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# ASYMPTOTIC BEHAVIOUR OF DISCONJUGATE $n$TH ORDER DIFFERENTIAL EQUATIONS

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**1. Introduction.** An ordered set $(u_1, \ldots, u_n)$ of positive $C^n(a, b)$-solutions of the linear differential equation

$$(1.1) \qquad Lu = u^{(n)} + p_1(t)u^{(n-1)} + \ldots + p_n(t)u = 0$$

will be called a *fundamental principal system* on $[a, b)$ provided that

$$(1.2) \qquad \lim_{t \to b^-} \frac{u_k(t)}{u_{k+1}(t)} = 0, \qquad k = 1, \ldots, n-1,$$

and

$$(1.3) \quad u_k^{(k-1)}(a^+) = 1, \quad u_k^{(m)}(a^+) = 0, \quad m = 0, \ldots, k-2, \quad k = 1, \ldots, n.$$

A system $(u_1, \ldots, u_n)$ satisfying just (1.2) will be called a *principal system on* $[a, b)$. In any principal system $(u_1, \ldots, u_n)$, the solution $u_1$ will be called a *minimal solution.*

Clearly, if there exists a fundamental principal system for (1.1), then it is unique. This follows from the fact that principal systems are linearly independent sets.

Equation (1.1) is said to be *disconjugate* on an interval $I$ if no non-trivial $C^n(I)$-solution (hereafter, simply called "solution") has more than $n-1$ zeros, each zero being counted in accordance with its multiplicity, in $I$. Finally, (1.1) is said to be *normal* on $I$ if $p_k \in C(I)$, $k = 1, \ldots, n$.

Hartman [**9**] has recently shown that (1.1) has a principal system on $(a, b)$ $(a > -\infty)$, provided (1.1) is a normal disconjugate equation on $[a_0, b)$ for some $a_0$ such that $a_0 < a$. For any given set of functions $\xi_1, \ldots, \xi_n$, define

$$(1.4) \qquad \begin{aligned} I(t, s; \xi_1) &= \int_s^t \xi_1(\tau)\, d\tau, \\ I(t, s; \xi_1, \ldots, \xi_k) &= \int_s^t \xi_1(\tau) I(\tau, s; \xi_2, \ldots, \xi_k)\, d\tau, \qquad k = 2, \ldots, n. \end{aligned}$$

In § 2, we will prove the following results.

THEOREM 1.1. *Assume that* $-\infty < a_0 < a < b \leqq \infty$ *and that* $Lu = 0$ *is a normal disconjugate equation on* $[a_0, b)$. *Then there exists* $\xi_k$, $k = 1, \ldots, n$, *such that the following hold:*

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(i) $\xi_k \in C^{n-k+1}(a_0, b)$, $\xi_k > 0$, $\xi_k(a) = 1, k = 1, \ldots, n$;

(ii) $\int_a^b \xi_k(s)\,ds = \infty, k = 2, \ldots, n$

(iii) *The fundamental principal system* $(u_1, \ldots, u_n)$ *on* $[a, b)$ *of* $Lu = 0$ *exists and*

$$u_1(t) = \xi_1(t), u_k(t) = \xi_1(t)I(t, a; \xi_2, \ldots, \xi_k), \qquad k = 2, \ldots, n; \tag{1.5}$$

(iv) *The Cauchy function* $g(t, s)$ *for the initial-value problem at* $t = a$ *for* $Lu = 0$ *satisfies*

$$\xi_1^{-1}(t)\, g(t, s) \prod_{k=1}^{n} \xi_k(s) = I(t, s; \xi_2, \ldots, \xi_n) = (-1)^{n-1} I(s, t; \xi_n, \ldots, \xi_2).$$

THEOREM 1.2. *If* $Lu = 0$ *is a normal disconjugate equation on* $[a_0, b)$ *and if its formal adjoint equation* $L^*v = 0$ *is normal on* $[a_0, b)$, *then* $L^*v = 0$ *is disconjugate on* $[a_0, b)$, *and its fundamental principal system on* $[a, b)$ $(a > a_0)$ *is* $(v_n, \ldots, v_1)$, *where*

$$v_n(t) = \left(\prod_{k=1}^{n} \xi_k(t)\right)^{-1}, \tag{1.6}$$
$$v_k(t) = v_n(t) I(t, a; \xi_n, \ldots, \xi_{k+1}), \qquad k = 1, \ldots, n-1$$

*and* $\xi_k$ *is as in Theorem* 1.1. *Furthermore,*

$$g(t, s) = u_n(t)v_n(s) - \ldots + (-1)^{n-1}u_1(t)v_1(s). \tag{1.7}$$

Hartman [**9**, pp. 329–331] showed in the general disconjugate case the existence of a minimal solution as a rather complicated limit of a sequence of other solutions. Our development in this regard is simpler and along the lines of the original proof of Morse and Leighton [**12**] for the case $n = 2$. We rely heavily upon the classic results of Pólya [**13**], which are stated at the beginning of § 2.

The most interesting aspects of Theorems 1.1 and 1.2 are the possibilities that the representation (1.7) allows for the general development of an asymptotic theory for perturbed equations of the form

$$Ly = f(t, y, y', \ldots, y^{(n-1)}). \tag{1.8}$$

We carry out such a development in §§ 3, 4.

The equivalence of (1.8) to the integral equation

$$y = u(t) + \int_a^t g(t, s) f(s, y, \ldots, y^{(n-1)})\, ds, \tag{1.9}$$

where $Lu = 0$, is well known. The general asymptotic theory, which can be derived directly from (1.9), has been worked out in detail by Trench [**17**], Locke [**11**], and Katz [**10**]. Although quite adequate for equations where $Lu = 0$ is oscillatory on $[a, b)$, the resulting theory is inadequate for equations where $Lu = 0$ is disconjugate on $[a, b)$. This is clearly illustrated by the known results [**1–4**; **6**; **20**] for the cases when $L$ is either a constant coefficient operator or an Euler operator, $f$ is linear, and $b = \infty$. For example, the equation

$$y'' - y = f(t)y \tag{1.10}$$

has solutions

$$y_1 = e^t[1 + o(1)], \qquad y_2 = e^{-t}[1 + o(1)], \qquad \text{as } t \to \infty,$$

provided

$$\int^\infty |f(t)|\, dt < \infty. \tag{1.11}$$

However, the specialization to (1.10) of the general results obtained in [**17**; **11**; or **10**] requires that $\int^\infty e^{2t}|f(t)|\, dt < \infty$ to make the same conclusion. Our asymptotic results in § 4 require only (1.11) as the "smallness condition".

We obtain linearly independent solutions $y_j$ of (1.8) as solutions of an operator equation of the form

$$y = u_j + T_j y, \tag{1.12}$$

where $T_j$ is an integral operator with the property that

$$T_j u_j = o(u_j), \qquad \text{as } t \to b^-.$$

Equation (1.12) is a modification of (1.9) that utilizes the known relative behaviour at $a$ and $b$ of the fundamental systems $(u_1, \ldots, u_n)$ and $(v_n, \ldots, v_1)$. The only assumption on $L$ is that $Lu = 0$ be disconjugate. Previous results of a similar generality have always assumed that $L = D^n$ or $n = 2$. For a more precise comparison of the asymptotic results in §§ 3 and 4 with previous results, see § 4.

**2. The disconjugate $n$th order linear equation.** In this section, we will consider equation (1.1), $Lu = 0$, on intervals $(a_0, b)$ or $[a, b)$, where $b$ may be finite or infinite. For two functions $f$ and $g$ defined on $[a, b)$, we will write $f = o(g)$, if $g(t) \neq 0$ for $t < b$ in some neighbourhood of $b$ and

$$\lim_{t \to b^-} \frac{f(t)}{g(t)} = 0.$$

Let

$$W_k(u_1, \ldots, u_k) = \det \begin{pmatrix} u_1 & \cdots & u_k \\ u_1' & \cdots & u_k' \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ u_1^{(k-1)} & \cdots & u_k^{(k-1)} \end{pmatrix}$$

be the Wronskian determinant of the $k$ functions $u_1, \ldots, u_k$. Essential to our proof of the existence of principal systems of $Lu = 0$ are the following fundamental results of Pólya [**13**].

