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Beltrami–Bernoulli equilibria in weakly rotating self-gravitating fluid

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In the presence of a stationary gravitomagnetic field, a weakly rotating self-gravitating fluid relaxes into equilibrium flow configurations which admit both large- and small-scale structures. Unlike the traditional Beltrami equilibria in fluid, the equilibrium states in a rotating self-gravitating fluid in a gravitomagnetic field have structure similar to that of double curl Beltrami equilibria in plasma where the generalized vortical lines are aligned with the flow field. Different equilibrium flow configurations in the rotating fluid can be distinguished by the ratio between total energy and helicity. However, these fluid equilibria do not exhibit diamagnetic behaviours as observed in multi-species plasma equilibria.

Key words: astrophysical plasmas, plasma properties

1. Introduction

Vorticity is a measure of the rotation of the velocity field at any point in a fluid. It is defined as the curl of the fluid velocity field $\boldsymbol{v}$:

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v}. \tag{1.1}$$

Vortex sizes in a fluid can range from thousands of light-years (galaxies) to a few hundred metres (tornadoes) in nature and are often observed to self-organize into equilibrium configurations (Nitsche 2006). The presence of these ordered vortex structures in fluid and plasma is quite important in understanding the formation and evolution of galaxies, accretion discs, stars, etc. (Brahic 1982; Acosta-Pulido *et al.* 1990; Abramowicz *et al.* 1992; Shapiro 1996; Klahr & Bodenheimer 2003; Porter, Jones & Ryu 2015; Jelic-Cizmek *et al.* 2018). One particularly important class of equilibria in an ideal fluid can be identified with the Beltrami flow and expressed as

$$\boldsymbol{\omega} = \alpha \boldsymbol{v}, \tag{1.2}$$

where α is an arbitrary function. When α is taken as a constant, it is also known as Trkalian flow.

An analogue of this condition can be derived in the context of force-free single-fluid magnetohydrodynamics when the flow velocity $\boldsymbol{v}$ is replaced by magnetic field $\boldsymbol{B}$ (Woltjer 1958):

$$\nabla \times \boldsymbol{B} = \alpha \boldsymbol{B}, \tag{1.3}$$

where α is a scalar field and must satisfy $\boldsymbol{B} \cdot \nabla \alpha = 0$. This state, first discussed by Woltjer and Taylor, has been successful in modelling fusion and astrophysical plasmas (Woltjer

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1958; Taylor 1974). It can be derived by the minimization of the magnetic field energy

$$E = \int \frac{B^2}{8\pi}\,\mathrm{d}V, \tag{1.4}$$

with magnetic helicity ($\boldsymbol{A}$ being the magnetic vector potential)

$$h = \int \boldsymbol{A}\boldsymbol{\cdot}\boldsymbol{B}\,\mathrm{d}V \tag{1.5}$$

as a global constraint which leads to the ‘relaxed state’ with spatially homogeneous α, also known as constant-α-Beltrami field.

Later, this state was extended to multi-species plasmas where the energy now consists of both kinetic and magnetic parts (Mahajan & Yoshida 1998; Steinhauer & Ishida 1998; Yoshida & Mahajan 2002):

$$E = \int \left(\Sigma_i \frac{1}{2}\rho v_i^2 + \frac{B^2}{8\pi}\right)\mathrm{d}V, \tag{1.6}$$

with a generalized helicity, not the magnetic helicity,

$$H_i = \int \boldsymbol{P}_i\boldsymbol{\cdot}\boldsymbol{\varOmega}_i\,\mathrm{d}V \tag{1.7}$$

playing the role of constraint in the minimization, where the new canonical momentum $\boldsymbol{P} = \boldsymbol{A} + mc/q\,\boldsymbol{v}$ and its curl $\boldsymbol{\varOmega} = \nabla\times\boldsymbol{P} = \boldsymbol{B} + mc/q\,\nabla\times\boldsymbol{v}$. Here, plasma density $\rho = mn$, with m the mass of species, n the number density, q the charge, c the speed of light and i the individual plasma species. The collinearity condition, now defined as the generalized Beltrami condition, can be succinctly written as

$$\boldsymbol{\varOmega}_i = \mu_i\boldsymbol{v}_i, \tag{1.8}$$

where μ_i is the Lagrange multiplier. The generalized Beltrami condition implies an alignment of generalized vorticity and flow field in a multi-species charged fluid. To fully describe such equilibrium states, the Beltrami condition must be supplemented by the appropriate Bernoulli condition which indicates homogeneity in energy distribution in the plasma. These equilibrium states are called Beltrami–Bernoulli states, and are usually characterized by the number of independent single Beltrami systems needed to construct them.

