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NON-EXTENDABILITY OF BOUNDED CONTINUOUS FUNCTIONS

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If X is a dense subspace of Y, much is known about the question of when every bounded continuous real-valued function on X extends to a continuous function on Y. Indeed, this is one of the central topics of [**5**]. In this paper we are interested in the opposite question: When are there continuous bounded real-valued functions on X which extend to no point of $Y - X$? (Of course, we cannot hope that every function on X fails to extend since the restrictions to X of continuous functions on Y extend to Y.) In this paper, we show that if Y is a compact metric space and if X is a dense subset of Y, then X admits a bounded continuous function which extends to no point of $Y - X$ if and only if X is completely metrizable. We also show that for certain spaces Y and dense subsets X, the set of bounded functions on X which extend to a point of $Y - X$ form a first category subset of $C^*(X)$. Several examples relevant to these results are given. Some combinatorial consequences (all of which were earlier proven by Hechler using different methods) of the construction of one of these examples are given. Furthermore, one of the examples is used to prove that if X is not pseudocompact, then there is a space Y which contains X as a proper dense subset and which has the following properties: (i) For each $p \in Y - X$, there is a continuous bounded function on X which does not extend to p, but (ii) for each continuous bounded function f on X, there is a point $p \in Y - X$ such that f extends to p. In the last section, we briefly discuss the υX analogue of the material in the earlier sections.

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1. Preliminaries. We will use the notation and conventions of [**5**]. In particular, all given spaces are assumed to be completely regular and Hausdorff. **N**, **Q**, and **R** denote the sets (or spaces) of natural numbers, rational numbers, and real numbers respectively. If X is a space, $C^*(X)$ is the Banach space of real-valued continuous functions on X endowed with the uniform norm $\|\cdot\|$. If X is dense in Y, $p \in Y$, and $f \in C^*(X)$, then the *oscillation* of f at p is defined by

$$\mathrm{osc}_f(p) = \inf \{\sup|f(x) - f(y)| : x, y \in U \cap X\}$$

where U ranges over all neighborhoods of p.

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If X is a dense subset of Y and $A \subseteq Y - X$, then if $f \in C(X)$, we say that f *extends to* A (or to p if $A = \{p\}$) if there is an $\hat{f} \in C(X \cup A)$ such that the restriction $\hat{f}|X$ of $\hat{f}$ to X is f. We note that for $p \in Y - X$, f extends to p if and only if $\operatorname{osc}_f(p) = 0$. If X is dense in Y, we write $Y \cap \beta X = X$ for the statement "for each $p \in Y - X$, there is an $f_p \in C^*(X)$ such that f does not extend to p." If a single element of $C^*(X)$ serves as every f_p, we write $Y \cap \beta X = X$ singly. Thus $Y \cap \beta X = X$ singly if and only if there is an element of $C^*(X)$ which extends to no point of $Y - X$. If $Y \cap \beta X = X$, but it is not the case that $Y \cap \beta X = X$ singly, then we say that $Y \cap \beta X = X$ multiply. For example, $\mathbf{R} \cap \beta\mathbf{Q} = \mathbf{Q}$ multiply. To see this we note that if $p \in \mathbf{R} - \mathbf{Q}$, the function $f_p \in C^*(\mathbf{Q})$ defined by

$$f_p(x) = \frac{x - p}{|x - p|}$$

does not extend to p, so $\mathbf{R} \cap \beta\mathbf{Q} = \mathbf{Q}$. But since every continuous function on a subset of a space extends to a function defined on a G_δ-set and since $\mathbf{Q}$ is not a G_δ-set of $\mathbf{R}$, every $f \in C^*(\mathbf{Q})$ extends to some point of $\mathbf{R} - \mathbf{Q}$. Thus it is not the case that $\mathbf{R} \cap \beta\mathbf{Q} = \mathbf{Q}$ singly.

We close this section with a simple and well-known proposition which was essentially used in the preceding example.

1.1. PROPOSITION. *If $Y \cap \beta X = X$ singly, then X is a G_δ-set in Y.*

Proof. It is well-known (see [**2**], for example) that every $f \in C^*(X)$ extends to a function $g \in C^*(G)$ where G is a G_δ-set in Y. Therefore, if $Y \cap \beta X = X$ singly, $G = X$, that is, X is a G_δ-set of Y.