LEMMA 2.1. (i) *If* $v \neq 0$, *then*

$$W_k(vu_1, \ldots, vu_k) = v^k W_k(u_1, \ldots, u_k). \tag{2.1}$$

(ii) *If* $u_1 \neq 0$, *then*

$$W_k(u_1, \ldots, u_k) = u_1^k W_{k-1}(v_1, \ldots, v_{k-1}), \tag{2.2}$$

*where*

$$v_j = \left(\frac{u_{j+1}}{u_1}\right)', \qquad j = 1, \ldots, k-1.$$

THEOREM 2.1. *Assume that* $Lu = 0$ *is normal on* $[a, b)$. *Then* $Lu = 0$ *is disconjugate on* $[a, b)$, *if and only if there exist solutions* $u_1, \ldots, u_n$ *of* $Lu = 0$ *such that*

$$W_k(u_1, \ldots, u_k) > 0 \quad on\ (a, b), \qquad k = 1, \ldots, n. \tag{2.3}$$

Actually, Theorem 2.1 is an improvement of Pólya's original result. We have utilized the recent result of Sherman [**15**; **16**], which is that a normal linear equation on a half-open interval $I$ is disconjugate on $I$ if and only if it is disconjugate on the interior of $I$, to obtain Theorem 2.1.

In what follows, an ordered set $(u_1, \ldots, u_n)$ of solutions will be called a *Pólya system* on the interval $(a, b)$ if (2.3) holds.

THEOREM 2.2. *If* $Lu = 0$ *is normal on* $(a, b)$ *and if there exists a Pólya system* $(u_1, \ldots, u_n)$ *on* $(a, b)$ *for* $Lu = 0$, *then there exists a minimal solution* $\xi$ *and solutions* $w_2, \ldots, w_n$ *such that* $(\xi, w_2, \ldots, w_n)$ *is also a Pólya system on* $(a, b)$ *for* $Lu = 0$.

*Proof.* The proof is by induction on the order $n$ of the operator $L$. Clearly the theorem is true for all first-order equations. Suppose that the theorem is true for all $(n-1)$st order equations having Pólya systems of solutions. Let $Lu = 0$ be any normal $n$th order equation with a Pólya system $(u_1, \ldots, u_n)$. Then, $u_1 > 0$ on $(a, b)$. Let $u = u_1(t)z$ in $Lu = 0$. Then, there exists a normal $(n-1)$st order linear operator $M$ on $(a, b)$ such that

$$0 = L[u_1(t)z] = u_1(t)Mz' \qquad (z' \equiv dz/dt). \tag{2.4}$$

Let

$$v_k = \left(\frac{u_{k+1}}{u_1}\right)', \qquad k = 1, \ldots, n-1.$$

Then, (2.4) implies that $Mv_k = 0$. Furthermore, Lemma 2.1 implies that

$$W_k(v_1, \ldots, v_k) = u_1^{-k-1} W_{k+1}(u_1, \ldots, u_{k+1}) > 0 \text{ on } (a, b), \quad k = 1, \ldots, n-1.$$

Thus, $(v_1, \ldots, v_{n-1})$ is a Pólya system on $(a, b)$ for $Mv = 0$. The induction hypothesis now implies that $Mv = 0$ has a minimal solution $\zeta \in C^{n-1}(a, b)$ and solutions $\zeta_2, \ldots, \zeta_{n-1}$ such that $(\zeta, \zeta_2, \ldots, \zeta_{n-1})$ is a Pólya system on $(a, b)$.

At this point, the proof separates into two cases. First, suppose that

$$\int^{b^-} \zeta(t)\, dt = \infty. \tag{2.5}$$

In this case, we will show that $\xi = u_1$ is a minimal solution of $Lu = 0$. Clearly, $\xi$ is positive on $(a, b)$. Suppose that $L\varphi = 0$ and that $\varphi$ is linearly independent of $\xi$. Then $\lambda = (\varphi/\xi)'$ is a non-trivial solution of $Mv = 0$. If $\lambda$ and $\zeta$ are linearly independent in $C^{n-1}(a, b)$, then $\zeta = o(\lambda)$ because $\zeta$ is a minimal solution. Thus $\lambda(t)$ must eventually be of one sign as $t \to b^-$, and

$$\left| \int^{b^-} \lambda(t)\, dt \right| = \infty, \tag{2.6}$$

because of (2.5). On the other hand, if $\lambda$ and $\zeta$ are linearly dependent functions in $C^{n-1}(a, b)$, then $\lambda = c\zeta$ for some non-zero constant $c$ and (2.5) again implies (2.6). Therefore, in any case,

$$\left| \frac{\xi(t)}{\varphi(t)} \right| = \frac{1}{\left| \beta + \int_\alpha^t \lambda(s)\, ds \right|} \leqq \frac{1}{\left| \int_\alpha^t \lambda(s)\, ds \right| - |\beta|} \to 0 \quad \text{as } t \to b^-,$$

where $a < \alpha < b$ and $\beta = \varphi(\alpha)\xi^{-1}(\alpha)$. Thus, $\xi$ is a minimal solution, and furthermore, $(\xi, u_2, \ldots, u_n) = (u_1, u_2, \ldots, u_n)$ is a Pólya system on $(a, b)$ in this case.

Next, suppose that (2.5) does not hold. In this case, we will show that

$$\xi(t) = u_1(t) \int_t^b \zeta(s)\, ds$$

is a minimal solution of $Lu = 0$. Clearly $\xi$ is a positive solution of $Lu = 0$ on $(a, b)$. Let $\varphi$ be a solution of $Lu = 0$ linearly independent of $\xi$. Let $\lambda = (\varphi/u_1)'$. Then, $M\lambda = 0$. Let $a < \alpha < b$ and $\beta = \varphi(\alpha)u_1^{-1}(\alpha)$. Then,

$$\varphi(t) = u_1(t)\left[\beta + \int_\alpha^t \lambda(s)\, ds\right], \qquad \alpha < t < b. \tag{2.7}$$

At the beginning of this proof, we showed that $(v_1, \ldots, v_{n-1})$ was a Pólya system on $(a, b)$ for $Mv = 0$. Hence, Theorem 2.1 implies that $Mv = 0$ is disconjugate on $[\alpha, b)$. Thus, $\lambda(t)$ is eventually of one sign, as $t \to b^-$, and $\int_\alpha^b \lambda(s)\, ds$ exists in the extended real numbers. If

$$\int_\alpha^b \lambda(s)\, ds \neq -\beta,$$

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then

$$\left|\frac{\xi(t)}{\varphi(t)}\right| = \frac{\int_t^b \zeta(s)\,ds}{\left|\beta + \int_\alpha^t \lambda(s)\,ds\right|} \to 0, \qquad \text{as } t \to b^-.$$

Suppose that $\int_\alpha^b \lambda(s)\,ds = -\beta$. Then, (2.7) implies that

$$\varphi(t) = -u_1(t) \int_t^b \lambda(s)\,ds.$$

Now, if $\zeta$ and $\lambda$ are linearly dependent functions in $C^{n-1}(a, b)$, then $\lambda = c\zeta$ for some non-zero constant $c$. Hence, $\varphi = -c\xi$, which contradicts that $\varphi$ and $\xi$ are linearly independent. Thus, $\zeta$ and $\lambda$ must be linearly independent. Since $\zeta$ is a minimal solution, $\zeta = o(\lambda)$. Thus, by L'Hôpital's Rule,

$$\lim_{t\to b^-} \frac{\xi(t)}{\varphi(t)} = \lim_{t\to b^-} \frac{\int_t^b \zeta(s)\,ds}{-\int_t^b \lambda(s)\,ds} = \lim_{t\to b^-} \frac{-\zeta(t)}{\lambda(t)} = 0.$$

This completes the proof that $\xi$ is a minimal solution in this case.