In this paper, we explore the possibility of the formation of Beltrami–Bernoulli states in a weakly rotating self-gravitating neutral (uncharged) fluid. The significance of rotation within the context of frame-dependent effects such as Coriolis force in vortical fluid dynamics has been explored in the stellar structure, turbulence, zonal flows, dynamos, etc. (Singh & Singh 1984; Hopfinger & Van Heijst 1993; Shukla & Stenflo 2003; Itoh *et al.* 2006; González, Costa & Santini 2010; Shatashvili & Yoshida 2011; González 2014). Here, we consider the effect of fluid rotation in the context of the weak-field limit of general relativity, where the rotation of a self-gravitating fluid can twist the background space–time surrounding it, also known as frame dragging. This frame-dragging phenomenon can be identified with a magnetic-type gravitational field, i.e. gravitomagnetic field, and its governing equations can be derived by taking the weak-field, slow-velocity limit of Einstein’s equation. As moving charges create a magnetic field, the gravitomagnetic field exists due to the mass currents in the rotating

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fluid. The role of gravitomagnetic field in various astrophysical phenomena such as jet collimation, pulsar beam precession, gyroscope precession, vorticity generation, etc., has been explored in detail (Bardeen & Petterson 1975; Nelson & Papaloizou 2000; Lei, Zhang & Gao 2012; McKinney, Tchekhovskoy & Blandford 2013; Nealon, Price & Nixon 2015; Krishnan *et al.* 2020; Bhattacharjee & Stark 2021). By defining generalized vorticity in the rotating fluid as a combination of flow vorticity and gravitomagnetic field, we obtain equilibrium flow configurations which are similar to a unique class of Beltrami–Bernoulli states known as double curl Beltrami states.

We present a brief overview of the Einstein–Maxwell equation based on the analogue of the electromagnetic Maxwell equation. Then, we construct the generalized vortical dynamics of a rotating fluid followed by an analysis of the equilibrium solution of the vorticity transport equation. We compare our results with the plasma equilibrium states and delineate features that are unique to uncharged self-gravitating rotating fluid. Next, we present an analysis of helicity and energy, which is helpful in understanding the physical meaning of the Lagrange multiplier. Finally, we discuss the limiting cases of these states, possible implications and future work.

2. Einstein–Maxwell equation

To study the gravitational dynamics of a weakly rotating self-gravitating fluid, we need to explore the linearized limit of the Einstein equation. When the space–time metric is almost Minskowskian, $g_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}$, and terms of $\mathcal{O}(c^{-4})$ or higher are neglected, we can write the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{16\pi G}{c^4} T_{\mu\nu} \tag{2.1}$$

as a set of linearized equations almost identical to the electromagnetic Maxwell equation as (Braginsky, Caves & Thorne 1977; Thorne 1988; Manfredi 2015)

$$\nabla \cdot \boldsymbol{E}_g = -4\pi\rho, \tag{2.2}$$

$$\nabla \times \boldsymbol{E}_g = 0, \tag{2.3}$$

$$\nabla \cdot \boldsymbol{B}_g = 0, \tag{2.4}$$

$$\nabla \times \boldsymbol{B}_g = -\frac{16\pi G}{c} \rho \boldsymbol{v} + \frac{1}{c} \frac{\partial \boldsymbol{E}_g}{\partial t}, \tag{2.5}$$

where $T_{\mu\nu}$ is the stress energy tensor, $\boldsymbol{E}_g = -\nabla\phi$ is the Newtonian gravitational field, ϕ is the gravitational potential and $\boldsymbol{B}_g = \nabla \times \boldsymbol{A}_g$ is the gravitomagnetic field, with $\boldsymbol{A}_g$ being the corresponding vector potential. Also, G is the gravitational constant and ρ is the matter density. It should be noted that the right-hand side of (2.3) does not have any $\partial \boldsymbol{B}_g/\partial t$ to this order (Thorne 1988).

3. Vortical fluid dynamics

In the weak-field, slow-velocity limit of general relativity, the Euler hydrodynamics equation for a rotating ideal fluid can be written as (Thorne, Price & MacDonald 1986)

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{\nabla p}{\rho} + \boldsymbol{E}_g + \left(\boldsymbol{v} \times \frac{\boldsymbol{B}_g}{c}\right), \tag{3.1}$$

where $\mathrm{d}/\mathrm{d}t = (\partial/\partial t + \boldsymbol{v} \cdot \nabla)$, $\boldsymbol{v}$ is the fluid velocity and p is the fluid pressure.