2. Separable metric spaces. In this section, we prove the converse of 1.1 for the case that Y is a compact metric space. We first handle the case where X is an open dense subset of Y.

2.1. LEMMA. *Suppose Y is a compact metric space. If X is a dense open subset of Y, then there is a continuous function $F: X \to [0, 1]$ such that for each $p \in Y - X$, $\operatorname{osc}_F(p) = 1$.*

Proof. For each $n \in \mathbf{N}$, let $\{S_{\frac{1}{n}}(y_k^n): k = 1, \ldots, r_n\}$ be a finite cover of $Y - X$ where $S_\epsilon(p)$ is the open ϵ-sphere in Y centered at p. For $k = 1, \ldots, r_1$, choose $x_k^1 \in S_1(y_k^1 \cap X)$. If x_k^m is chosen for $m < n$ and $k = 1, \ldots, r_m$, choose

$$x_k^n \in (S_{1_n}(y_k^n) - \{x_k^m: m < n, 1 \leqq k \leqq r_m\}) \cap X.$$

Let

$$D = \{x_k^n: n \in \mathbf{N}, k = 1, \ldots, r_n\}.$$

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Then it is easy to show that D is a closed discrete subset of X and that $\mathrm{Cl}_Y D - D = Y - X$. Let $f: D \to [0, 1]$ be defined by

$$f(x_k^n) = \begin{cases} 0 \text{ if } n \text{ is even} \\ 1 \text{ if } n \text{ is odd} \end{cases}.$$

Since X is normal, f extends to a continuous function $F: X \to [0, 1]$. If $p \in Y - X$, then $\mathrm{osc}_F(p) = 1$. To see this, let V be an open neighborhood of p. Then

$$V \supseteq \{p\} = \bigcap \{\mathrm{Cl}_Y S_{1/2n}(y_k^{2n}) : p \in S_{1/2n}(y_k^{2n})\}$$

so there is an even n_0 and a k_0 such that

$$S_{1/n_0}(y_{k_0}^{n_0}) \subseteq V.$$

Similarly there is an odd n_1 and a k_1 such that

$$S_{1/n_1}(y_{k_1}^{n_1}) \subseteq V.$$

Then

$$F(x_{k_0}^{n_0}) = 0 \text{ and } F(x_{k_1}^{n_1}) = 1,$$

so F assumes both the values 0 and 1 in V. But V is an arbitrary neighborhood of p so $\mathrm{osc}_F(p) = 1$.

Remark. The construction of D in the proof of 2.1 is basically just the well-known proof that every nowhere dense subset of a metric space is contained in the closure of a discrete subset of the metric space.

2.2. LEMMA. *Suppose X is a dense subset of the space Y, and that $X = \bigcap_{n=1}^{\infty} U_n$ where $\{U_n\}_{n=1}^{\infty}$ is a decreasing sequence of open sets. If for each n, there is a continuous $f_n: U_n \to [0, 1]$ such that $\mathrm{osc}_{f_n}(p) = 1$ for each $p \in Y - U_n$, then $Y \cap \beta X = X$ singly.*

Proof. We may assume that $U_1 = X$. Let $f: X \to [0, 1]$ be defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{f_n(x)}{3^n}.$$

Then $f \in C^*(X)$. We claim that if $p \in Y - X$, then $\mathrm{osc}_f(p) > 0$. Suppose $p \in U_{n_0} - U_{n_0+1}$ and $\hat{V}$ is a neighborhood of p. Each of the functions $f_1, \ldots, f_{n_0}$ is continuous on U_{n_0} so $\hat{V}$ contains a neighborhood V of p such that for each $x, y \in V$,

$$(*) \quad \sum_{k=1}^{n_0} \frac{|f_k(x) - f_k(y)|}{3^k} < \frac{1}{8 \cdot 3^{n_0+1}}.$$

Now choose $x, y \in V \cap X$ such that $f_{n_0+1}(x) \leqq 1/8$, $f_{n_0+1}(y) \geqq 7/8$.

