S (u_g \otimes \mu_h \otimes r_g).\end{aligned}$$

3. From (2.6), we deduce that $S\mathcal{A} \otimes \mathcal{K}(\mathfrak{H}, L^2(G)) \subseteq \mathcal{A} \otimes \mathcal{K}(\mathfrak{H})$. Combining this once more with (2.6) and the fact that $W_G \in \mathcal{L}(L^2(G)) \otimes \mathcal{L}(L^2(G))$ yields

$$W_S \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K}(\mathfrak{H}) \otimes \mathcal{K}(L^2(G))).$$

But since $C_r^*(G)$ is the fixed point algebra of $\mathcal{K}(L^2(G))$ under $\operatorname{Ad}[\lambda_g]$, $g \in G$, and W_G satisfies (2.3), it may further be concluded that $W_S \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K}(\mathfrak{H}) \otimes C_r^*(G))$. ∎

Here and subsequently, we do not distinguish between $\mathcal{L}(\mathfrak{H}_\mathcal{A} \otimes \mathfrak{H} \otimes L^2(G))$ and $\mathcal{L}(\mathfrak{H}_\mathcal{A} \otimes \mathfrak{H}) \otimes \mathcal{L}(L^2(G))$ for simplicity of notation.

Lemma 3.2 *For the map*

$$\delta : \mathcal{L}(\mathfrak{H}_\mathcal{A} \otimes \mathfrak{H}) \to \mathcal{L}(\mathfrak{H}_\mathcal{A} \otimes \mathfrak{H} \otimes L^2(G)), \quad \delta(x) := \operatorname{Ad}[W_S](x \otimes 1_G) := W_S (x \otimes 1_G) W_S^*$$

the following assertions hold:

1. *δ is equivariant w.r.t. $\operatorname{Ad}[u_g \otimes \mu_h]$, $g, h \in G$ on $\mathcal{L}(\mathfrak{H}_\mathcal{A} \otimes \mathfrak{H})$ and $\operatorname{Ad}[u_g \otimes \mu_h \otimes r_h]$, $g, h \in G$, on $\mathcal{L}(\mathfrak{H}_\mathcal{A} \otimes \mathfrak{H} \otimes L^2(G))$.*
2. *$\delta(x) \in \mathcal{M}(\mathcal{B} \otimes \mathcal{K}(\mathfrak{H}) \otimes C_r^*(G))$ for all $x \in \mathcal{B} \otimes \mathcal{K}(\mathfrak{H})$.*
3. *δ satisfies the coaction identity $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta_G) \circ \delta$.*

Proof 1. Let $g, h \in G$ and let $x \in \mathcal{L}(\mathfrak{H}_\mathcal{A} \otimes \mathfrak{H})$. Then

$$\begin{aligned}\delta(\operatorname{Ad}[u_g \otimes \mu_h](x)) &= \operatorname{Ad}[W_S](\operatorname{Ad}[u_g \otimes \mu_h \otimes 1_G](x \otimes 1_G)) \\ &= \operatorname{Ad}[W_S (u_g \otimes \mu_h \otimes 1_G)](x \otimes 1_G) \\ &= \operatorname{Ad}[W_S (u_g \otimes \mu_h \otimes r_g)](x \otimes 1_G) \\ &= \operatorname{Ad}[(u_g \otimes \mu_h \otimes r_h) W_S](x \otimes 1_G) \\ &= \operatorname{Ad}[u_g \otimes \mu_h \otimes r_h](\operatorname{Ad}[W_S](x \otimes 1_G)) \\ &= \operatorname{Ad}[u_g \otimes \mu_h \otimes r_h](\delta(x)),\end{aligned}$$

where we have used Lemma 3.1.2 for the third-to-last equality.

2. Let $x \in \mathcal{B} \otimes \mathcal{K}(\mathfrak{H})$ and let $y \in \mathcal{B} \otimes \mathcal{K}(\mathfrak{H}) \otimes C_r^*(G)$. Applying Lemma 3.1.3 yields $\delta(x) y \in \mathcal{A} \otimes \mathcal{K}(\mathfrak{H}) \otimes C_r^*(G)$. Moreover, the equivariance of δ implies that

$$\begin{aligned}\operatorname{Ad}[u_g \otimes 1_\mathfrak{H} \otimes 1_G](\delta(x) y) &= \operatorname{Ad}[u_g \otimes 1_\mathfrak{H} \otimes 1_G](\delta(x)) \operatorname{Ad}[u_g \otimes 1_\mathfrak{H} \otimes 1_G](y) \\ &= \delta(\operatorname{Ad}[u_g \otimes 1_\mathfrak{H}](x)) y = \delta(x) y\end{aligned}$$

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for all $g \in G$, i. e., $\delta(x)y \in \mathcal{B} \otimes \mathcal{K}(\mathfrak{H}) \otimes C_r^*(G)$. Likewise, we obtain $y\delta(x) \in \mathcal{B} \otimes \mathcal{K}(\mathfrak{H}) \otimes C_r^*(G)$. It follows that $\delta(x) \in \mathcal{M}(\mathcal{B} \otimes \mathcal{K}(\mathfrak{H}) \otimes C_r^*(G))$ as required.