To complete the induction proof, we still need to show the existence of a Pólya system $(\xi, w_2, \ldots, w_n)$ in the second case considered above. Recall that the induction hypothesis implied the existence of a Pólya system $(\zeta, \zeta_2, \ldots, \zeta_{n-1})$ for $Mv = 0$. Let

$$w_k(t) = u_1(t) \int_\alpha^t \zeta_{k-1}(s)\,ds \qquad (k = 3, \ldots, n; a < \alpha < b).$$

Then, $Lw_k = M\zeta_{k-1} = 0$. Lemma 2.1 implies that

$$\begin{aligned} W_k(\xi, u_1, w_3, \ldots, w_k) &= -W_k(u_1, \xi, w_3, \ldots, w_k) \\ &= -u_1^k W_{k-1}(-\zeta, \zeta_2, \ldots, \zeta_{k-1}) \\ &= u_1^k W_{k-1}(\zeta, \zeta_2, \ldots, \zeta_{k-1}) > 0, \qquad k = 3, \ldots, n. \end{aligned}$$

Since $W_2(\xi, u_1) = \zeta u_1^2 > 0$, it is clear that $(\xi, u_1, w_3, \ldots, w_n)$ is a Pólya system for $Lu = 0$ in this case.

To prove Theorems 1.1 and 1.2 and to establish the development in subsequent sections, we will need the identities contained in the following two lemmas. These lemmas can be proved by induction, and their proofs have been omitted. In what follows, we use the abbreviation

$$I(t; \xi_1, \ldots, \xi_k) \equiv I(t, a; \xi_1, \ldots, \xi_k).$$

LEMMA 2.2. *Assume that* $\zeta_j \in C[a, b)$, $j = 1, \ldots, k$. *Let*

$$J_k(t, s) = \sum_{j=0}^{k} (-1)^j I(t; \zeta_k, \ldots, \zeta_{k-j+1}) I(s; \zeta_1, \ldots, \zeta_{k-j}), \qquad t, s \in [a, b),$$

*where* $I(t; \zeta_k, \zeta_{k+1}) \equiv 1 \equiv I(t; \zeta_1, \zeta_0)$ *by definition. Then*

$$J_k(t, s) = I(s, t; \zeta_1, \ldots, \zeta_k) = (-1)^k I(t, s; \zeta_k, \ldots, \zeta_1). \tag{2.8}$$

LEMMA 2.3. *Assume that* $\zeta_j \in C^{k-j-1}[a, b)$, $j = 1, \ldots, k-1$, $\zeta_k \in C[a, b)$. *Let*

$$I_j(t) = I(t; \zeta_1, \ldots, \zeta_j), \qquad j = 1, \ldots, k.$$

*Then*

$$W_k(I_1, \ldots, I_k) = I(t; \zeta_k, \ldots, \zeta_1) \prod_{j=1}^{k} [\zeta_j(t)]^{k-j}. \tag{2.9}$$

*Proof of Theorem* 1.1. Since $Lu = 0$ is disconjugate on $[a_0, b)$, Theorem 2.1 implies the existence of a Pólya system on $(a_0, b)$ for $Lu = 0$. Hence, Theorem 2.2 implies the existence of a Pólya system $(\xi_1, w_2, \ldots, w_n)$ with $\xi_1$ a minimal solution. Without loss of generality, we can assume that $\xi_1(a) = 1$, since otherwise $\xi_1(t)$ can be replaced by $\xi_1(t)\xi_1^{-1}(a)$. Let $u_1(t) = \xi_1(t)$.

Let $u = \xi_1(t)z$ in $Lu = 0$. Then, there exists an $(n-1)$st order normal operator $M_1$ on $(a_0, b)$ such that

$$0 = Lu = \xi_1(t)M_1z'.$$

Let

$$v_k = \left(\frac{w_{k+1}}{\xi_1}\right)', \qquad k = 1, \ldots, n-1.$$

Lemma 2.1 implies that

$$W_k(v_1, \ldots, v_k) = \xi_1^{-k-1} W_{k+1}(\xi_1, w_2, \ldots, w_{k+1}) > 0, \qquad k = 1, \ldots, n-1.$$

Hence, $(v_1, \ldots, v_{n-1})$ is a Pólya system on $(a_0, b)$ for $M_1 v = 0$. Thus, Theorem 2.2 implies the existence of a minimal solution $\xi_2 \in C^{n-1}(a_0, b)$ and a Pólya system $(\xi_2, \bar{w}_3, \ldots, \bar{w}_n)$ of $M_1 v = 0$. Let $\xi_2(a) = 1$. Let

$$u_2(t) = \xi_1(t) \int_a^t \xi_2(s)\, ds = \xi_1(t) I(t; \xi_2).$$

Lemma 2.1 implies that

$$W_2(u_1, u_2) = \xi_1^2 W_1(\xi_2) = \xi_1^2 \xi_2 > 0.$$

Hence, $u_1$ and $u_2$ are linearly independent solutions of $Lu = 0$. Since $u_1$ is a minimal solution,

$$\infty = \lim_{t \to b^-} \left|\frac{u_2(t)}{u_1(t)}\right| = \int_a^b \xi_2(s)\, ds.$$

Thus, the following statement for $j = 2$ has been established:

($\mathfrak{F}_j$) There exist functions $\xi_1, \ldots, \xi_j$ such that the following are valid:

(i) $\xi_j$ is the minimal solution on $(a_0, b)$ of a normal $(n - j + 1)$st order linear equation $M_{j-1}u = 0$ with $\xi_j(a) = 1$;

(ii) Solutions $w_{j+1}, \ldots, w_n$ of $M_{j-1}u = 0$ exist such that $(\xi_j, w_{j+1}, \ldots, w_n)$ is a Pólya system on $(a_0, b)$;

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(iii) If $M_{j-1}v = 0$, then $L[\xi_1(t)I(t; \xi_2, \ldots, \xi_{j-1}, v)] = 0$ ($L[\xi_1(t)I(t; v)] = 0$ if $j = 2$);

(iv) $\int_a^b \xi_k(s)\, ds = \infty, k = 2, \ldots, j.$

Assume that $(\mathfrak{F}_j)$ $(j \geqq 2)$ is true. Then, there exists an $(n - j)$th order normal linear operator $M_j$ on $(a_0, b)$ such that

$$0 = M_{j-1}[\xi_j(t)z] = \xi_j(t)M_jz'. \tag{2.10}$$

Let

$$y_k = \left(\frac{w_k}{\xi_j}\right)', \qquad k = j + 1, \ldots, n.$$

Then, Lemma 2.1 and $(\mathfrak{F}_j)$ (ii) imply that

$$W_k(y_{j+1}, \ldots, y_{j+k}) = \xi_j^{-k-1}W_{k+1}(\xi_j, w_{j+1}, \ldots, w_{j+k}) > 0, \quad k = 1, \ldots, n - j.$$

Thus, $(y_{j+1}, \ldots, y_n)$ is a Pólya system on $(a_0, b)$ for $M_jy = 0$. Hence, Theorem 2.2 implies $(\mathfrak{F}_{j+1})$ (i) and $(\mathfrak{F}_{j+1})$ (ii).

Let $\xi_{j+1}$ be the minimal solution of $M_jv = 0$ with $\xi_{j+1}(a) = 1$. Since

$$\lambda(t) = \xi_j(t) \int_a^t \xi_{j+1}(s)\, ds$$

is a solution of $M_{j-1}u = 0$ which is linearly independent of the minimal solution $\xi_j$, we conclude that

$$\infty = \lim_{t \to b^-} \left|\frac{\lambda(t)}{\xi_j(t)}\right| = \int_a^b \xi_{j+1}(s)\, ds,$$

which establishes $(\mathfrak{F}_{j+1})$ (iv).