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Now

$$
\begin{aligned}
f(x) &= \sum_{k=1}^{\infty} \frac{f_k(x)}{3^k} = \sum_{k=1}^{n_0} \frac{f_k(x)}{3^k} + \frac{f_{n_0+1}(x)}{3^{n_0+1}} + \sum_{k=n_0+2}^{\infty} \frac{f_k(x)}{3^k} \\
&\leqq \sum_{k=1}^{n_0} \frac{f_k(x)}{3^k} + \frac{f_{n_0+1}(x)}{3^{n_0+1}} + \sum_{k=n_0+2}^{\infty} \frac{1}{3^k} \\
&\leqq \sum_{k=1}^{n_0} \frac{f_k(x)}{3^k} + \frac{1}{8 \cdot 3^{n_0+1}} + \frac{1}{2 \cdot 3^{n_0+1}} \\
&= \sum_{k=1}^{n_0} \frac{f_k(x)}{3^k} + \frac{5}{8 \cdot 3^{n_0+1}}.
\end{aligned}
$$

Also

$$
\begin{aligned}
f(y) &= \sum_{k=1}^{\infty} \frac{f_k(y)}{3^k} = \sum_{k=1}^{n_0} \frac{f_k(y)}{3^k} + \frac{f_{n_0+1}(y)}{3^{n_0+1}} + \sum_{k=n_0+2}^{\infty} \frac{f_k(y)}{3^k} \\
&\geqq \sum_{k=1}^{n_0} \frac{f_k(y)}{3^k} + \frac{7}{8 \cdot 3^{n_0+1}}.
\end{aligned}
$$

Hence

$$
f(y) - f(x) \geqq \sum_{k=1}^{n_0} \frac{f_k(y) - f_k(x)}{3^k} + \frac{1}{4 \cdot 3^{n_0+1}}
$$

and so by (*)

$$
f(y) - f(x) \geqq \frac{1}{8 \cdot 3^{n_0+1}}.
$$

Thus, any neighborhood of p contains elements x and y of X such that $|f(x) - f(y)| \geqq 1/8 \cdot 3^{n_0+1}$. Therefore, $\operatorname{osc}_f(p) > 0$ so f cannot be extended to p. But p is an arbitrary element of $Y - X$, and so $Y \cap \beta X = X$ singly.

Remark. A consequence of 2.2 is that if $Y \cap \beta X = X$ and $Y - X$ is countable, then $Y \cap \beta X = X$ singly. In Section 3 we give an entirely different proof of this result. In fact we show that in this case the elements of $C^*(X)$ which extend beyond X are very rare.

2.3. PROPOSITION. (i) *If Y is a compact metric space and X is a dense subspace, then $Y \cap \beta X = X$ singly if and only if X is completely metrizable.*

(ii) *If X is a separable metric space, then X is completely metrizable if and only if $Y \cap \beta X = X$ singly for every compact metric space Y such that X is a dense subspace of Y.*

Proof. Combine 1.1, 2.1, 2.2, and the fact that complete metric spaces coincide with absolute G_δ metric spaces.

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Remarks. (i) We note that 2.3 does not state that $Y - X$ is the set of discontinuities of a bounded function $f: Y \to \mathbf{R}$. The characterization of sets of points of discontinuities of functions on compact metric spaces is given, for example, in [**1**]. For the non-metric case, there are compact spaces Y and open dense subsets X such that $Y \cap \beta X \neq X$ but such that $Y - X$ is the set of points of discontinuities of a function on Y (just take $X = [0, \omega_1)$, $Y = [0, \omega_1]$). The characteristic function of X is continuous exactly on X, but $\beta X = Y$.

(ii) Proposition 2.3 (i) does not hold if Y is not metrizable even if X is separable and open in X. To see this, let $Y = \beta\mathbf{N}$ and $X = \beta\mathbf{N} - \{p\}$ where $p \in \beta\mathbf{N} - \mathbf{N}$. Then X is C^*-embedded in Y and so $Y \cap \beta X \neq X$. In Section 4 an example is given of a compactification Y of $\mathbf{N}$ such that $Y \cap \beta\mathbf{N} = \mathbf{N}$ multiply.

(iii) The referee has pointed out that the following is proven in [**4**], 3.10(2): If Y is locally connected and X is a dense intersection of countably many cozero sets of Y, then $Y \cap \beta X = X$ singly.