3. To demonstrate that $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta_G) \circ \delta$, we first evaluate both sides of the identity. Indeed, for each $x \in \mathcal{L}(\mathfrak{H}_\mathcal{A} \otimes \mathfrak{H})$ we have

$$\begin{aligned} \delta \otimes \mathrm{id}(\delta(x)) &= \delta \otimes \mathrm{id}(\mathrm{Ad}[W_S](x \otimes 1_G)) \\ &= \mathrm{Ad}[(W_S)_{123}(W_S)_{124}](x \otimes 1_G) \\ &= \mathrm{Ad}[S_{12}(W_G)_{23}(W_G)_{24}S_{12}^*](x \otimes 1_G). \end{aligned}$$

Moreover, the right hand side of the identity becomes

$$\begin{aligned} \mathrm{id} \otimes \delta_G(\delta(x)) &= \mathrm{id} \otimes \delta_G(\mathrm{Ad}[W_S](x \otimes 1_G)) \\ &= \mathrm{Ad}[(W_G)_{34}^*(W_S)_{123}](x \otimes 1_G) \\ &= \mathrm{Ad}[(W_G)_{34}^*(W_S)_{123}(W_G)_{34}](x \otimes 1_G) \\ &= \mathrm{Ad}[(W_G)_{34}^*S_{12}(W_G)_{23}S_{12}^*(W_G)_{34}](x \otimes 1_G) \\ &= \mathrm{Ad}[S_{12}(W_G)_{34}^*(W_G)_{23}(W_G)_{34}S_{12}^*](x \otimes 1_G). \end{aligned}$$

Comparing these two expressions, we see that the claim will be proved once we show that $(W_G)_{12}(W_G)_{13} = (W_G)_{23}^*(W_G)_{12}(W_G)_{23}$. But this is clear from (2.1). ∎

Remark 3.3 We stress that the map δ is, in general, not a *-homomorphism.

To proceed, we put $\mathfrak{H}_P := P(\mathfrak{H}_\mathcal{A} \otimes \mathfrak{H})$, identify $\mathcal{L}(\mathfrak{H}_P)$ with $P\mathcal{L}(\mathfrak{H}_\mathcal{A} \otimes \mathfrak{H})P$ by means of the map $P\mathcal{L}(\mathfrak{H}_\mathcal{A} \otimes \mathfrak{H})P \to \mathcal{L}(\mathfrak{H}_P)$, $x \mapsto x|_{\mathfrak{H}_P}^{\mathfrak{H}_P}$, and note that $\mathcal{D} \subseteq \mathcal{L}(\mathfrak{H}_P)$ is nondegenerate.

Lemma 3.4 *Restricting δ to $\mathcal{D}$ (see (3.1)) yields a map*

$$\delta_\mathcal{D} : \mathcal{D} \to \mathcal{M}(\mathcal{D} \otimes C_r^*(G)) \subseteq \mathcal{L}(\mathfrak{H}_P \otimes L^2(G)), \tag{3.2}$$

for which the following assertions hold:

1. *$\delta_\mathcal{D}$ is a *-homomorphism.*
2. *$\delta_\mathcal{D}$ is faithful.*
3. *$\delta_\mathcal{D}$ is nondegenerate.*
4. *$\delta_\mathcal{D}$ satisfies the coaction identity $(\delta_\mathcal{D} \otimes \mathrm{id}) \circ \delta_\mathcal{D} = (\mathrm{id} \otimes \delta_G) \circ \delta_\mathcal{D}$.*
5. *$\delta_\mathcal{D}$ is equivariant w.r.t. $\mathrm{Ad}[1_\mathcal{A} \otimes \mu_g]$, $g \in G$, on $\mathcal{D}$ and $\mathrm{Ad}[1_\mathcal{A} \otimes \mu_g \otimes r_g]$, $g \in G$, on $\mathcal{M}(\mathcal{D} \otimes C_r^*(G))$.*

Proof 0. We first show that $\delta_\mathcal{D}$ is well defined. For this purpose, let $x = PyP \in \mathcal{D}$ for some $y \in \mathcal{B} \otimes \mathcal{K}(\mathfrak{H})$ and let $z \in \mathcal{D} \otimes C_r^*(G) \subseteq \mathcal{B} \otimes \mathcal{K}(\mathfrak{H}) \otimes C_r^*(G)$. By Lemma 3.2.2, $\delta(y)z \in \mathcal{B} \otimes \mathcal{K}(\mathfrak{H}) \otimes C_r^*(G)$. Moreover,

$$\delta(x)z = (P \otimes 1_G)\delta(y)(P \otimes 1_G)z = (P \otimes 1_G)\delta(y)z(P \otimes 1_G).$$

Consequently,

$$\delta(x)z \in (P \otimes 1_G)\mathcal{B} \otimes \mathcal{K}(\mathfrak{H}) \otimes C_r^*(G)(P \otimes 1_G) = \mathcal{D} \otimes C_r^*(G).$$

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Because the same conclusion can be drawn for the element $z\delta(x)$, it follows that $\delta(x) \in \mathcal{M}(\mathcal{D} \otimes C_r^*(G))$.