Finally, assume that $M_j\zeta = 0$. Let

$$\rho(t) = \xi_j(t) \int_a^t \zeta(s)\, ds.$$

Thus, $M_{j-1}\rho = 0$. Hence, $(\mathfrak{F}_j)$ (iii) implies that

$$0 = L[\xi_1(t)I(t; \xi_2, \ldots, \xi_{j-1}, \rho)]. \tag{2.11}$$

However,

$$L[\xi_1(t)I(t; \xi_2, \ldots, \xi_{j-1}, \rho)] = L[\xi_1(t)I(t; \xi_2, \ldots, \xi_{j-1}, \xi_j, \zeta)].$$

Hence, (2.11) implies $(\mathfrak{F}_{j+1})$ (iii).

We conclude by the Principle of Finite Induction that $(\mathfrak{F}_j)$ is true for $j = 2, \ldots, n$. Thus, we obtain functions $\xi_1, \ldots, \xi_n$ satisfying parts (i) and (ii) of the theorem. Furthermore, it is clear that the system $(u_1, \ldots, u_n)$, where $u_k$ is defined by (1.5), is the fundamental principal system on $[a, b)$ of $Lu = 0$.

It is well known that the Cauchy function $g(t, s)$ for initial-value problems for $Lu = 0$ is given by the formula

(2.12) $$g(t, s) = u_n(t)z_n(s) - \ldots + (-1)^{n-1}u_1(t)z_1(s),$$

where

(2.13) $$z_k(t) = \frac{W_{n-1}(u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_n)}{W_n(u_1, \ldots, u_n)}, \qquad k = 1, \ldots, n.$$

Repeated applications of Lemma 2.1 imply that

(2.14) $$W_n(u_1, \ldots, u_n) = \xi_1^{\,n} W_{n-1}\left(\left(\frac{u_2}{u_1}\right)', \ldots, \left(\frac{u_n}{u_1}\right)'\right) = \xi_1^{\,n}\xi_2^{\,n-1} \cdot \ldots \cdot \xi_n.$$

Thus, $W_n(u_1, \ldots, u_n)(a) = 1$, and so Abel's formula implies that

(2.15) $$W_n(u_1, \ldots, u_n) = \exp\left(-\int_a^t p_1(s)\, ds\right) \equiv P(t).$$

Lemma 2.1 also implies that

$$W_{n-1}(u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_n) = \xi_1^{\,n-1} \cdot \ldots \cdot \xi_k^{\,n-k} W_{n-k}(I_1, \ldots, I_{n-k})$$
$$(k = 1, \ldots, n;\ W_0 \equiv 1),$$

where

$$I_j = I(t; \xi_{k+1}, \ldots, \xi_{k+j}), \qquad j = 1, \ldots, n-k.$$

But Lemma 2.3 implies that

$$W_{n-k}(I_1, \ldots, I_{n-k}) = I(t; \xi_n, \ldots, \xi_{k+1})\xi_{k+1}^{\,n-k-1} \cdot \ldots \cdot \xi_{n-1}.$$

Hence, $z_k(t) = v_k(t)$, $k = 1, \ldots, n$, where $v_k$ is defined in (1.6). The two formulas in part (iv) of the theorem now follow directly from Lemma 2.2 after substituting from (1.5) and (1.6) into (2.12).

*Proof of Theorem* 1.2. The general theory of linear differential equations implies that $L^*z_k = 0$, where $z_k$ is defined in (2.13). But in the proof of Theorem 1.1, we established that $z_k = v_k$. Hence, $L^*v_k = 0$.

Lemma 2.1 and (1.6) imply that

$$W_k(v_n, \ldots, v_{n-k+1}) = v_n^{\,k}\xi_n^{\,k-1} \cdot \ldots \cdot \xi_{n-k+2} > 0 \quad \text{on } (a_0, b), \qquad k = 2, \ldots, n.$$

Thus, $(v_n, \ldots, v_1)$ is a Pólya system on $(a_0, b)$. Theorem 2.1 accordingly implies that $L^*v = 0$ is disconjugate on $[a_0, b)$. Finally, a simple application of L'Hôpital's Rule implies that $(v_n, \ldots, v_1)$ is the fundamental principal system on $[a, b)$ of $L^*v = 0$.

**3. The non-homogeneous equation.** In this section, we will consider the equation

(3.1) $$Ly = y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_n(t)y = f(t), \quad -\infty < a_0 \leqq t < b \leqq \infty.$$

Assume that $Lu = 0$ is disconjugate and normal on $[a_0, b)$ and let $a_0 < a < b$.

Then, Theorems 1.1 and 1.2 imply that $Lu = 0$ has a fundamental principal set $(u_1, \ldots, u_n)$ of solutions on $[a, b)$ and $L^*v = 0$ has a fundamental principal set $(v_n, \ldots, v_1)$ of solutions on $[a, b)$. In what follows, let $j$, $1 \leqq j \leqq n$, be fixed. We will be concentrating on determining when $Ly = f$ has a solution $y_j$ such that $y_j - u_j = o(u_j)$ (as $t \to b^-$).

With $h_1 = 0$ if $j = 1$, $h_2 = 0$ if $j = n$, and

$$h_1(t, s) = \sum_{k=1}^{j-1} (-1)^{n-k} u_k(t) \left(\frac{v_k(s)}{v_j(s)}\right)', \qquad j = 2, \ldots, n,$$

$$h_2(t, s) = \sum_{k=j+1}^{n} (-1)^{n-k+1} u_k(t) \left(\frac{v_k(s)}{v_j(s)}\right)', \qquad j = 1, \ldots, n-1,$$

define

$$h(t, s) = \begin{cases} h_1(t, s) & \text{for } a < s \leqq t < b, \\ h_2(t, s) & \text{for } a \leqq t < s < b. \end{cases} \tag{3.2}$$

Let

$$\Omega = \{(t, s) : a \leqq t < b, a < s < b, s \neq t\}.$$

LEMMA 3.1. *The system* $(v_n, \ldots, v_1)$ *given by* (1.6) *satisfies*

$$\left[\frac{v_k(t)}{v_j(t)}\right]' \begin{cases} >0 & \textit{for } k < j, \\ <0 & \textit{for } k > j, \end{cases} \qquad a < t < b.$$

*Proof*. Since $f' > 0$ implies $(1/f)' < 0$ in general, we need to prove the lemma for just one of the cases, say $k < j$. Let

$$w_n(t) = \frac{v_k(t)}{v_j(t)} = \frac{I(t; \xi_n, \ldots, \xi_{k+1})}{I(t; \xi_n, \ldots, \xi_{j+1})} \qquad (I(t; \xi_n, \xi_{n+1}) \equiv 1).$$

The proof is by induction on $n$. For $n = 2$, clearly, $w_2'(t) = (v_1/v_2)' > 0$. Suppose that $w_{n-1}'(t) > 0$. Then,

$$w_n'(t) = \left(\frac{\int_a^t \xi_n(s) u(s)\, ds}{\int_a^t \xi_n(s) v(s)\, ds}\right)' \tag{3.3}$$

$$= \frac{\xi_n(t) v(t)}{\left(\int_a^t \xi_n(s) v(s)\, ds\right)^2} \int_a^t w_{n-1}'(s) \left(\int_a^s \xi_n(\tau) v(\tau)\, d\tau\right) ds > 0.$$

THEOREM 3.1. *Let* $h(t, s)$ *be given by* (3.2). *Then* $\partial^k h/\partial t^k \in C(\Omega)$, $k = 0, \ldots, n$. *Furthermore, for* $a < \tau < b$,

$$\text{(i)}\ \frac{\partial^k h}{\partial t^k}(\tau, \tau^-) - \frac{\partial^k h}{\partial t^k}(\tau, \tau^+) = \begin{cases} 0 & \textit{for } k = 0, \ldots, n-3, \\ -v_j^{-1}(\tau) & \textit{for } k = n-2, \\ P(\tau)[P^{-1}(\tau) v_j^{-1}(\tau)]' & \textit{for } k = n-1; \end{cases}$$

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(ii) $$u_j(t) = (-1)^{n-j-1} \int_a^b h(t,s)\,ds, \qquad a \leqq t < b;$$

(iii) $$(-1)^{n-j-1} h(t,s) > 0, \qquad (t,s) \in \Omega.$$

THEOREM 3.2. *Let $h(t,s)$ be given by* (3.2). *If*

$$\int_a^b v_j(s)|f(s)|\,ds < \infty$$

*and*

(3.4) $$y_j(t) = \int_a^b h(t,s)\left(\int_s^b v_j(\tau) f(\tau)\,d\tau\right) ds,$$

*then $Ly_j = f$ on $[a,b)$ and*

(3.5) $$y_j^{(k)}(t) = o(\mu_{k+1}(t)), \qquad k = 0, \ldots, n-1,$$

*where*

(3.6) $$\begin{cases} \mu_1(t) = u_j(t), \\ \mu_k(t) = \sum_{\substack{m=1;\\ m\neq j}}^{n} |u_m^{(k-1)}(t)| v_m(t) v_j^{-1}(t), & k = 2, \ldots, n-1, \\ \mu_n(t) = \sum_{\substack{m=1;\\ m\neq j}}^{n} |u_m^{(n-1)}(t)| v_m(t) v_j^{-1}(t) + v_j^{-1}(t). \end{cases}$$

In order to prove Theorems 3.1 and 3.2, we will need the following lemma.