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The continuity equation for the corresponding fluid is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0. \tag{3.2}$$

Now using the vector identity $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \nabla(v^2/2) - \boldsymbol{v} \times (\nabla \times \boldsymbol{v})$ and the expressions $E_g = -\nabla\phi$ and $\boldsymbol{B}_g = \nabla \times \boldsymbol{A}_g$, we can rewrite (3.1) as

$$\frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{v} \times \left(\nabla \times \boldsymbol{v} + \frac{\boldsymbol{B}_g}{c}\right) - \nabla\left(\frac{p}{\rho} + \frac{v^2}{2} + \phi\right), \tag{3.3}$$

where we have assumed a barotropic equation of state for pressure, i.e. $p(\rho)$. Since the vector potential $\boldsymbol{A}_g$ does not depend on time, we can rewrite (3.3) as

$$\frac{\partial}{\partial t}\left(\boldsymbol{v} + \frac{\boldsymbol{A}_g}{c}\right) = \boldsymbol{v} \times \left(\nabla \times \boldsymbol{v} + \frac{\boldsymbol{B}_g}{c}\right) - \nabla\Phi, \tag{3.4}$$

where we identify the quantity $\boldsymbol{P}_g = (\boldsymbol{v} + \boldsymbol{A}_g/c)$ as a new canonical momentum and Φ contains all the potentials for the gradient forces.

Now, we take the curl of (3.4) and obtain the vorticity transport equation:

$$\frac{\partial}{\partial t}\left(\nabla \times \boldsymbol{v} + \frac{\boldsymbol{B}_g}{c}\right) - \nabla \times \boldsymbol{v} \times \left(\nabla \times \boldsymbol{v} + \frac{\boldsymbol{B}_g}{c}\right) = 0, \tag{3.5}$$

where we identify $\boldsymbol{\Omega}_g = \nabla \times \boldsymbol{P}_g = \nabla \times \boldsymbol{v} + \boldsymbol{B}_g/c$ as the generalized vorticity.

It should be noted here that (3.5) does not contain any source terms, which implies that if the vorticity is zero at any time, it remains so for all times in an ideal barotropic fluid.

4. Equilibrium state

In this paper, we are interested in large-scale equilibrium structures in a self-gravitating and weakly rotating fluid. The stationary solution of (3.5) can be written as

$$\boldsymbol{\Omega}_g \equiv \nabla \times \boldsymbol{v} + \boldsymbol{\omega}_g = \mu \frac{16\pi G\rho}{c^2}\boldsymbol{v}, \tag{4.1}$$

where we have defined $\boldsymbol{\omega}_g = \boldsymbol{B}_g/c$ and satisfies the requirement of vanishing divergence of generalized vorticity and time-independent continuity equation, i.e. $\nabla \cdot (\rho \boldsymbol{v}) = 0$.

To fully solve the equilibrium state, we need to supplement the Beltrami condition with the time-independent gravitomagnetic Ampère law rewritten in terms of the new quantity $\boldsymbol{\omega}_g$, which has the following form:

$$\nabla \times \boldsymbol{\omega}_g = -\frac{16\pi G\rho}{c^2}\boldsymbol{v}. \tag{4.2}$$

The separation constant μ in (4.1) can be identified as the Lagrange multiplier when the Beltrami condition is derived via the variational principle.

For (4.1) to be defined as the stationary solution of (3.5), we need to impose the Bernoulli constraint which is an expression of the balance of all remaining potential forces, i.e. $\nabla\Phi = 0$.

Combining (4.1) and (4.2), we obtain the following equation:

$$\nabla \times \nabla \times \boldsymbol{v} - 4\boldsymbol{v} = \frac{\mu}{\tilde{\lambda}_J}\nabla \times \boldsymbol{v}, \tag{4.3}$$

where we have normalized $|\nabla|$ to the inverse of skin depth $\tilde{\lambda}_J = \tilde{\alpha}\lambda_J$ with Jean's length $\lambda_J = c_{s0}/\omega_J$, Jean's frequency $\omega_J = (4\pi G\rho_0\hat{\rho})^{1/2}$ and $\tilde{\alpha} = c/c_{s0}$. Here we have defined

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sound speed c_{s0} in terms of some ambient mass density ρ_0 and $\hat{\rho}$ is the density envelope which we take to be a constant of order unity for the rest of this paper. Equation (4.3) is known as the double curl Beltrami equation and has been studied thoroughly in the context of Hall magnetohydrodynamics (Mahajan & Yoshida 1998; Mahajan *et al.* 2001; Ohsaki *et al.* 2001).