1. It is immediate that $\delta_{\mathcal{D}}$ is a *-map. To see that $\delta_{\mathcal{D}}$ is an algebra map, let $x, y \in \mathcal{D}$. Then

$$\begin{aligned}\delta_{\mathcal{D}}(xy) &= W_S(xy \otimes 1_G)W_S^* = W_S(xPy \otimes 1_G)W_S^* \\ &= W_S(x \otimes 1_G)(P \otimes 1_G)(y \otimes 1_G)W_S^* \\ &= \underbrace{W_S(x \otimes 1_G)W_S^*}_{=\delta_{\mathcal{D}}(x)} \underbrace{W_S(y \otimes 1_G)W_S^*}_{=\delta_{\mathcal{D}}(y)}.\end{aligned}$$

2. Let $x \in \mathcal{D}$ such that $\delta(x) = 0$. Multiplying the latter equation by W_S^* from the left and by W_S from the right gives $PxP \otimes 1_G = x \otimes 1_G = 0$. Hence $x = 0$ as required.
3. Let $\{T_i\}$ be a bounded approximate identity for $\mathcal{K}(\mathfrak{H})$. Then $\{\delta(P(1_{\mathcal{B}} \otimes T_i)P)\}$ is a bounded approximate identity for $\mathcal{D} \otimes C_r^*(G)$, i. e., $\delta(P(1_{\mathcal{B}} \otimes T_i)P)$ converges to 1 in the strict topology of $\mathcal{M}(\mathcal{D} \otimes C_r^*(G))$. In particular, $\delta(\mathcal{D})(\mathcal{D} \otimes C_r^*(G))$ is dense in $\mathcal{D} \otimes C_r^*(G)$ which amounts to saying that $\delta_{\mathcal{D}}$ is nondegenerate. As a matter of fact, we even have equality by the Cohen factorization theorem (see, e. g., [18, Proposition 2.33]).
4. Since $\delta_{\mathcal{D}}$ is nondegenerate, $\delta_D \otimes \mathrm{id} : \mathcal{D} \otimes C_r^*(G) \to \mathcal{M}(\mathcal{D} \otimes C_r^*(G) \otimes C_r^*(G))$ uniquely extends to a *-homomorphisms with domain $\mathcal{M}(\mathcal{D} \otimes C_r^*(G))$ which, by uniqueness, must agree with the restriction–corestriction

$$\delta \otimes \mathrm{id} \restriction_{\mathcal{M}(\mathcal{D}\otimes C_r^*(G))}^{\mathcal{M}(\mathcal{D}\otimes C_r^*(G)\otimes C_r^*(G))}.$$

   This establishes the statement when combined with Lemma 3.2.3
5. By Lemma 3.2.1, it suffices to note that $\mathcal{D}$ is invariant under $\mathrm{Ad}[1_{\mathcal{A}} \otimes \mu_g]$, $g \in G$, and that $\mathcal{M}(\mathcal{D} \otimes C_r^*(G))$ is invariant under $\mathrm{Ad}[1_{\mathcal{A}} \otimes \mu_g \otimes r_g]$, $g \in G$. ∎

Remark 3.5 Much the same proof as above also works for the restriction of δ to $\mathcal{L}(\mathfrak{H}_P)$ in domain and codomain, i. e., the map $\mathcal{L}(\mathfrak{H}_P) \to \mathcal{L}(\mathfrak{H}_P \otimes L^2(G))$, $x \mapsto \delta(x)$.

Corollary 3.6 *$\delta_{\mathcal{D}}$ is a coaction of G on $\mathcal{D}$ that is equivariant w.r.t.* $\mathrm{Ad}[1_{\mathcal{A}} \otimes \mu_g]$, *$g \in G$, on $\mathcal{D}$ and* $\mathrm{Ad}[1_{\mathcal{A}} \otimes \mu_g \otimes r_g]$*, $g \in G$, on $\mathcal{M}(\mathcal{D} \otimes C_r^*(G))$.*

The task is now to recover $(\mathcal{A}, G, \alpha)$ from $\delta_{\mathcal{D}}$. For this purpose, we recall that $\mathcal{M}(\mathcal{D})$ may be identified with $P\mathcal{M}(\mathcal{B} \otimes \mathcal{K}(\mathfrak{H}))P$ (see [3, Section II.7.3.14]) and consider $\delta_{\mathcal{D}}$'s fixed point algebra:

$$\mathrm{Fix}(\delta_{\mathcal{D}}) := \{x \in \mathcal{M}(\mathcal{D}) : \delta_{\mathcal{D}}(x) = x \otimes 1_G\} \subseteq \mathcal{L}(\mathfrak{H}_P).$$