LEMMA 3.2. *Assume that $\zeta_k \in C[a,b)$, $\zeta_k > 0$, $k = 1, \ldots, n$. Then*

(3.7) $$R_n(s,\tau) = I(s,\tau;\zeta_n,\ldots,\zeta_1) I(s,a;\zeta_{n-1},\ldots,\zeta_1) - I(s,a;\zeta_n,\ldots,\zeta_1) I(s,\tau;\zeta_{n-1},\ldots,\zeta_1) < 0 \quad \textit{for } a < \tau < s.$$

*Proof.* Let $\tau$ be fixed, $a < \tau < b$, and let

$$H_{jk}(s) = I(s,\tau;\zeta_j,\ldots,\zeta_1) I(s;\zeta_k,\ldots,\zeta_1) - I(s;\zeta_j,\ldots,\zeta_1) I(s,\tau;\zeta_k,\ldots,\zeta_1),$$
$$\tau < s < b, \quad k = 1,\ldots,j, \quad j = 1,2,\ldots,n.$$

One can show by double induction on $j$ and $k$ that

$$H_{jk}(s) < 0, \qquad \tau < s < b, \quad k = 1,\ldots,j-1.$$

*Proof of Theorem* 3.1. Let

$$\Delta_s(t) = h_1(t,s) - h_2(t,s).$$

Then,

$$\Delta_\tau^{(k)}(\tau) = \frac{\partial^k h_1}{\partial t^k}(\tau,\tau) - \frac{\partial^k h_2}{\partial t^k}(\tau,\tau)$$
$$= \frac{\partial^k h}{\partial t^k}(\tau,\tau^-) - \frac{\partial^k h}{\partial t^k}(\tau,\tau^+), \qquad k = 0,\ldots,n-1,$$

and

$$\Delta_s(t) = \frac{\partial}{\partial s} \cdot \frac{g(t, s)}{v_j(s)},$$

where $g(t, s)$ is the Cauchy function for $L$. Thus,

$$\Delta_s^{(k)}(t) = \frac{1}{v_j(s)} \frac{\partial^{k+1} g}{\partial t^k \partial s}(t, s) + \left(\frac{1}{v_j(s)}\right)' \frac{\partial^k g}{\partial t^k}(t, s). \tag{3.8}$$

It is well known that

$$\frac{\partial^k g}{\partial t^k}(\tau, \tau) = \begin{cases} 0 & \text{for } k = 0, \ldots, n-2, \\ 1 & \text{for } k = n-1, \end{cases}$$

and

$$\frac{\partial^{k+1} g}{\partial t^k \partial s}(\tau, \tau) = \begin{cases} 0 & \text{for } k = 0, \ldots, n-3 \\ -1 & \text{for } k = n-2, \\ -P^{-1}P' & \text{for } k = n-1. \end{cases}$$

Substituting into (3.8) with $t = s = \tau$, we obtain (i) after some minor computations.

Since $(v_n, \ldots, v_1)$ is a fundamental principal system on $[a, b)$,

$$v_k(t) = o(v_j(t)), \quad \text{as } t \to b^-, \text{ for } k > j,$$

and

$$v_k(t) = o(v_j(t)), \quad \text{as } t \to a^+, \text{ for } k < j.$$

Hence, for any $t \in (a, b)$, $[v_k(s)/v_j(s)]'$ is integrable on $[t, b)$ for $k > j$ and is integrable on $(a, t]$ for $k < j$. Thus,

$$\begin{aligned} \int_a^b h(t, s)\, ds &= \int_a^t h_1(t, s)\, ds + \int_t^b h_2(t, s)\, ds \\ &= [g(t, t) - (-1)^{n-j} u_j(t) v_j(t)] v_j^{-1}(t) \\ &= (-1)^{n-j-1} u_j(t), \end{aligned}$$

which establishes (ii).

To establish (iii), we note that

$$h_1(t, s) = (-1)^{n-1} \xi_1(t) Q_j^n(t, s), \tag{3.9}$$

where

$$Q_j^n(t, s) = \sum_{i=1}^{j-1} (-1)^{i+1} \left[\frac{I(s; \xi_n, \ldots, \xi_{i+1})}{I(s; \xi_n, \ldots, \xi_{j+1})}\right]' I(t; \xi_2, \ldots, \xi_i), \qquad a < s < t,$$

$$(I(s; \xi_n, \xi_{n+1}) \equiv 1 \equiv I(t; \xi_2, \xi_1)).$$

For $n = 2, 3, \ldots,$

$$\begin{aligned} Q_n^n(t, s) &= \sum_{i=1}^{n-1} (-1)^{i+1} I'(s; \xi_n, \ldots, \xi_{i+1}) I(t; \xi_2, \ldots, \xi_i) \\ &= \xi_n(s) \sum_{i=1}^{n-1} (-1)^{i+1} I(s; \xi_{n-1}, \ldots, \xi_{i+1}) I(t; \xi_2, \ldots, \xi_i) \end{aligned}$$

$$(I(s; \xi_{n-1}, \xi_n) \equiv 1).$$

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Thus, Lemma 2.2 implies that

$$(-1)^n Q_n^n(t, s) = \xi_n(s) I(t, s; \xi_2, \ldots, \xi_{n-1}) > 0, \qquad a < s < t.$$

Consider the statement

$$(\mathfrak{M}_n) \qquad (-1)^j Q_j^n(t, s) > 0 \quad \text{for } a < s < t, \qquad j = 2, \ldots, n.$$

We have already shown that $(\mathfrak{M}_2)$ is true and $(\mathfrak{M}_n)$, for $j = n$, is true. Consider $Q_j^n$, $2 \leqq j \leqq n - 1$. Apply (3.3) in the proof of Lemma 3.1 to each term in the sum making up $Q_j^n$. Then,

$$Q_j^n(t, s) = \frac{\xi_n(s) I(s; \xi_{n-1}, \ldots, \xi_{j+1})}{I^2(s; \xi_n, \ldots, \xi_{j+1})} \int_a^s I(\tau; \xi_n, \ldots, \xi_{j+1}) Q_j^{n-1}(t, \tau)\, d\tau.$$

