TOURNAMENTS AND IDEAL CLASS GROUPS

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ABSTRACT.   Results are given for a class of square $\{0, 1\}$-matrices which provide information about the 4-rank of the ideal class group of certain quadratic number fields.

1. **Tournaments and ideal class groups.**   An antisymmetric, square $\{0, 1\}$-matrix can be thought of as the adjacency matrix for a *tournament*, a directed graph with exactly one edge connecting each pair of vertices. We wish to consider such matrices in which each column has an even number of ones. We will call such matrices *adjusted tournament matrices*.

These matrices enable us to determine some of the structure of the ideal class groups of a family of quadratic number fields. Let $E = \mathbf{Q}(d^{\frac{1}{2}})$, where $d = p_1 \cdots p_n$, and each $p_i \equiv 3 \pmod 4$ is a distinct prime. Let $A = (a_{ij}) \in \mathbf{M}_{n \times n}(\mathbf{F}_2)$ be defined by:

$$(-1)^{a_{ij}} = \begin{cases} (\frac{p_j}{p_i}) & \text{if } i \neq j \\ (-1)^{n+1}(\frac{d/p_i}{p_i}) & \text{if } i = j \end{cases}.$$

Then by quadratic reciprocity, A is an adjusted tournament matrix. Let $C(E)$ denote the ideal class group of E. We find the following relationship between the 4-rank of $C(E)$ and the matrix A:

THEOREM 1.   *Let, E, d and A be as defined above, and let $r_4(d)$ denote 4-rank $C(E)$. Then:*

  (i) *If n is even, then $r_4(d) = (n-1) - \mathrm{rank}_{\mathbf{F}_2}(A)$, and $r_4(-d) = n - \mathrm{rank}_{\mathbf{F}_2}(A)$ or $(n-1) - \mathrm{rank}_{\mathbf{F}_2}(A)$.*
  (ii) *If n is odd, then $r_4(d) = (n-1) - \mathrm{rank}_{\mathbf{F}_2}(A)$ or $(n-2) - \mathrm{rank}_{\mathbf{F}_2}(A)$, and $r_4(-d) = (n-1) - \mathrm{rank}_{\mathbf{F}_2}(A)$.*

Furthermore, define the column vector $\vec{v}_2$ as follows:

$$\vec{v}_2 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},$$

where $(-1)^{c_i} = (\frac{2}{p_i})$. Then in the cases where the formula is ambiguous, the greater value is taken when $\vec{v}_2 \in \mathrm{Im}(A + I)$.

PROOF.   This result can be easily derived from results in Rédei, [3].   ∎

It is an easy consequence of Dirichlet's Theorem on Primes in Arithmetic Progressions that a given adjusted tournament matrix will occur for infinitely many fields. Thus, we can learn about infinite classes of number fields by studying the properties of finite classes of matrices; for instance, the matrices of one particular size.

2. **Values occuring as $r_4(d)$.** To determine which values may occur as $r_4(d)$, it suffices to determine which values occur as ranks for adjusted tournament matrices.

A 1984 result of Gerth's [1] provides us with a bound on the size of $r_4(d)$:

THEOREM. *Let $A \in \mathbf{M}_{n \times n}(\mathbf{F}_2)$ be an antisymmetric matrix. Let $r = \mathrm{rank}_{\mathbf{F}_2}(A)$. Let $c(A)$ denote the column space of A. If n is even, then:*

$$\dim[c(A) + c(A^T)] = n \quad and \quad \dim[c(A) \cap c(A^T)] = 2r - n.$$

If n is odd, then:

$$\dim[c(A) + c(A^T)] \geq n - 1 \quad and \quad \dim[c(A) \cap c(A^T)] \leq 2r - n + 1. \qquad \blacksquare$$

When A is the adjusted tournament matrix corresponding to a field $E = \mathbf{Q}(d^{\frac{1}{2}})$, n is the number of prime divisors of d. Clearly, $\mathrm{rank}(A) \leq n - 1$. Thus, we have the following bounds on $r_4(d)$:

$$
\begin{array}{cc}
\underline{n \text{ even}} & \underline{n \text{ odd}} \\
0 \leq r_4(-d) \leq \frac{n}{2} & 0 \leq r_4(-d) \leq \frac{n-1}{2} \\
0 \leq r_4(d) \leq \frac{n}{2} - 1 & 0 \leq r_4(d) \leq \frac{n-1}{2}
\end{array}
$$

Furthermore, any value in these ranges can occur. For example, consider the case $n = 5$. Then we have a lower bound of 0 and an upper bound of 2 for $r_4(-d)$ and $r_4(d)$. To achieve these 4-ranks, we begin with an adjusted tournament matrix with rows occurring in "pairs", and, through successive manipulations of the diagonal, the rightmost column, and the bottommost row, obtain the following matrices with ranks 2, 3, and 4, respectively:

$$
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0
\end{bmatrix}
$$

By similar adjustments, adjusted tournament matrices of any rank in the possible range can be constructed for any size.