Since $P \in \mathcal{M}(\mathcal{D})$ and $\delta_{\mathcal{D}}(P) = P \otimes 1_G$, we see that $\mathrm{Fix}(\delta_{\mathcal{D}})$ is a unital C*-subalgebra of $\mathcal{M}(\mathcal{D})$. Furthermore, $\mathrm{Fix}(\delta_{\mathcal{D}})$ is invariant under $\mathrm{Ad}[1_{\mathcal{A}} \otimes \mu_g]$, $g \in G$, as is easy to check. Unfortunately, the resulting action $G \to \mathrm{Aut}(\mathrm{Fix}(\delta_{\mathcal{D}}))$ is unlikely to be continuous for the norm topology on $\mathcal{M}(\mathcal{D})$. The best we can do is say that it is continuous for the strict topology. This inconvenience can be avoided by only taking into account those elements $x \in \operatorname{Fix}(\delta_{\mathcal{D}})$ for which the map $G \to \operatorname{Fix}(\delta_{\mathcal{D}})$, $g \mapsto \operatorname{Ad}[1_{\mathcal{A}} \otimes \mu_g](x)$ is norm continuous, i. e.,

$$\tilde{\mathcal{A}} := \{x \in \operatorname{Fix}(\delta_{\mathcal{D}}) : G \ni g \mapsto \operatorname{Ad}[1_{\mathcal{A}} \otimes \mu_g](x) \in \operatorname{Fix}(\delta_{\mathcal{D}}) \text{ norm continuous}\}.$$

Now, a straightforward verification shows that $\tilde{\mathcal{A}}$ is a unital C*-subalgebra of $\operatorname{Fix}(\delta_{\mathcal{D}})$ with unit $1_{\tilde{\mathcal{A}}} := P$ on which $\operatorname{Ad}[1_{\mathcal{A}} \otimes \mu_g]$, $g \in G$, acts strongly continuous. In particular, writing $\tilde{\alpha}_g$, $g \in G$, for the restriction of $\operatorname{Ad}[1_{\mathcal{A}} \otimes \mu_g]$, $g \in G$, to $\tilde{\mathcal{A}}$ in both domain and codomain, we conclude that $(\tilde{\mathcal{A}}, G, \tilde{\alpha})$ is a concrete C*-dynamical system on $\mathfrak{H}_P$.

Lemma 3.7 *For the element $\tilde{S} := S_{12}S_{13}(W_G)^*_{23}S^*_{12} \in \mathcal{L}(\mathfrak{H}_P \otimes L^2(G), \mathfrak{H}_P \otimes \mathfrak{H})$ the following assertions hold:*

1. *$\tilde{S}$ is an isometry satisfying $\delta \otimes \operatorname{id}(\tilde{S}) = \tilde{S}_{124}$.*
2. *$(u_g \otimes \mu_h \otimes \mu_k)\tilde{S} = \tilde{S}(u_g \otimes \mu_h \otimes r_h\lambda_k)$ for all $g, h, k \in G$.*
3. *$\tilde{S}\mathcal{D} \otimes \mathcal{K}(L^2(G)) \subseteq \mathcal{D} \otimes \mathcal{K}(L^2(G), \mathfrak{H})$ and $\mathcal{D} \otimes \mathcal{K}(\mathfrak{H})\tilde{S} \subseteq \mathcal{D} \otimes \mathcal{K}(L^2(G), \mathfrak{H})$, i.e., $\tilde{S} \in \mathcal{M}(\mathcal{D} \otimes \mathcal{K}(L^2(G) \oplus \mathfrak{H}))$ such that $(1_{\mathfrak{H}_P} \otimes p_{L^2(G)})\tilde{S} = 0 = \tilde{S}(1_{\mathfrak{H}_P} \otimes p_{\mathfrak{H}})$, where $p_{\mathfrak{H}}$ and $p_{L^2(G)}$ denote the canonical projections onto $\mathfrak{H}$ and $L^2(G)$, respectively (see the discussion below Lemma 2.3).*
4. *$\tilde{S}\mathcal{M}(\mathcal{D}) \otimes \mathcal{K}(L^2(G)) \subseteq \mathcal{M}(\mathcal{D}) \otimes \mathcal{K}(L^2(G), \mathfrak{H})$.*

Proof

1. We have $\tilde{S}^*\tilde{S} = S_{12}(W_G)_{23}S^*_{13}S^*_{12}S_{12}S_{13}(W_G)^*_{23}S^*_{12} = S_{12}S^*_{12} = 1_{\tilde{\mathcal{A}}} \otimes 1_G$, i. e., $\tilde{S}$ is an isometry. Furthermore, it is easily deduced that

$$\begin{aligned}\delta \otimes \operatorname{id}(\tilde{S}) &= \operatorname{Ad}[S_{12}(W_G)_{23}S^*_{12}](\tilde{S}_{124}) \\ &= S_{12}(W_G)_{23}S^*_{12}S_{12}S_{14}(W_G)^*_{24}S^*_{12}S_{12}(W_G)^*_{23}S^*_{12} \\ &= S_{12}S_{14}(W_G)_{23}(W_G)^*_{24}(W_G)^*_{23}S^*_{12} \\ &= S_{12}S_{14}(W_G)^*_{24}S^*_{12} = \tilde{S}_{124},\end{aligned}$$

where the second-to-last equality is due to the fact that $W_G \in L^\infty(G) \otimes \mathcal{L}(L^2(G))$.