Next, (4.3) is written as

$$(\nabla \times -\lambda_+)(\nabla \times -\lambda_-)\boldsymbol{v} = 0, \tag{4.4}$$

where

$$\lambda_\pm = \frac{1}{2}\left[\frac{\mu}{\tilde{\lambda}_J} \pm \left[\left(\frac{\mu}{\tilde{\lambda}_J}\right)^2 + 16\right]^{1/2}\right]. \tag{4.5}$$

It should also be noted that (4.4) is the combination of two Beltrami fields $\mathbb{G}_+$ and $\mathbb{G}_-$, i.e.

$$\nabla \times \mathbb{G}_\pm = \lambda_\pm \mathbb{G}_\pm, \tag{4.6}$$

with the final solution

$$\boldsymbol{v} = C_+\mathbb{G}_+ + C_-\mathbb{G}_-, \tag{4.7}$$

where the constant amplitudes $C_\pm$ are determined from initial conditions and characterize the double curl Beltrami states along with their corresponding eigenvalues $\lambda_\pm$. The explicit solution for the Beltrami condition in (4.6) is provided by the Chandrashekar–Kendall function in cylindrical coordinates whereas it takes the form of the Arnold–Beltrami–Childress solution in Cartesian coordinates (Chandrasekhar & Kendall 1957).

One can obtain the solution for gravitomagnetic field by using (4.1) which can be written as follows:

$$\boldsymbol{\omega}_g = \left(\frac{\mu}{\tilde{\lambda}_J^2} - \frac{\lambda_+}{\tilde{\lambda}_J}\right) C_+\mathbb{G}_+ \left(\frac{\mu}{\tilde{\lambda}_J^2} - \frac{\lambda_-}{\tilde{\lambda}_J}\right) C_-\mathbb{G}_-. \tag{4.8}$$

Finally, it should be emphasized here that the coupling between gravitomagnetic and flow fields has enabled us to uncover a far richer equilibrium structure in the fluid compared with the traditional Trkalian flow.

4.1. *Comparison with the double curl Beltrami states in plasmas*

Though double Beltrami states emerge in both plasma and self-gravitating rotating fluid, the physical characteristics of these states might not be the same in both types of systems. In this section, we study possible differences between the two systems. For a single-species dynamic charged fluid with constant density in an appropriate neutralizing background, the generalized Beltrami condition can be written as

$$\boldsymbol{\Omega} \equiv \boldsymbol{B} + \frac{mc}{q}\nabla \times \boldsymbol{v} = \frac{4\pi nq}{c}\boldsymbol{v}, \tag{4.9}$$

which, combined with the Ampère law, gives us the plasma counterpart of (4.3) (Mahajan 2008):

$$\nabla \times \nabla \times \boldsymbol{v} + \boldsymbol{v} = \frac{\mu}{\lambda_s}\nabla \times \boldsymbol{v}, \tag{4.10}$$

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with the following roots:

$$\lambda_{\pm} = \frac{1}{2}\left[\frac{\mu}{\lambda_s} \pm \left[\left(\frac{\mu}{\lambda_s}\right)^2 - 4\right]^{1/2}\right], \tag{4.11}$$

where species skin depth $\lambda_s = c/\omega_p$ and plasma frequency $\omega_p = \sqrt{4\pi n q^2/m}$. The roots are real for $(\mu/\lambda_s)^2 > 4$ but form a complex conjugate pair when $(\mu/\lambda_s)^2 < 4$.

Apart from a factor of 4, there is a sign difference between the left-hand sides of (4.3) and (4.10) and the origin of this can be attributed to the fact that gravity is always attractive as reflected in (4.2). Contrary to the plasma equilibrium states, the roots $\lambda_{\pm}$ in (4.5) for double curl Beltrami states in self-gravitating fluid are always real. Moreover, the presence of an inherent length scale, i.e. Jean's length, has introduced a singular perturbation term $\nabla \times \nabla \times \boldsymbol{v}$ in (4.3). This implies equilibrium states in self-gravitating fluid are also endowed with two length scales in a gravitomagnetic field. This can have major consequences for the formation of large- and small-scale flow configurations in the fluid as demonstrated in charged fluid (Mahajan *et al.* 2001; Kagan & Mahajan 2010).

