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REMARKS ON MODULE-FINITE PAIRS

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Let $R \subsetneq T$ be an extension of commutative rings having the same identity. A. Wadsworth (**10**) studies the situation when R and T are integral domains, and all rings between R and T are Noetherian. In this case (R, T) is called a Noetherian pair. In a similar vein, E. Davis (**4**) studies normal pairs and I. Papick (**8**) shows when coherent pairs are Noetherian pairs.

These papers are the motivation for this article. We study the concept of a *module-finite pair*—that is a pair (R, T) in which each intermediate ring (including T) is module-finite over R. It is clear that in the category of Noetherian rings such pairs abound, and it is our intention (Theorem 2) to show that in a very general setting this is the only place these pairs exist. However, in complete generality, module-finite pairs do exist in non-Noetherian categories. (Remark (a)). Note that we do not necessarily assume that R and T are domains.

Let (R, T) be a module-finite pair, $I = (R \underset{R}{:} T)$, $P \in \mathrm{Spec}(R)$, and $S = R + PT$. By integrality, $PT \in \mathrm{Spec}(S)$. With this notation we have:

Lemma 1. *If $I \subseteq P$, then $(S \underset{S}{:} T) = PT$.*

Proof. For notational purposes, let sub U denote all localizations at $R \backslash P$; e.g., $S_U = S_{R \backslash P}$ and $P_U = PR_P$. By the ring theory version of (**7**, Exercise 41(c), p. 46) $R_U \subsetneq T_U$, and so (R_U, T_U) is a proper module-finite pair. Hence by Nakayama's Lemma $R_U + P_U T_U \neq T_U$. But $S_U = R_U + P_U T_U$, and so we may choose a $t \in T_U \backslash S_U$. Notice that $(S_U \underset{S_U}{:} t) = P_U T_U$, since $S_U / P_U T_U$ is a field. Hence,

$$P_U T_U = (S_U \underset{S_U}{:} T_U) = (S \underset{S}{:} T)_U \text{ (\textbf{1}, Corollary 3.15).}$$

From this it follows that $PT \supset (S \underset{S}{:} T)$. Therefore $PT = (S \underset{S}{:} T)$ and the proof is now complete.

Theorem 2. *Let (R, T) be a module-finite pair with R an integral domain, and set $I = (R \underset{R}{:} T)$.*

(a) *If $I = (0)$, then R is a Noetherian domain.*

(b) *If $I \neq (0)$, T a domain, and R is a coherent domain, then each ideal of R containing I is finitely generated. In particular, R/I is a Noetherian ring.*

Proof. (a) The proof of this may be found in (**8**, p. 561). However, we include it here for completeness. Let J be an ideal of R. Since T is not contained in the quotient field of R, we may choose an element $t \in T \backslash R$ such that $R[t]$ is a free R-module with basis $\{1, t, \ldots, t^{n-1}\}$, $n \geqq 2$. Let $S = R + Jt + J^2t^2 + \ldots + J^{n-1}t^{n-1}$. Observe that S is a subring of T, and moreover, $S = R \oplus Jt \oplus J^2t^2 \oplus \ldots \oplus J^{n-1}t^{n-1}$. As S is module-finite over R, so is Jt module-finite over R. Hence, J is a finitely generated ideal of R.

(b) To show that each ideal of R containing I is finitely generated, it suffices to show that each prime ideal of R containing I is finitely generated (**7**, Exercise 24, p. 65).

Let $P \in \operatorname{Spec}(R)$ such that $I \subseteq P$, and set $S = R + PT$. Note that since $I \neq (0)$, T is a domain contained in the quotient field of R. Hence, as R is coherent and $R \subseteq S$ is a module-finite extension of domains, then S coherent (**6**, Corollary 1.5). By Lemma 1, $PT = (S \underset{S}{:} T)$. Then since $T = \sum_{i=1}^{m} S(a_i/b_i)$ for some $a_i, b_i \in S$, we have $(S \underset{S}{:} T) = \bigcap_{i=1}^{m} (b_{i_S} \underset{S}{:} a_i)$; and so the coherence of S forces PT to be finitely generated in S (**3**, Theorem 2.2). Finally, as $PT \cap R = P$, it follows that P is finitely generated in R (**2**, Exercise 11(d), p. 44). The proof of Theorem 2 is complete.