2. Let $g, h, k \in G$. Then

$$\begin{aligned}(u_g \otimes \mu_h \otimes \mu_k)\tilde{S} &= (u_g \otimes \mu_h \otimes \mu_k)S_{12}S_{13}(W_G)^*_{23}S^*_{12} \\ &\overset{(2.7),(2.8)}{=} S_{12}S_{13}(u_g \otimes r_g\lambda_h \otimes r_g\lambda_k)(W_G)^*_{23}S^*_{12} \\ &\overset{(2.2),(2.3),(2.4)}{=} S_{12}S_{13}(W_G)^*_{23}(u_g \otimes r_g\lambda_h \otimes r_h\lambda_k)S^*_{12} \\ &\overset{(2.7),(2.8)}{=} S_{12}S_{13}(W_G)^*_{23}S^*_{12}(u_g \otimes \mu_h \otimes r_h\lambda_k) = \tilde{S}(u_g \otimes \mu_h \otimes r_h\lambda_k).\end{aligned}$$

3. Using successively the identity $\tilde{S} = S_{12}S_{13}S^*_{12}W^*_S$, the inclusion $\mathcal{D} \subseteq \mathcal{A} \otimes \mathcal{K}(\mathfrak{H})$, Lemma 3.1.3, and three times (2.6) yields

$$\begin{aligned}\tilde{S}\mathcal{D} \otimes \mathcal{K}(L^2(G)) &= S_{12}S_{13}S^*_{12}W_S\mathcal{D} \otimes \mathcal{K}(L^2(G)) \\ &\subseteq S_{12}S_{13}S^*_{12}\mathcal{A} \otimes \mathcal{K}(\mathfrak{H}) \otimes \mathcal{K}(L^2(G)) \\ &\subseteq \mathcal{A} \otimes \mathcal{K}(\mathfrak{H}) \otimes \mathcal{K}(L^2(G), \mathfrak{H}).\end{aligned}$$

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As $(u_g \otimes 1_{\mathfrak{H}} \otimes 1_{\mathfrak{H}})\tilde{S}x = \tilde{S}x(u_g \otimes 1_{\mathfrak{H}} \otimes 1_G)$ for all $g \in G$ and $x \in \mathcal{D} \otimes \mathcal{K}(L^2(G))$, which is due to the second part of the lemma and the fact that $\mathcal{D}$ is fixed under $\mathrm{Ad}[u_g \otimes 1_{\mathfrak{H}}]$, $g \in G$, we further conclude that

$$\tilde{S}\mathcal{D} \otimes \mathcal{K}(L^2(G)) \subseteq \mathcal{B} \otimes \mathcal{K}(\mathfrak{H}) \otimes \mathcal{K}(L^2(G), \mathfrak{H}),$$

and hence that

$$\begin{aligned}\tilde{S}\mathcal{D} \otimes \mathcal{K}(L^2(G)) &= (P)_{12}\tilde{S}\mathcal{D} \otimes \mathcal{K}(L^2(G))(P)_{12} \\ &\subseteq (P)_{12}\mathcal{B} \otimes \mathcal{K}(\mathfrak{H}) \otimes \mathcal{K}(L^2(G), \mathfrak{H})(P)_{12} = \mathcal{D} \otimes \mathcal{K}(L^2(G), \mathfrak{H}).\end{aligned}$$

In the same manner, we can see that $\mathcal{D} \otimes \mathcal{K}(\mathfrak{H})\tilde{S} \subseteq \mathcal{D} \otimes \mathcal{K}(L^2(G), \mathfrak{H})$. From this it follows that $\tilde{S} \in \mathcal{M}(\mathcal{D} \otimes \mathcal{K}(L^2(G) \oplus \mathfrak{H}))$ such that $(1_{\mathfrak{H}_P} \otimes p_{L^2(G)})S = 0 = S(1_{\mathfrak{H}_P} \otimes p_{\mathfrak{H}})$ as claimed.

4. This is clear from Lemma 2.1, because

$$\tilde{S}\mathcal{M}(D) \otimes \mathcal{K}(L^2(G)) \subseteq \mathcal{L}(\mathfrak{H}_P) \otimes \mathcal{K}(L^2(G) \oplus \mathfrak{H})$$

and

$$\begin{aligned}\tilde{S}\mathcal{M}(D) \otimes \mathcal{K}(L^2(G)) &\subseteq \mathcal{M}(\mathcal{D} \otimes \mathcal{K}(L^2(G) \oplus \mathfrak{H}))\mathcal{M}(\mathcal{D} \otimes \mathcal{K}(L^2(G) \oplus \mathfrak{H})) \\ &\subseteq \mathcal{M}(\mathcal{D} \otimes \mathcal{K}(L^2(G) \oplus \mathfrak{H}))\end{aligned}$$

by the third part of the lemma. ∎

Corollary 3.8 *The C^*-dynamical system $(\tilde{A}, G, \tilde{\alpha})$ is free.*