Thus, $(\mathfrak{M}_{n-1})$ implies $(\mathfrak{M}_n)$. This induction implies that $(\mathfrak{M}_n)$ is true for $n = 2, 3, \ldots$. We conclude from (3.9) that

$$(-1)^{n-j-1} h(t, s) > 0 \quad \text{for } a < s < t.$$

We note next that

$$h_2(t, s) = -\xi_1(t) P_j^n(t, s), \qquad a \leqq t < s < b, \tag{3.10}$$

where

$$P_j^n(t, s) = \sum_{i=1}^{n-j} (-1)^{i+1} \left[ \frac{I(s; \xi_n, \ldots, \xi_{n+2-i})}{I(s; \xi_n, \ldots, \xi_{j+1})} \right]' I(t; \xi_2, \ldots, \xi_{n+1-i})$$

$$(I(s; \xi_n, \xi_{n+1}) \equiv 1).$$

Let

$$I^*(s) = [1/I(s; \xi_n, \ldots, \xi_{j+1})]'$$

and assume that $s$ is fixed and that

$$I(s; \xi_{n-1}, \xi_n) \equiv 1 \equiv I(s; \xi_n, \xi_{n+1})$$

in the following. Then,

$$\begin{aligned} P_j^n(t, s) &= I^*(s) \sum_{i=1}^{n-j} (-1)^{i+1} I(s; \xi_n, \ldots, \xi_{n+2-i}) I(t; \xi_2, \ldots, \xi_{n+1-i}) \\ &\quad + \xi_n(s) I^{-1}(s; \xi_n, \ldots, \xi_{j+1}) \sum_{i=2}^{n-j} (-1)^{i+1} I(s; \xi_{n-1}, \ldots, \xi_{n+2-i}) \\ &\qquad \cdot I(t; \xi_2, \ldots, \xi_{n+1-i}) \\ &= I^*(s) I(t; \xi_2, \ldots, \xi_j R), \end{aligned} \tag{3.11}$$

where

$$R(s,\tau) = \sum_{i=1}^{n-j} (-1)^{i+1} I(\tau;\xi_{j+1},\ldots,\xi_{n+1-i}) I(s;\xi_n,\ldots,\xi_{n+2-i})$$
$$+ I(s;\xi_n,\ldots,\xi_{j+1}) I^{-1}(s;\xi_{n-1},\ldots,\xi_{j+1})$$
$$\cdot \sum_{i=1}^{n-j-1} (-1)^{i+1} I(s;\xi_{n-1},\ldots,\xi_{n+1-i}) I(\tau;\xi_{j+1},\ldots,\xi_{n-i})$$
$$= (-1)^{n-j}\{I(s,\tau;\xi_n,\ldots,\xi_{j+1}) - I(s;\xi_n,\ldots,\xi_{j+1})$$
$$- I(s;\xi_n,\ldots,\xi_{j+1}) I^{-1}(s;\xi_{n-1},\ldots,\xi_{j+1})$$
$$\cdot [I(s,\tau;\xi_{n-1},\ldots,\xi_{j+1}) - I(s;\xi_{n-1},\ldots,\xi_{j+1})]\}$$
$$= (-1)^{n-j} I^{-1}(s;\xi_{n-1},\ldots,\xi_{j+1})[I(s,\tau;\xi_n,\ldots,\xi_{j+1})$$
$$\cdot I(s;\xi_{n-1},\ldots,\xi_{j+1}) - I(s;\xi_n,\ldots,\xi_{j+1}) I(s,\tau;\xi_{n-1},\ldots,\xi_{j+1})]$$

by Lemma 2.2. Lemma 3.2 now implies that

$$(-1)^{n-j} R(s,\tau) < 0 \quad \text{for } \tau < s.$$

Since $I^*(s) < 0$, we conclude from (3.11) that

$$(-1)^{n-j} P_j^n(t,s) > 0 \quad \text{for } t < s < b,$$

which implies by (3.10) that

$$(-1)^{n-j-1} h_2(t,s) > 0 \quad \text{for } t < s < b.$$

*Proof of Theorem* 3.2. Let $y_j$ be given by (3.4). Theorem (3.1) (i) implies that

$$(3.12)\quad y_j^{(k)}(t) - \int_a^b \frac{\partial^k h}{\partial t^k}(t,s)\left(\int_s^b v_j(\tau) f(\tau)\,d\tau\right) ds$$
$$= \begin{cases} 0 & \text{for } k = 0,\ldots,n-2 \\ -v_j^{-1}(t)\displaystyle\int_t^b f(s) v_j(s)\,ds & \text{for } k = n-1, \\ -\dfrac{P'(t)}{P(t)v_j(t)}\displaystyle\int_t^b f(s)v_j(s)\,ds + f(t) & \text{for } k = n. \end{cases}$$

Since $L_t h(t,s) = 0$ and $P'(t) = -p_1(t)P(t)$, it is an easy matter to verify that $Ly_j = f$.

We will show next that (3.12) implies (3.5). Let $T$ be fixed, $a < T < b$. Lemma 3.1 implies that

$$(3.13)\quad \int_a^T \left|\frac{\partial^k h_1}{\partial t^k}(t,s)\right| ds \leqq \sum_{m=1}^{j-1} |u_m^{(k)}(t)| v_m(T) v_j^{-1}(T), \quad k = 0,\ldots,n-1.$$

Since $u_m = o(u_j)$ for $m < j$, (3.13) implies that

$$\int_a^T |h_1(t,s)|\,ds = o(u_j).$$

Since $v_j = o(v_m)$ for $m < j$, (3.13) further implies that

$$\int_a^T \left|\frac{\partial^k h_1}{\partial t^k}(t, s)\right| ds = o\left(\sum_{m=1}^{j-1} |u_m^{(k)}(t)| v_m(t) v_j^{-1}(t)\right), \quad k = 1, \ldots, n-1.$$

Next, Theorem 3.1 implies that

$$\text{(3.14)} \qquad \int_a^b |h(t, s)|\, ds = (-1)^{n-j-1} \int_a^b h(t, s)\, ds = u_j(t) = \mu_1(t),$$

and Lemma 3.1 implies that

$$\int_a^b \left|\frac{\partial^k h}{\partial t^k}(t, s)\right| ds = \int_a^t \left|\frac{\partial^k h_1}{\partial t^k}(t, s)\right| ds + \int_t^b \left|\frac{\partial^k h_2}{\partial t^k}(t, s)\right| ds \leqq \mu_{k+1}(t),$$

$$\text{(3.15)} \qquad k = 1, \ldots, n-2.$$

$$\int_a^b \left|\frac{\partial^{n-1} h}{\partial t^{n-1}}(t, s)\right| ds + \frac{1}{v_j(t)} \leqq \mu_n(t).$$

Let $\epsilon > 0$ be given. Then, there exists $T$, $a < T < b$, such that

$$\int_T^b |f(s)| v_j(s)\, ds < \epsilon.$$

Thus, for $t \geqq T$ and $k = 0, \ldots, n-2$,

$$|y^{(k)}(t)| \leqq \int_a^b \left|\frac{\partial^k h}{\partial t^k}(t, s)\right| \left(\int_s^b |f(\tau)| v_j(\tau)\, d\tau\right) ds = \int_a^T \cdot + \int_T^b \cdot$$

$$\leqq \int_a^T \left|\frac{\partial^k h_1}{\partial t^k}(t, s)\right| ds \int_a^b |f(s)| v_j(s)\, ds$$

$$+ \int_a^b \left|\frac{\partial^k h}{\partial t^k}(t, s)\right| ds \int_T^b |f(s)| v_j(s)\, ds$$

$$\leqq o(\mu_{k+1}(t)) + \mu_{k+1}(t)\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude that $y^{(k)} = o(\mu_{k+1})$ for $k = 0, \ldots, n-2$. Similarly, (3.12) and (3.15) imply that

$$|y^{(n-1)}(t)| \leqq o(\mu_n(t)) + \mu_n(t)\epsilon;$$

hence, $y^{(n-1)} = o(\mu_n)$.

**4. Perturbations of linear equations.** In this section, we will consider the equation

$$\text{(4.1)} \qquad Ly = f(t, y, y', \ldots, y^{(n-1)}), \qquad a_0 \leqq t < b \leqq \infty,$$

where $L$ is a normal disconjugate $n$th order linear operator on $[a_0, b)$.

Let $a_0 < a < b$. Theorem 1.1 implies that $Lu = 0$ has the fundamental principal system $(u_1, \ldots, u_n)$ on $[a, b)$, which can be represented by (1.5). Let

$v_k$, $k = 1, \ldots, n$, be the functions subsequently defined by (1.6), or equivalently, by (2.13) ($v_k = z_k$). Define

$$h^j(t, s) = h(t, s) \quad \text{and} \quad \mu_k^j(t) = \mu_k(t), \qquad k = 1, \ldots, n,$$

where $h(t, s)$ is defined in (3.2) and $\mu_k(t)$ is defined in (3.6). In this section, we wish to emphasize the dependence of these functions on $j$. Otherwise, the situation with respect to $L$ and the notation used in this section are similar to that of the previous sections; hence, the results there are valid in the present setting when interpreted properly.