2.4. COROLLARY. *Suppose Y is a separable metric space and X is a completely metrizable dense subset of Y. Then $Y \cap \beta X = X$ singly.*

Proof. Y has a metric compactification Y^*. Since X is completely metrizable, X is a G_δ-set in Y^*. By 2.3, $Y^* \cap \beta X = X$ singly. Therefore, $Y \cap \beta X = X$ singly.

3. Spaces with countable remainders. In this section we show that if $Y \cap \beta X = X$ and $Y - X$ is countable, then the set of elements of $C^*(X)$ which extend to some point of $Y - X$ forms a first category subset of $C^*(X)$. The technique we use is much easier than the methods in Section 2 and gives more functions which fail to extend.

The following lemma was proved by M. Rice in a private conversation.

3.1. LEMMA. *Suppose X is a dense subset of Y. Let*

$$A(X) = \{f \in C^*(X): f \text{ extends to } Y - X\}.$$

If $A(X) \neq C^*(X)$, then $A(X)$ is nowhere dense in $C^*(X)$.

Proof. We first show that $A(X)$ is closed in $C^*(X)$. If $f_n \in A(X)$ for each $n \in \mathbf{N}$ and $f_n \to f \in C^*(X)$, let $\hat{f}_n \in C^*(Y)$ be an extension of f_n for each n. Then the sequence $\{\hat{f}_n\}$ converges to an element $\hat{f}$ of $C^*(Y)$. $\hat{f}$ is an extension of f so $f \in A(X)$. This shows that $A(X)$ is closed in $C^*(X)$. Now suppose $\operatorname{Int} A(X) \neq \emptyset$. Then there is an $f \in A(X)$ and $\epsilon > 0$ such that the open sphere $S_\epsilon(f)$ of $C^*(X)$ is a subset of $A(X)$. Choose $g \in C^*(X)$. Since g is bounded, there is a $\delta > 0$ such that $\|\delta g\| < \epsilon$. Then

$$\delta g + f \in S_\epsilon(f) \subseteq A(X),$$

so there is an extension $h \in C^*(X)$ of $\delta g + f$. If $\hat{f} \in C^*(Y)$ is the extension of f, the function $\hat{g} = (h - \hat{f})/\delta$ is an element of $C^*(Y)$ and if $x \in X$,

$$\hat{g}(x) = (h(x) - \hat{f}(x))/\delta = (\delta g(x) + f(x) - f(x))/\delta = g(x),$$

so $\hat{g}$ is an extension of g. Therefore, $g \in A(X)$. But g is arbitrary. Hence, if $\operatorname{Int} A(X) \neq \emptyset$, then $C^*(X) = A(X)$.

It is not difficult to show that if $Y \cap \beta X = X$ singly, then $\{f \in C^*(X): f$ extends to a point of $Y - X\}$ has dense complement in $C^*(Y)$. The next proposition shows that if $Y - X$ is countable, then this set is a first category subset of the Banach space $C^*(X)$.

3.2. PROPOSITION. *Suppose $Y - X$ is countable and $Y \cap \beta X = X$. Then $Y \cap \beta X = X$ singly. Furthermore, $\{f \in C^*(X)$: f extends to some element of $Y - X\}$ is a meager subset of $C^*(X)$.*

Proof. Write $Y - X = \{y_k: k \in \mathbf{N}\}$. For each $k \in \mathbf{N}$, let

$$F_k = \{f \in C^*(X): f \text{ extends to } y_k\}.$$

Since $Y \cap \beta X = X$, $F_k \neq C^*(X)$ for each k. Hence, by 3.1, each F_k is nowhere dense in $C^*(X)$. By the Baire Category Theorem,

$$C^*(X) - \bigcup_{k=1}^{\infty} F_k \neq \emptyset.$$

Therefore, $Y \cap \beta X = X$ singly.

3.3. COROLLARY. *If $Y \cap \beta X = X$, then $(X \cup A) \cap \beta X = X$ singly for every countable subset A of Y.*

4. Some examples. In this section we present several applications and examples relevant to the material of Sections 2 and 3. One of the examples (4.2) will then be used in Section 5.