Proof Our strategy of proof is to apply Lemma 2.3. Due to the first two statements of Lemma 3.7, we can already assert that $\tilde{S} \in \mathcal{L}(\mathfrak{H}_P \otimes L^2(G), \mathfrak{H}_P \otimes \mathfrak{H})$ is an isometry satisfying (2.7) and (2.8). What is left is to establish that

$$\tilde{S}\tilde{A} \otimes \mathcal{K}(L^2(G)) \subseteq \tilde{A} \otimes \mathcal{K}(L^2(G), \mathfrak{H}).$$

For this purpose, let $x \in \tilde{A} \otimes \mathcal{K}(L^2(G))$. By Lemma 3.7, we have $\tilde{S}x \in \mathcal{M}(\mathcal{D}) \otimes \mathcal{K}(L^2(G), \mathfrak{H})$, and hence we may apply the map $\delta_{\mathcal{D}} \otimes \mathrm{id}$ which gives

$$\delta_{\mathcal{D}} \otimes \mathrm{id}(\tilde{S}x) = \delta_{\mathcal{D}} \otimes \mathrm{id}(\tilde{S})\delta_{\mathcal{D}} \otimes \mathrm{id}(x) = \tilde{S}_{124}x_{124} = (\tilde{S}x)_{124}$$

when combined with Lemma 3.7.1 That is, $\tilde{S}x \in \mathrm{Fix}(\delta_{\mathcal{D}}) \otimes \mathcal{K}(L^2(G), \mathfrak{H})$. To see that, in fact, $\tilde{S}x \in \tilde{A} \otimes \mathcal{K}(L^2(G), \mathfrak{H})$, we prove that the map $G \ni g \mapsto \mathrm{Ad}[1_{\mathcal{A}} \otimes \mu_g] \otimes \mathrm{id}(\tilde{S}x) \in \mathrm{Fix}(\delta_{\mathcal{D}}) \otimes \mathcal{K}(L^2(G), \mathfrak{H})$ is norm continuous. For the latter assertion, we use the identity

$$\mathrm{Ad}[1_{\mathcal{A}} \otimes \mu_g] \otimes \mathrm{id}(\tilde{S}x) = \tilde{S}(1_{\mathcal{A}} \otimes 1_{\mathfrak{H}} \otimes r_g)\,\mathrm{Ad}[1_{\mathcal{A}} \otimes \mu_g \otimes 1_G](x), \qquad g \in G,$$

which is a consequence of Lemma 3.7.2, together with the immediate norm continuity of the maps $G \ni g \mapsto \mathrm{Ad}[1_{\mathcal{A}} \otimes \mu_g \otimes 1_G](x)$ and $G \ni g \mapsto (1_{\mathcal{A}} \otimes 1_{\mathfrak{H}} \otimes r_g)y$, $y \in \mathcal{L}(\mathfrak{H}_P) \otimes \mathcal{K}(L^2(G))$. ∎

The fixed point algebra of $(\tilde{A}, G, \tilde{\alpha})$ is characterized by our next result.

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Lemma 3.9 *The fixed point algebra of the* C^**-dynamical system* $(\tilde{\mathcal{A}}, G, \tilde{\alpha})$ *is the image of the faithful unital* **-homomorphism* $\gamma : \mathcal{B} \to \tilde{\mathcal{A}}$, $\gamma(b) := S(b \otimes 1_G)S^*$.

Proof We first establish that γ is well defined. To do this, let $b \in \mathcal{B}$ and let $x \in \mathcal{D}$. By (2.6) and (2.9), we have $\gamma(b)x \in \mathcal{A} \otimes \mathcal{K}(\mathfrak{H})$. Combining (2.7) with the fact that $\mathcal{D}$ is fixed under $\mathrm{Ad}[u_g \otimes 1_{\mathfrak{H}}]$, $g \in G$, further yields $\gamma(b)x \in \mathcal{B} \otimes \mathcal{K}(\mathfrak{H})$. In consequence, $\gamma(b)x \in \mathcal{D}$, because $\gamma(b)x = S^*S\gamma(b)xS^*S$. In the same way we see that $x\gamma(b) \in \mathcal{D}$. Hence $\gamma(b) \in \mathcal{M}(\mathcal{D})$. Since

$$\begin{aligned}\delta_{\mathcal{D}}(\gamma(b)) &= \mathrm{Ad}[S_{12}(W_G)_{23}S_{12}^*](S_{12}(b \otimes 1_G \otimes 1_G)S_{12}^*)\\ &= \mathrm{Ad}[S_{12}(W_G)_{23}S_{12}^*S_{12}](b \otimes 1_G \otimes 1_G)\\ &= \mathrm{Ad}[S_{12}](b \otimes 1_G \otimes 1_G) = \gamma(b) \otimes 1_G,\end{aligned}$$