THEOREM 4.1. *If there exists $\delta$, $1 \leqq \delta \leqq \infty$, and $j$, $1 \leqq j \leqq n$, such that for*

$$(4.2) \qquad S = \{y \in C^n[a, b): |y^{(k)}(t)| < \delta\mu_{k+1}^j(t), a \leqq t < b, \quad k = 0, \ldots, n-1\},$$

*the function $f(t, y(t), \ldots, y^{(n-1)}(t))$ is continuous on $[a, b)$ for $y \in S$, the function*

$$(4.3) \qquad M_j(t) = \sup\{|f(t, y(t), \ldots, y^{(n-1)}(t))|: y \in S\}$$

*is measurable on $[a, b)$, and*

$$(4.4) \qquad \int_a^b v_j(s) M_j(s)\, ds < \infty,$$

*then for any solution $y_j \in S$ of (4.1), there exist $c_m$, $m = 1, \ldots, n$, such that*

$$(4.5) \qquad y_j^{(k)}(t) = \sum_{m=1}^{n} c_m u^{(k)}(t) + o(\mu_{k+1}^j(t)), \qquad k = 0, \ldots, n-1.$$

*Furthermore, if $\delta < \infty$, then*

$$(4.6) \qquad c_{j+1} = \ldots = c_n = 0.$$

*Proof.* If $y_j \in S$ is a solution of (4.1), then $y_j$ is a solution of the non-homogeneous linear equation $Ly = F(t)$, where $F(t) = f(t, y_j(t), \ldots, y_j^{(n-1)}(t))$. Furthermore, (4.4) implies that

$$\int_a^b v_j(s)|F(s)|\, ds < \infty.$$

Thus, by Theorem 3.2 and the general theory of linear differential equations there exist constants $c_1, \ldots, c_n$ such that

$$y_j(t) = \sum_{m=1}^{n} c_m u_m(t) + \int_a^b h^j(t, s)\left(\int_s^b v_j(\tau)\, F(\tau)\, d\tau\right) ds.$$

Theorem 3.2 also implies (4.5).

If $\delta < \infty$, then (4.6) follows from the fact that

$$|y_j(t)| \leqq \delta\mu_1^j(t) = \delta\mu_j(t)$$

and $u_m(t)u_j^{-1}(t) \to \infty$, as $t \to b^-$, when $m > j$.

THEOREM 4.2. *Let the assumptions of Theorem* 4.1 *hold. If*

$$\int_a^b v_j(s)M_j(s)\,ds < \delta - 1, \tag{4.7}$$

*then there exists a solution* $y_j$ *of* (4.1) *such that* $y_j \in S$ *and*

$$y_j^{(k)}(t) = u_j^{(k)}(t) + o(\mu_{k+1}{}^j(t)), \qquad k = 0, \ldots, n-1. \tag{4.8}$$

COROLLARY 4.1. *Assume that* $f(t, y_1, \ldots, y_n)$ *is continuous on* $[a, b) \times R^n$ *and that there exist* $r_k \in C[a, b)$, $k = 0, \ldots, n$, *and there exist constants* $\lambda_k$, $k = 1, \ldots, n$, *such that*

$$|f(t, y_1, \ldots, y_n)| \leqq r_0(t) + \sum_{k=1}^n r_k(t)|y_k|^{\lambda_k} \quad \textit{for } a \leqq t < b, \tag{4.9}$$

$|y_k| < \infty$, $k = 1, \ldots, n$. *Let* $\lambda = \max(\lambda_1, \ldots, \lambda_n)$ *and assume that*

$$c_0 = \int_a^b v_j(s)r_0(s)\,ds < \infty \tag{4.10}$$

*and*

$$c = \sum_{k=1}^n \int_a^b r_k(s)v_j(s)[\mu_k{}^j(s)]^{\lambda_k}\,ds < \infty.$$

*If any one of the following holds:*

(i) $\lambda < 1$,

(ii) $\lambda = 1$ *and* $c < 1$,

(iii) $\lambda > 1$ *and* $c \leqq \dfrac{(\lambda-1)^{\lambda-1}}{\lambda^\lambda}\dfrac{1}{(1+c_0)^{\lambda-1}}$,

*then* $Ly = f$ *has a solution* $y_j \in C^n[a, b)$ *satisfying* (4.8).

THEOREM 4.3. *Let the assumptions of Theorem* 4.1 *hold and assume that*

$$\gamma = \int_a^b v_j(s)|f(s, 0, \ldots, 0)|\,ds < \infty. \tag{4.11}$$

*If there exist* $r_k \in C[a, b)$, $k = 1, \ldots, n$, *such that*

$$|f(t, x_1, \ldots, x_n) - f(t, y_1, \ldots, y_n)| \leqq \sum_{k=1}^n r_k(t)|x_k - y_k|, \tag{4.12}$$

$$a \leqq t < b,\ |x_k|, |y_k| < \delta,$$

*and*

$$\nu = \sum_{k=1}^n \int_a^b r_k(s)\mu_k{}^j(s)v_j(s)\,ds \leqq 1 - (1+\gamma)\delta^{-1} \tag{4.13}$$

(*strict inequality if* $\delta = \infty$),

*then there exists a unique solution* $y_j$ *of* (4.1) *in* $S$. *Furthermore,* (4.8) *holds.*

COROLLARY 4.2. *Assume that* $Lu = 0$ *is a disconjugate normal equation on* $[a_0, b)$ *and* $r_k \in C[a_0, b)$, $k = 0, \ldots, n$. *For any* $j$, $1 \leqq j \leqq n$, *such that*

$$(4.14)\quad \int^{b^-} v_j(s)|r_0(s)|\,ds + \int^{b^-} u_j(s)v_j(s)|r_n(s)|\,ds + \int^{b^-} |r_1(s)|\,ds + \sum_{k=2}^{n} \int^{b^-} |r_{n-k+1}(s)| \sum_{\substack{i=1;\\ i\neq j}}^{n} |u_i^{(k-1)}|v_i(s)\,ds < \infty,$$

*there exists a unique solution* $y_j \in C^n[a_0, b)$ *of*

$$(4.15)\quad My = y^{(n)} + [p_1(t) + r_1(t)]y^{(n-1)} + \ldots + [p_n(t) + r_n(t)]y = r_0(t)$$

*satisfying* (4.8).

Theorem 4.2 includes the main results of Hale and Onuchic [**8**, Theorem 2 and subsequent corollaries], who considered the special case of (4.1) given by assuming $L = D^n$ and there exist non-decreasing functions $L_k$ such that

$$|f(t, y_1, \ldots, y_n)| \leqq \sum_{k=1}^{n} h_k(t)|y_k|L_k(|y_1|, \ldots, |y_n|),$$

$$\sum_{k=1}^{n} \int^{\infty} t^{k-1}h_k(t)L_k(\delta t^j, \delta t^{j-1}, \ldots, \delta t^{j-n+1})\,dt < \infty \qquad (b = \infty).$$

Corollary 4.1 includes the special results of Waltman [**18**; **19**], who considered the equation $y^{(n)} = f(t, y)$ with

$$|f(t, y)| \leqq r(t)|y|^{\lambda} \quad \text{and} \quad \int^{\infty} t^{\lambda(n-1)}r(t)\,dt < \infty.$$

Corollary 4.2 includes the well-known result of Dunkel [**2**] for constant coefficient operators $L$ with distinct characteristic numbers and the not so well-known result of Faedo [**3**; **4**] for constant coefficient operators $L$ with multiple characteristic numbers. Faedo's result is as follows. Assume that $L$ has characteristic numbers $k_1, \ldots, k_\sigma$ of multiplicity $\gamma_1, \ldots, \gamma_\sigma$, respectively. Let $\nu = \max(\nu_1, \ldots, \nu_\sigma)$. Then, $My = 0$ has a fundamental set $y_1, \ldots, y_n$ of solutions asymptotic, as $t \to \infty$, to a fundamental set $u_1, \ldots, u_n$ of solutions of $Lu = 0$, provided that

$$(4.16)\quad \int^{\infty} t^{\nu-1}|r_m(t)|\,dt < \infty, \qquad m = 1, \ldots, n.$$

Condition (4.16) is equivalent to (4.14) in this setting. Ghizzetti [**6**] and Zlámal [**20**] have also considered the linear situation with $L = D^n$. Halanay [**7**] seems to be the only one who has obtained results of the same scope, when applied to a specific equation, as Corollary 4.2. Halanay essentially proved Corollary 4.2 for the special case of the second-order equation

$$y'' + [p(t) + r(t)]y = 0.$$

See the introduction for a comparison of some of the results of this section with the results of Katz [**10**] and Locke [**11**] for equation (1.10).