4.1. *Example.* $\beta\mathbf{Q} \cap \beta(\beta\mathbf{Q} - \mathbf{Q}) = \beta\mathbf{Q} - \mathbf{Q}$ singly. In fact, the set of elements of $C^*(\beta\mathbf{Q} - \mathbf{Q})$ which extend to any point of $\mathbf{Q}$ is a first category subset of $C^*(\beta\mathbf{Q} - \mathbf{Q})$. This is immediate from the fact that $\beta\mathbf{Q} \cap \beta(\beta\mathbf{Q} - \mathbf{Q}) = \beta\mathbf{Q} - \mathbf{Q}$ ([**5**], 60.4) and Proposition 3.2.

We will need the following theorem of Magill to construct the next example.

THEOREM. ([**7**]). *Suppose A is locally compact and $f: (\beta A - A) \to C$ is a continuous surjection. Then the quotient topology on $A \cup C$ induced by the map $g: \beta A \to A \cup C$ defined by*

$$g(x) = \begin{cases} x & \text{if } x \in A \\ f(x) & \text{if } x \in \beta A - A \end{cases} \quad \textit{is (compact) Hausdorff.}$$

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4.2. *Example.* There is a scattered compactification $B\mathbf{N}$ of $\mathbf{N}$, such that $B\mathbf{N} \cap \beta\mathbf{N} = \mathbf{N}$ multiply. This should be compared to 2.3. The compactification $B\mathbf{N}$ will be constructed as a quotient of $\beta\mathbf{N}$ using Magill's theorem. Let $\{C_f: f \in C^*(\mathbf{N})\}$ be a family of pairwise disjoint non-empty open-and-closed subsets of $\beta\mathbf{N} - \mathbf{N}$. (We can index any family of $2^{\aleph_0}$ pairwise disjoint clopen subsets of $\beta\mathbf{N} - \mathbf{N}$ by $C^*(\mathbf{N})$ since $|C^*(\mathbf{N})| = 2^{\aleph_0}$.) For each $f \in C^*(\mathbf{N})$ let K_f be a proper non-empty open-and-closed subset of C_f such that the restriction of f to K_f is constant. Let

$$L = (\beta\mathbf{N} - \mathbf{N}) - \bigcup_{f \in C^*(\mathbf{N})} K_f.$$

Let B be the quotient of $\beta\mathbf{N} - \mathbf{N}$ obtained by identifying L to a single point l and each K_f to a single point k_f. Let $B\mathbf{N}$ be the induced quotient of $\beta\mathbf{N}$; that is, $B\mathbf{N}$ is obtained from $\beta\mathbf{N}$ by identifying L and each K_f to a single point. The space B is Hausdorff; in fact, it is the one point compactification of the discrete space of cardinal $2^{\aleph_0}$. Hence, by Magill's theorem, $B\mathbf{N}$ is a compact Hausdorff space. Clearly, $\mathbf{N}$ is dense in $B\mathbf{N}$. We first show that $B\mathbf{N} \cap \beta\mathbf{N} = \mathbf{N}$. Suppose $y \in B\mathbf{N} - \mathbf{N}$ $(=B)$. Then y is obtained from $\beta\mathbf{N} - \mathbf{N}$ identifying a set S to a point; S is either L or K_f for some f. Let p_0, p_1 be distinct elements of S. Let $g \in C^*(\beta\mathbf{N})$ be a function such that $g(p_0) = 0$, $g(p_1) = 1$. If $\hat{g}$ is the restriction of g to $\mathbf{N}$, then $\hat{g}$ does not extend to y. Hence $B\mathbf{N} \cap \beta\mathbf{N} = \mathbf{N}$. On the other hand, if $f \in C^*(\mathbf{N})$, the extension f^β of f to $\beta\mathbf{N}$ is constant on K_f, so we may define $\hat{f}:\mathbf{N} \cup \{k_f\} \to [0, 1]$ by

$$f(x) = \begin{cases} f(x) & \text{if } x \in \mathbf{N} \\ f^\beta(p) & \text{if } x = k_f \text{ and } p \in K_f. \end{cases}$$

Then $\hat{f} \in C^*(N \cup \{k_f\})$ and $\hat{f}$ is an extension of f. Therefore, it is not the case that $B\mathbf{N} \cap \beta\mathbf{N} = \mathbf{N}$ singly.

4.3. *Example.* If D is an infinite discrete space, D has a scattered compactification BD such that $BD \cap \beta D = D$ multiply.