3. **Minimal ranks for adjusted tournament matrices.** From the explanation above, we see that any result limiting the rank of an adjusted tournament matrix yields information about large values for $r_4(d)$, for appropriate fields.

If $n \leq 5$, the only minimal rank adjusted tournament matrix of size n is the matrix for the *transitive* tournament; that is, the unique (up to isomorphism) tournament of size n in which one vertex "defeats" all of the others, a second "defeats" all but the first, and so on. This can be verified merely by consulting the list of the distinct tournaments of size n given in [2].

The first non-transitive tournament with a minimal rank adjusted tournament matrix occurs at rank 6:

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$
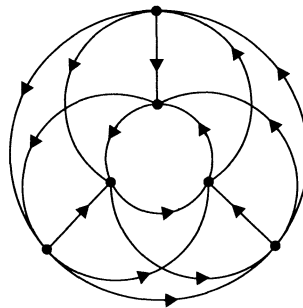

FIGURE 1

Note the degree of symmetry in the graph of this tournament.

Minimal rank adjusted tournament matrices may be classified as follows:

THEOREM 2. *An adjusted tournament matrix A, of given size n, has minimal rank if and only if it is idempotent.*

PROOF. We will make extensive use of the following relation, true for any antisymmetric $\{0, 1\}$-matrix:

(1) $A = A^T + I + J,$

where I is the $n \times n$ identity matrix and J is the $n \times n$ matrix consisting entirely of ones.

Suppose, first, that A is idempotent. We consider A as a linear transformation on the (column) vector space of $\mathbf{F}_2^n$. Then $A|_{c(A)} = I|_{c(A)}$. Since each column of A has even weight, and each vector in $c(A)$ is a sum of columns of A, each member of $c(A)$ has even weight. Therefore, $J|_{c(A)} = 0$. Thus, restricting each map in equation (1) to $c(A)$, we find

that $A|_{c(A)} = A|_{c(A)} + A^T|_{c(A)}$; in other words, $A^T|_{c(A)} = 0$. Therefore, $c(A) \subseteq \ker(A^T)$, so rank$(A^T) \leq n - $rank$(A)$. But rank$(A) = $rank$(A^T)$, so rank$(A) \leq \frac{n}{2}$.

Suppose conversely that A has minimal rank; *i.e.*, that $2 \cdot$rank$(A) \leq n$. Then by Gerth's result in [2], $c(A) \cap c(A^T) = \{0\}$. Once again, restrict equation (1) to $c(A)$. Then again $J|_{c(A)} = 0$, so $A|_{c(A)} + I|_{c(A)} = A^T|_{c(A)}$. The image of the map on the left is contained within $c(A)$, and certainly $c(A^T|_{c(A)}) \subseteq c(A^T)$, so $c(A^T|_{c(A)}) \subseteq c(A) \cap c(A^T) = \{0\}$. Thus $A|_{c(A)} = I|_{c(A)}$, but this implies that A is idempotent.   ∎

While checking a matrix this way is equal in computational complexity to Gauss-Jordan reduction, it is an intuitively easier task. The following corollary details the implications of this theorem for the values $r_4(d)$ and $r_4(-d)$:

COROLLARY 1.   *Suppose* $d = q_1 \cdots q_n$, *with each* $q_i \equiv 3 \pmod 4$ *prime. Let* A *be the adjusted tournament matrix associated with* $q_1, \ldots, q_n$, *and let the column vector* $\vec{v}_2$ *be defined as above. Then:*

  (i) *If* n *is even, then* $r_4(d) = \frac{n}{2} - 1$ *if and only if* A *is idempotent.* $r_4(-d) = \frac{n}{2}$ *if and only if* A *is idempotent and* $\vec{v}_2 \in \text{Im}(A + I)$.
  (ii) *If* n *is odd, then* $r_4(d) = \frac{n-1}{2}$ *if and only if* A *is idempotent and* $\vec{v}_2 \in \text{Im}(A + I)$. $r_4(-d) = \frac{n-1}{2}$ *if and only if* A *is idempotent.*   ∎

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