i.e., $\gamma(b) \in \mathrm{Fix}(\delta_{\mathcal{D}})$, and $\mathrm{Ad}[1_{\mathcal{A}} \otimes \mu_g](\gamma(b)) = \gamma(b)$ for all $g \in G$, which is due to (2.8), it may finally be concluded that $\gamma(b) \in \tilde{\mathcal{A}}$. Note that we have actually proved more, namely that $\gamma(b)$ is fixed under $\tilde{\alpha}_g$, $g \in G$, which amounts to saying that $\gamma(\mathcal{B}) \subseteq \tilde{\mathcal{A}}^G$. We thus proceed to show that $\tilde{\mathcal{A}}^G \subseteq \gamma(\mathcal{B})$. For this, let $x \in \tilde{\mathcal{A}}^G$. We put $y := S^*xS$ and observe that $(W_G)_{23}(y \otimes 1_G) = (y \otimes 1_G)(W_G)_{23}$ or, equivalently, $\mathrm{id} \otimes \delta_G(y \otimes 1_G) = y \otimes 1_G$. As the coaction δ_G is ergodic, we can assert that $y = b \otimes 1_G$ for some $b \in \mathcal{B}$. It follows that $x = PxP = SyS^* = S(b \otimes 1_G)S^* = \gamma(b)$, i. e., $\tilde{\mathcal{A}}^G \subseteq \gamma(\mathcal{B})$ as required. That γ is a faithful unital *-homomorphism is clear, and so the proof is complete. ■

We proceed with a technical no-go result which is also of independent interest. For its proof we make use of the fact that each C*-dynamical system $(\mathcal{A}, G, \alpha)$ can be decomposed into its isotypic components, say $A(\sigma)$, $\sigma \in \mathrm{Irr}(G)$, which amounts to saying that their algebraic direct sum forms a dense *-subalgebra of $\mathcal{A}$ (see, e. g., [13, Theorem 4.22]).

Lemma 3.10 *Let* $(\mathcal{A}, G, \alpha)$ *be a free* C^**-dynamical system with fixed point algebra* $\mathcal{B}$. *Furthermore, let* $\mathcal{A}_0$ *be a G-invariant unital* C^**-subalgebra. If the induced* C^**-dynamical system* $(\mathcal{A}_0, G, \alpha)$ *is free and* $\mathcal{A}_0^G = \mathcal{B}$, *then* $\mathcal{A}_0 = \mathcal{A}$.

Proof We shall have established the lemma if we prove that the respective isotypic components are equal, i. e., $A_0(\sigma) = A(\sigma)$ for all $\sigma \in \mathrm{Irr}(G)$. To prove the latter assertion, we fix $\sigma \in \mathrm{Irr}(G)$ and recall that the freeness of $(\mathcal{A}, G, \alpha)$ implies that $A(\sigma)$ is a Morita equivalence bimodule between the unital C*-algebras $\mathcal{B} \otimes \mathcal{L}(V_\sigma)$ and

$$\mathcal{C} := \{x \in \mathcal{A} \otimes \mathcal{L}(V_\sigma) : (\forall g \in G)\,(1_{\mathcal{A}} \otimes \sigma_g)\alpha_g(x) = x(1_{\mathcal{A}} \otimes \sigma_g)\}$$

(see [20, Section 3]). Likewise, the freeness of $(\mathcal{A}_0, G, \alpha)$ implies that $A_0(\sigma)$ is a Morita equivalence bimodule between the unital C*-algebras $\mathcal{B} \otimes \mathcal{L}(V_\sigma)$ and

$$\mathcal{C}_0 := \{x \in \mathcal{A}_0 \otimes \mathcal{L}(V_\sigma) : (\forall g \in G)\,(1_{\mathcal{A}} \otimes \sigma_g)x = x(1_{\mathcal{A}} \otimes \sigma_g)\}.$$

This makes it possible to apply [22, Lemma A.1] which gives elements $s_1, \dots, s_n \in A_0(\sigma)$ such that $\sum_{k=1}^n \langle s_k, s_k \rangle_{\mathcal{C}_0} = 1_{\mathcal{C}}$. Since the inner products of $A_0(\sigma)$ are exactly the restrictions in domain and codomain of the respective inner products of $A(\sigma)$,

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we conclude that $\langle s_k, s_k \rangle_{\mathcal{C}_0} = \langle s_k, s_k \rangle_{\mathcal{C}}$ for all $1 \le k \le n$. Hence for each $x \in A(\sigma)$ we have

$$x = x \,.\, 1_{\mathcal{C}} = \sum_{k=1}^{n} x \,.\, \langle s_k, s_k \rangle_{\mathcal{C}_0} = \sum_{k=1}^{n} x \,.\, \langle s_k, s_k \rangle_{\mathcal{C}} = \sum_{k=1}^{n} {}_{\mathcal{B} \otimes \mathcal{L}(V_\sigma)} \langle x, s_k \rangle \,.\, s_k \in A_0(\sigma),$$

i. e., $A(\sigma) \subseteq A_0(\sigma)$. This completes the proof, as the other inclusion is obvious. ∎

Now, we are almost in a position to state and prove the main result of this section. The only preparatory point remaining concerns the following map:

$$j_\alpha : \mathcal{A} \to C(G, \mathcal{A}) = \mathcal{A} \otimes C(G) \subseteq \mathcal{L}(\mathfrak{H}_{\mathcal{A}} \otimes L^2(G)),$$
$$j_\alpha(x)(g) = \alpha_{g^{-1}}(x) = u_g^* x u_g.$$

It is clear that j_α is injective. Moreover, straightforward computations reveal that

$$\mathrm{Ad}[u_g \otimes r_g] \circ j_\alpha = j_\alpha \qquad \forall g \in G, \tag{3.3}$$

$$\mathrm{Ad}[1_{\mathcal{A}} \otimes \lambda_g] \circ j_\alpha = j_\alpha \circ \alpha_g \qquad \forall g \in G. \tag{3.4}$$

Theorem 3.11 *The map*

$$\pi_S : \mathcal{A} \to \tilde{\mathcal{A}} \subseteq \mathcal{M}(\mathcal{D}) \subseteq \mathcal{L}(\mathfrak{H}_P), \qquad \pi_S(x) := S j_\alpha(x) S^* \tag{3.5}$$

is a G-equivariant *-isomorphism.