Proof. If $D = \mathbf{N}$, we may take BD to be the space $B\mathbf{N}$ of Example 4.2. For the general case, write D as a disjoint union $\bigcup_{\lambda \in \Lambda} N_\lambda$ where each N_λ is a copy of $\mathbf{N}$. Let BD be the one-point compactification of the topological union $\bigcup_{\lambda \in \Lambda} BN_\lambda$. Then clearly $BD \cap \beta D = D$ multiply.

Remark. The construction of 4.3 actually gives that each $f \in C^*(D)$ extends to a point p_f which is an isolated point of $BD - D$. Since $BD - D$ has $2^{\aleph_0} \cdot |D|$ isolated points, and since isolated points of $BD - D$ correspond to open-and-closed subsets of $\beta D - D$, we get the following:

4.5. Corollary. *Suppose D is the discrete space of infinite cardinal γ. Then there is a collection $\mathscr{C}$ of $\gamma \cdot 2^{\aleph_0}$ pairwise disjoint non-empty open-and-closed subsets of $\beta D - D$ such that for each $f \in C^*(\beta D)$, there is a $C_f \in \mathscr{C}$ such that the restriction of f to C_f is constant.*

5. Some combinatorial consequences. In this section we use the technique of Example 4.3 to obtain some combinatorial results which were first obtained by Hechler [**6**] by entirely different methods (see also [**3**].) If A and B are subsets of $\mathbf{N}$, then $A \subseteq^* B$ (A is almost contained in B) if $A - B$ is finite. If $\mathscr{E}$ is an infinite family of infinite subsets of $\mathbf{N}$, then $\mathscr{E}$ is almost-disjoint if distinct elements of $\mathscr{E}$ have finite intersections. If $\mathscr{E}$ is a maximal almost-disjoint family, we denote by $\Psi_{\mathscr{E}}$ the version of Ψ constructed from the family $\mathscr{E}$ where Ψ is the space described in problem 51 of [**5**].

5.1. Lemma. *The following statements are equivalent for a maximal almost-disjoint family $\mathscr{E}$ of subsets of* $\mathbf{N}$:

(i) *If $f \in C^*(\mathbf{N})$ and $|f(\mathbf{N})| \leqq n$ (where $n \in \mathbf{N}$), then f extends to a point of $\Psi_{\mathscr{E}} - \mathbf{N}$.*

(ii) *For every partition $\{A_1, \ldots, A_n\}$ of $\mathbf{N}$, there is an $E \in \mathscr{E}$ and a $k \leqq n$ such that $E \subseteq^* A_k$.*

Proof. (i) $\Rightarrow$ (ii). Let $\{A_1, \ldots, A_n\}$ be a partition of $\mathbf{N}$ and define $f \in C^*(\mathbf{N})$ by $f|A_k = k$. By assumption f extends to a point $\omega_E \in \Psi_{\mathscr{E}} - \mathbf{N}$. Call the extension $\tilde{f}$. If $\tilde{f}(\omega_E) = k_0$, then since $E \cup \{\omega_E\}$ is the one-point compactification of E and since f assumes only finitely many values, $f|E$ must be k_0 except on a finite subset of E so $E \subseteq^* A_{k_0}$.

(ii) $\Rightarrow$ (i). If $f:\mathbf{N} \rightarrow \{1, \ldots, n\}$ is any function, let $A_k = f^{-1}(k)$ for $k = 1, \ldots, n$. By hypothesis, there is a $k \leqq n$ and an $E \in \mathscr{E}$ such that $E \subseteq^* A_k$. If $\tilde{f}:\mathbf{N} \cup \{\omega_E\} \rightarrow \mathbf{R}$ is defined by $\tilde{f}|\mathbf{N} = f$ and $\tilde{f}(\omega_E) = k$, then $\tilde{f}$ is a continuous extension of f.