*Proof of Theorem* 4.2. Let $j$, $1 \leqq j \leqq n$, be fixed and chosen so that the assumptions hold. Let $C_*[a, b)$ denote the set of vector-valued functions $X(t) = (x_1(t), \ldots, x_n(t))$ with $x_k(t)$, $k = 1, \ldots, n$, continuous on $[a, b)$. For $X \in C_*[a, b)$, define

$$|X(t)| = \max\left(\frac{|x_1(t)|}{\mu_1^j(t)}, \ldots, \frac{|x_n(t)|}{\mu_n^j(t)}\right).$$

Since $(\mu_1^j, \ldots, \mu_n^j) \in C_*[a, b)$ and $\mu_k^j > 0$ on $(a, b)$, $|X(t)|$ is a continuous real-valued function on $(a, b)$. We consider $C_*[a, b)$ as a Fréchet space by choosing convergence in $C_*[a, b)$ to be uniform convergence on compact subintervals of $(a, b)$ as follows:

$$X_n \to X \text{ if for any compact } J \subset (a, b), \sup_{t \in J} |X_n(t) - X(t)| \to 0.$$

For the number $\delta$ given in the assumptions, let

$$C_\delta = \{X \in C_*[a, b): |X(t)| \leqq \delta, a \leqq t < b\}.$$

Clearly $C_\delta$ is a closed convex subset of $C_*$.

Consider the scalar integral equation

$$y(t) = u_j(t) + \int_a^b h^j(t, s)\left(\int_s^b v_j(\tau) f(\tau, y(\tau), \ldots, y^{(n-1)}(\tau))\, d\tau\right) ds. \tag{4.17}$$

Theorem 3.2 implies that any solution $y$ of (4.17) is a solution of (4.1). Let

$$E_n = (0, \ldots, 0, 1), \quad f(s, X) = f(s, x_1, \ldots, x_n) \qquad (X = (x_1, \ldots, x_n)),$$

$$U_j(t) = (u_j(t), u_j'(t), \ldots, u_j^{(n-1)}(t)),$$

$$H^j(t, s) = \left(h^j(t, s), \frac{\partial h^j}{\partial t}(t, s), \ldots, \frac{\partial^{n-1} h^j}{\partial t^{n-1}}(t, s)\right).$$

Then, (4.17) is equivalent to the system

$$Y = TY,$$

where

$$TY = U_j(t) - E_n v_j^{-1}(t) \int_t^b v_j(s) f(s, Y(s))\, ds + \int_a^b H^j(t, s)\left(\int_s^b v_j(\tau) f(\tau, Y(\tau))\, d\tau\right) ds. \tag{4.18}$$

That is, if there exist $Y = (y_1, \ldots, y_n) \in C_\delta$ such that $TY = Y$, then $y_1$ is a solution of (4.17), and thus (4.1), and $y_1^{(k)} = y_{k+1}$. Furthermore, Theorem 4.1 implies that $y_1$ satisfies (4.8).

We will show that $T$ has a fixed point in $C_\delta$ by using the Schauder-Tychonoff theorem. This requires showing that $TC_\delta \subset C_\delta$, since $T$ is clearly a completely continuous operator on $C_\delta$, that is, $T$ is a continuous operator with respect to uniform convergence on compact subsets of $(a, b)$ and $TC_\delta$ is a uniformly bounded and equicontinuous set.

Theorem 3.1 implies that

$$|u_j^{(k)}(t)| \leqq \int_a^b \left|\frac{\partial^k h^j}{\partial t^k}(t,s)\right| ds + \begin{cases} 0 & \text{for } k = 0, \ldots, n-2, \\ v^{-1}(t) & \text{for } k = n-1. \end{cases}$$

Thus, (3.15) implies that

$$|u_j^{(k)}(t)| \leqq \mu_{k+1}{}^j(t), \qquad k = 0, \ldots, n-1.$$

Since

$$\left| \int_a^b \frac{\partial^k h^j}{\partial t^k}(t,s)\left(\int_s^b v_j(\tau) f(\tau, Y(\tau))\, d\tau\right) ds\right|$$

$$\leqq \int_a^b \left|\frac{\partial^k h^j}{\partial t^k}(t,s)\right| ds \int_a^b v_j(\tau)|f(\tau, Y(\tau))|\, d\tau,$$

we conclude from the assumptions that for any $Y \in C_\delta$,

$$|TY| \leqq 1 + \int_a^b v_j(s)|f(s, Y(s))|\, ds \leqq 1 + \int_a^b v_j(s) M_j(s)\, ds < \delta.$$

Thus, $TY \in C_\delta$, i.e., $TC_\delta \subset C_\delta$.

*Proof of Corollary* 4.1. For any $\delta$, $1 \leqq \delta < \infty$, let

$$M_j(t) = r_0(t) + \sum_{k=1}^n \delta^{\lambda_k} r_k(t)[\mu_k{}^j(t)]^{\lambda_k}.$$

Then,

$$\int_a^b v_j(s) M_j(s)\, ds = c_0 + \sum_{k=1}^n \delta^{\lambda_k} \int_a^b r_k(s) v_j(s)[\mu_k{}^j(s)]^{\lambda_k}\, ds \leqq c_0 + c\delta^\lambda.$$

The corollary follows from Theorem 4.2, provided there exists $\delta$ such that

(4.19) $$c_0 + c\delta^\lambda \leqq \delta - 1.$$

Clearly (4.19) can be achieved by taking $\delta$ sufficiently large in cases (i) and (ii), or if $c = 0$. If $\lambda > 1$ and $c > 0$, then (4.19) can be achieved for $\delta = \delta_0$, where $\delta_0$ is the value at which the function

$$g(\delta) = c_0 + c\delta^\lambda - \delta + 1$$

takes on its minimum in $1 \leqq \delta < \infty$. This minimum is non-positive in this case because of assumption (iii).

*Proof of Theorem* 4.3. Consider the mapping $T$ defined in (4.18) on the set $C_\delta$. Simple computations show that for any $X$, $Y \in C_\delta$,

$$|TX - TY| \leqq \nu|X - Y|,$$

where $\nu$ is defined in (4.13). Also, it is easy to show that

$$|TO| \leqq 1 + \gamma \leqq \delta(1 - \nu)$$

by (4.13). Since $\nu < 1$, the Principle of Contraction Mappings implies that $T$ has a unique fixed point in $C_\delta$. Subsequently, Theorem 4.1 implies that (4.8) holds.

*Proof of Corollary* 4.2. By choosing $a$ sufficiently close to $b$ (sufficiently large, if $b = \infty$), we can make $\gamma$ and $\nu$, defined in (4.11) and (4.13), arbitrarily small. Hence, let $\delta > 1$ be given. Choose $a$, $a_0 < a < b$, so that (4.13) holds. Then, Theorem 4.3 implies that there exists a solution $y_j \in C^n[a, b)$ of (4.15), which satisfies (4.8). Since (4.15) is a normal linear equation on $[a_0, b)$, $y_j(t)$ can be uniquely extended as a solution to $[a_0, b)$.

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