Proof We start again by proving that the map under consideration is well defined. For this purpose, let $x \in \mathcal{A}$. The verification of $\pi_S(x) \in \mathcal{M}(\mathcal{D})$ can be handled in much the same way as in the proof of Lemma 3.9, the only difference being that (2.7) needs to be combined with (3.3) to establish that $\pi_S(x)d, d\pi_S(x) \in \mathcal{B} \otimes \mathcal{K}(\mathfrak{H})$ for all $d \in \mathcal{D}$. Moreover, the identity $(W_G)_{23} j_\alpha(x)_{12} (W_G)_{23}^* = j_\alpha(x)_{12}$, which is easy to check, implies that

$$\begin{aligned} \delta_{\mathcal{D}}(\pi_S(x)) &= \mathrm{Ad}[S_{12}(W_G)_{23} S_{12}^*](S_{12} j_\alpha(x)_{12} S_{12}^*) \\ &= \mathrm{Ad}[S_{12}]((W_G)_{23} j_\alpha(x)_{12} (W_G)_{23}^*) \\ &= \mathrm{Ad}[S_{12}](j_\alpha(x)_{12}) = \pi_S(x)_{12}, \end{aligned}$$

i. e., $\pi_S(x) \in \mathrm{Fix}(\delta_{\mathcal{D}})$. That, in fact, $\pi_S(x) \in \tilde{\mathcal{A}}$ as claimed now follows from

$$\tilde{\alpha}_g(\pi_S(x)) \overset{(2.8)}{=} S(\mathrm{Ad}[1_{\mathcal{A}} \otimes \lambda_g](j_\alpha(x))S^* \overset{(3.4)}{=} \pi_S(\alpha_g(x)), \qquad g \in G. \tag{3.6}$$

Of course, (3.6) also establishes that π_S is G-equivariant, and so it remains to show that π_S is a *-isomorphism. Clearly, π_S is a *-homomorphism. Moreover, it is injective, because j_α is injective and S is an isometry. With this, we can assert that the induced C*-dynamical system $(\pi_S(\mathcal{A}), G, \tilde{\alpha})$ is free with fixed point algebra $\pi_S(\mathcal{B}) = \gamma(\mathcal{B}) = \tilde{\mathcal{A}}^G$ (see Lemma 3.9 for the latter equalities). Hence Lemma 3.10 implies that π_S is surjective, i. e., $\pi_S(\mathcal{A}) = \tilde{\mathcal{A}}$, and therefore the proof is complete. ∎

Remark 3.12 An alternative strategy for proofing Theorem 3.11 would be to establish that the isometries S and $\tilde{S}$ are conjugated in the sense of [22, Theorem 4.4]. With some technical effort this can be done.

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Corollary 3.13 (cf. [23, Corollary 4.2]) *The pair $(\pi_S, \mathrm{Ad}[1_{\mathcal{A}} \otimes \mu])$ is a faithful generalized covariant representation of $(\mathcal{A}, G, \alpha)$ on $\mathcal{M}(\mathcal{D}) = P\mathcal{M}(\mathcal{B} \otimes \mathcal{K}(\mathfrak{H}))P$. Moreover, any faithful *-representation $\pi_{\mathcal{B}} : \mathcal{B} \to \mathcal{L}(\mathfrak{H}_{\mathcal{B}})$ gives rise to an honest faithful covariant representation on $P(\mathfrak{H}_{\mathcal{B}} \otimes \mathfrak{H})$.*

Corollary 3.14 (cf. [27, Theorem 10]) *Let $(\mathcal{A}, G, \alpha)$ be a cleft C^*-dynamical system (see Section 2.4). Then $(\mathcal{A}, G, \alpha)$ can be realized as the invariants of an equivariant coaction of G on $\mathcal{B} \otimes \mathcal{K}(L^2(G))$. Moreover, any faithful *-representation $\pi_{\mathcal{B}} : \mathcal{B} \to \mathcal{L}(\mathfrak{H}_{\mathcal{B}})$ gives rise to an honest faithful covariant representation on $\mathfrak{H}_{\mathcal{B}} \otimes L^2(G)$.*

3 Outlook

As pointed out by the referee, a reformulation of our main result in terms of groups and their quantum duals could provide a pathway for generalizing our findings within the framework of quantum groups, potentially offering valuable insights in relation to Wassermann's work. We leave this extension as a direction for future research.

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