For $A \subset N$, let $A^* = (\mathrm{Cl}_{\beta\mathbf{N}} A) - \mathbf{N}$. Let $\{K_f: f \in C^*(\mathbf{N})\}$ be as in 4.2 and expand this family to a maximal disjoint family of clopen subsets of $\beta\mathbf{N} - \mathbf{N}$. This maximal family can be written $\{E^*: E \in \mathscr{E}\}$ where $\mathscr{E}$ is a maximal almost-disjoint family of subsets of $\mathbf{N}$. We can obtain a compactification $\gamma\mathbf{N}$ of $\mathbf{N}$ from $\beta\mathbf{N}$ by identifying each set E^* to a point and $\beta\mathbf{N} - \mathbf{N} - \cup\{E^*: E \in \mathscr{E}\}$ to a point ∞. Then $\gamma\mathbf{N}$ still contains all of the points k_f and $\gamma\mathbf{N} - \{\infty\} = \Psi_{\mathscr{E}}$. Since for each $f \in C^*(\mathbf{N})$, f extends to a point of $\Psi_{\mathscr{E}} - \mathbf{N}$, we get, by 5.1, the following:

5.2. Theorem. ([**6**]). *There is a maximal almost-disjoint family $\mathscr{E}$ of subsets of $\mathbf{N}$ such that for every finite partition $\{A_1, \ldots, A_n\}$ of $\mathbf{N}$, there is an $E \in \mathscr{E}$ such that $E \subseteq^* A_k$ for some $k \leqq n$.*

Remark. It is easy to construct families $\mathscr{E}$ which do not have the property described in 5.2. Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be maximal almost-disjoint families of subsets of the even integers A_1 and of the odd integers A_2 respectively. If

$$\mathscr{E} = \{E_1 \cup E_2: E_i \in \mathscr{E}_i\},$$

then $\mathscr{E}$ is a maximal almost-disjoint family of subsets of $\mathbf{N}$ but it is not the case that for some $E \in \mathscr{E}$, either $E \subseteq^* A_1$ or $E \subseteq^* A_2$. A more interesting example of the failure of 5.2 for families $\mathscr{E}$ is given in 5.4.

5.3. PROPOSITION. *Suppose $n \in \mathbf{N}$. There is a compactification $\delta\mathbf{N}$ of $\mathbf{N}$ with the following properties:* (i) *For every $f \in C^*(\mathbf{N})$ such that $|f(\mathbf{N})| \leqq n$, there is a point $p_f \in \delta\mathbf{N} - \mathbf{N}$ such that f extends to p_f.* (ii) *There is a function $f_0 \in C^*(\mathbf{N})$ such that $|f_0(\mathbf{N})| = n + 1$ and f_0 extends to no point of $\delta\mathbf{N} - \mathbf{N}$. Furthermore, $\delta\mathbf{N}$ may be chosen to be the one-point compactification of a version of Ψ.*

Proof. Let $\{A_1, \ldots, A_{n+1}\}$ be any partition of $\mathbf{N}$ into pairwise disjoint infinite subsets. Define $f_0 \in C^*(\mathbf{N})$ by $f_0|A_k = k$. Write each set A_k^* as $B_k^0 \cup B_k^1$ where B_k^0 and B_k^1 are disjoint non-empty clopen subsets of A_k^*. The pigeon-hole principle implies that for each $f \in C^*(\mathbf{N})$ such that $|f(\mathbf{N})| \leqq n$, there are two distinct integers $k_1, k_2 \in \{1, \ldots, n + 1\}$ and points x_f^1, x_f^2 with

$$x_f^1 \in B_{k_1}^0,\ x_f^2 \in B_{k_2}^0 \text{ and } f^\beta(x_f^1) = f^\beta(x_f^2).$$

For $k = 1, \ldots, n + 1$, let $\{C_f^k: f \in C^*(\mathbf{N})\}$ be a family of non-empty pairwise disjoint clopen (in $\beta\mathbf{N} - \mathbf{N}$) subsets of B_k^0. For each $f \in C^*(\mathbf{N})$, let K_f be a clopen (in $\beta\mathbf{N} - \mathbf{N}$) subset of $C_f^{k_1} \cup C_f^{k_2}$ such that $f^\beta|K_f$ is constant and

$$K_f \cap C_f^{k_1} \neq \emptyset \neq K_f \cap C_f^{k_2}$$

(for example, such that $f^\beta|K_f = f^\beta(x_f^1)$). For $k = 1, \ldots, n + 1$, let

$$D_k = A_k^* - \bigcup \{K_f: f \in C^*(\mathbf{N}), |f(\mathbf{N})| \leqq n\}.$$