Next, if we set $\mu = 0$ and reverse the normalization of the gradients, (4.3) can be rewritten as

$$\nabla^2 \boldsymbol{v} = -\frac{\boldsymbol{v}}{\tilde{\lambda}_J^2}, \tag{4.12}$$

which can be compared to the corresponding limiting case of (4.10) in plasma:

$$\nabla^2 \boldsymbol{v} = \frac{\boldsymbol{v}}{\lambda_s^2}, \tag{4.13}$$

which are nothing but the $\boldsymbol{\Omega} = 0$ solution of the vorticity transport equation for both plasma and fluid.

Though (4.12) and (4.13) have a similar structure, the physics exhibited by the two systems is completely opposite. Equation (4.13) is the superconducting limit in a plasma with a skin depth λ_s beyond which the magnetic flux is completely expelled from the interior of the plasma; a complete antithesis to the behaviour of gravitomagnetic flux in a neutral fluid. Another way to interpret (4.12) and (4.13) is that electric current is restricted to the skin depth in a diamagnetic plasma equilibrium, whereas the mass current is not confined to Jean's length in a fluid equilibrium.

5. Helicity and energy

From our analysis in the previous section, we notice that the values of Lagrange multiplier μ determine the characteristics of equilibrium states in a fluid. First, we take the following definition of helicity as one of the invariants of the system:

$$H = \frac{c^2}{32\pi G}\int \mathrm{d}^3 x\, \boldsymbol{P}_g \cdot \boldsymbol{\Omega}_g, \tag{5.1}$$

which is a measure of different topological features of vortical field lines such as knottedness and twists (Moffatt 1969).

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The physical interpretation of the Lagrange multiplier can then be obtained by computing the helicity from (5.1):

$$\begin{aligned} H &= \frac{c^2}{32\pi G}\left\langle\left(\boldsymbol{v}+\frac{\boldsymbol{A}_g}{c}\right)\cdot\left(\nabla\times\boldsymbol{v}+\frac{\boldsymbol{B}_g}{c}\right)\right\rangle \\ &= \frac{c^2}{32\pi G}\left\langle\left(\boldsymbol{v}+\frac{\boldsymbol{A}_g}{c}\right)\cdot\frac{\mu 16\pi G\rho}{c^2}\right\rangle \\ &= \frac{\mu c^2}{32\pi G}\left\langle\frac{16\pi G\rho v^2}{c^2}-\frac{\boldsymbol{A}_g\cdot\nabla\times\boldsymbol{B}_g}{c^2}\right\rangle \\ &= \mu\left\langle\frac{1}{2}\rho v^2-\frac{B_g^2}{32\pi G}\right\rangle = \mu E, \end{aligned} \tag{5.2}$$

yielding an expression for μ in terms of two invariants of the motion:

$$\mu = \frac{H}{E}, \tag{5.3}$$

where $\langle\rangle = \int \mathrm{d}^3 x$. Therefore, the Lagrange multiplier is a measure of generalized helicity as a fraction of total energy. It should be noted that, gravity being an attractive force, the energy density of the gravitomagnetic field has a negative sign in front of it (Sebens 2020).

6. Discussion

In light of our results, we notice that the equilibrium flow configurations of a rotating fluid in a stationary gravitomagnetic field allow for small-scale structures due to the singular perturbation term $\nabla\times\nabla\times\boldsymbol{v}$. These small-scale structures can be distinguished by comparing the ratio between Jean's length and system size. Moreover, two invariants of the motion, i.e. generalized helicity H and energy E, emerge as the determinants of different classes of flow configurations in the fluid. The interaction and evolution of structures at different length scales can provide new insights into turbulence in accretion discs, differential rotation patterns in stars and jet formations in various astrophysical objects (Tobias, Dagon & Marston 2011).

One can obtain the traditional Beltrami flow if the system length $L \gg \tilde{\lambda}_J$ and (4.3) reduces to $\nabla\times\boldsymbol{v} = -(\mu^{-1})\boldsymbol{v}$. Similar to the double Beltrami equilibria in a plasma, the double curl Beltrami states in a fluid can terminate and relax into single curl equilibria. Furthermore, such termination can result in catastrophic events. These events can have potential effects in astrophysics such as stability of a rotating star, formation of large-scale structure in galaxies, excitation of different wave modes, collimation of ionizing radiation in an accretion disc, etc.

The next step in this scheme will be to explore the gravitomagnetic effects in plasma equilibria. In that case, one has to appropriately incorporate the coupling between gravity and electromagnetic fields in Maxwell's equation which is beyond the scope of this paper. Finally, the consequence of double curl Beltrami flow in weakly rotating stars and accretion discs will be explored in future work.

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Declaration of interests

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