Let $\bar{R}$ (resp., $\bar{T}$) denote R/I (resp., T/I) where $I = (R \underset{R}{:} T)$.

Lemma 3. *Let T be a ring extension of R. Then (R, T) is a module-finite pair if and only if $(\bar{R}, \bar{T})$ is a module-finite pair.*

Proof. $(\rightarrow)$: Clear.
$(\leftarrow)$: Let $R \subset S \subset T$ where S is a subring of T. Since I is an ideal of S, $\bar{R} \subset \bar{S} \subset \bar{T}$ where $S/I = \bar{S}$. Write $\bar{S} = \sum_{i=1}^{n} \bar{R}\bar{s}_i$ for some $s_i \in S$. Then

$$S \subset \sum Rs_i + I \subset \sum Rs_i + R1 \subseteq S.$$

Hence, $S = Rs_1 + \ldots + Rs_n + R1$.

A version of Theorem 2(a) can now be established for rings with zero divisors.

Proposition 4. *Let (R, T) be a module-finite pair and assume that $I = (R \underset{R}{:} T) = (0)$. Then R/P is a Noetherian domain for each prime ideal P of R.*

Proof. For a fixed $P \in \operatorname{Spec}(R)$, let $S = R + PT$. By Lemma 1 $(S \underset{S}{:} T) = PT$, and by Lemma 3 $(S/PT, T/PT)$ is a module-finite pair. The conductor of S/PT in T/PT is (0), and S/PT is an integral domain. Applying Theorem 2(a) we see that $S/PT \cong R/P$ is a Noetherian domain.

If the ring R of Proposition 4 has only finitely many minimal prime divisors of (0), then $R/N(R)$ is a Noetherian ring ($N(R)$ = nilradical of R). Thus, if R is also reduced, then it is Noetherian.

Remarks

(a) There exist "non-coherent" module-finite pairs. For a particular example, let $k \subsetneq K$ be an extension of fields such that $[K : k] < \infty$. Let V be a valuation ring of the form

$K+M$ such that M is not finitely generated. Then, $R=k+M$ is not coherent (**5**, Theorem 3), but (R, V) is a module-finite pair. With respect to statement (b) of Theorem 2, it is interesting to note that $R/(R \underset{R}{:} V)$ is trivially Noetherian, yet $(R \underset{R}{:} V)=M$ is not finitely generated.

(b) In an attempt to determine general contexts for which the proof of Theorem 2(a) applies, we are naturally led to the following question: If $R \subseteq T$ is a module-finite extension of commutative rings and $(R \underset{R}{:} T)=(0)$, then does there exist an element $t \in T \backslash R$ such that $R[t]$ is a free R-module with basis $\{1, t, \ldots, t^{n-1}\}$, $n \geqq 2$? In general the answer is no, and we are thankful to Wolmer V. Vasconcelos for suggesting an appropriate counterexample.

Let (R, M) be a local non-Noetherian ring such that M is finitely generated, $M=Z(R)$ (zero divisors of R), and $\mathrm{Ann}_R(M)=(0)$ (**9**, Section 3). Set $T=R(+)M$, the idealization of M in R (**7**, Exercise 7, p. 63). We claim that $R \subsetneq T$ is the desired example. It is straightforward to see that T is module-finite over R, since M is finitely generated, and that $(R \underset{R}{:} T)=(0)$, since $\mathrm{Ann}_R(M)=(0)$. To complete the proof of the claim we will show that there does not exist a finitely generated free R-module S, such that S is a ring and $R \subsetneq S \subseteq T$. Assume such an S exists, and let 1, α be distinct R-linearly independent elements of S. Note that $\alpha=(r, m)$ for some $r \in R$, $m \in M$. Choose $0 \neq b \in R$ such that $bm=0$, and observe that $(br)1-b\alpha=0$, which is a contradiction.

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