Then $\operatorname{int}_{\beta\mathbf{N}-\mathbf{N}} D_k \neq \emptyset$ since $B_k^1 \subset D_k$. For $k = 1, 2, \ldots, n + 1$, let $\{R_\lambda^k: \lambda < c\}$ be a pairwise disjoint family of non-empty disjoint clopen and $|D_k - \bigcap_{\lambda<c} R_\lambda^k| \geqq 2$ (in $\beta\mathrm{N} - \mathrm{N}$) subsets of D_k such that $\bigcup_{\lambda<c} R_\lambda^k$ is dense in D_k. For each $\lambda < c$, let

$$R_\lambda = \bigcup_{k=1}^{n+1} R_\lambda^k.$$

Let $\delta\mathbf{N}$ be obtained from $\beta\mathbf{N}$ by identifying each R_λ to a point r_λ, each K_f to a point k_f, and $\beta\mathbf{N} - \mathbf{N} - \bigcup_{\lambda<c} R_\lambda - \bigcup_{f\in C^*(\mathbf{N})} K_f$ to a point ∞. It follows from Magill's theorem that $\delta\mathbf{N}$ is a Hausdorff compactification of $\mathbf{N}$. Then $f \in C^*(\mathbf{N})$, $|f(\mathbf{N})| \leqq n$ imply f extends to k_f since $f^\beta|K_f$ is constant. On the other hand, f_0 extends to no point of $\delta\mathbf{N} - \mathbf{N}$ because for each $y \in \delta\mathbf{N} - \mathbf{N}$,

$$|f_0^\beta(q^{-1}(y))| \geqq 2$$

where $q: \beta\mathbf{N} \to \delta\mathbf{N}$ is the Stone extension of the identity. Also, since $\{q^{-1}(p): p \notin \mathbf{N} \cup \{\infty\}\}$ is a maximal pairwise disjoint family of clopen subsets of $\beta\mathbf{N} - \mathbf{N}$, $\delta\mathbf{N} - \{\infty\}$ is a version of Ψ.

Combining 5.1 and 5.3, we get the following.

5.4. COROLLARY. ([**6**].) *Given $n \in \mathbf{N}$, there is a maximal almost-disjoint family $\mathscr{E}$ of subsets of* $\mathbf{N}$ *such that* (i) *given a partition* $\{A_1, \ldots, A_n\}$ *of* $\mathbf{N}$, *there is an $E \in \mathscr{E}$ such that $E \subseteq^* A_k$ for some $k \leqq n$, but* (ii) *there is a partition $\{A_1, \ldots, A_{n+1}\}$ of* $\mathbf{N}$ *such that there is no $k \leqq n + 1$ and no $E \in \mathscr{E}$ such that $E \subseteq^* A_k$.*

Remark. The proof of 5.3 actually shows that the partition $\{A_1, \ldots, A_{n+1}\}$ can be taken to be any partition of $\mathbf{N}$ into $n + 1$ infinite subsets. Of course, different partitions will yield different families $\mathscr{E}$.

We close this section with a modification of the construction of 5.3.

5.5. PROPOSITION. ([**6**]). *There is a compactification $\gamma\mathbf{N}$ of* $\mathbf{N}$ *such that $\gamma\mathbf{N} \cap \beta\mathbf{N} = \mathbf{N}$ singly but every $f \in C^*(\mathbf{N})$ such that $|f(\mathbf{N})| < \aleph_0$ extends to a point of $\gamma\mathbf{N} - \mathbf{N}$.*

Proof. Let $\{A_k: k \in \mathbf{N}\}$ be a partition of $\mathbf{N}$ into infinite subsets. Let $f_0 \in C^*(\mathbf{N})$ be given by $f|A_k = 1/k$. For each $k \in \mathbf{N}$, let $\{C_f^k: f \in C^*(\mathbf{N}), |f(\mathbf{N})| < \aleph_0\}$ be a collection of pairwise disjoint non-empty clopen (in $\beta\mathbf{N} - \mathbf{N}$) non-empty subsets of A_k^*. By the pigeon-hole principle, if $f \in C^*(\mathbf{N})$ and $F(\mathbf{N})$ is finite, there are distinct $k_1, k_2 \in \mathbf{N}$ and $x_{k_1}{}^f, x_{k_2}{}^f$ such that $x_{k_i}{}^f \in C_f^{k_i}$ for $i = 1, 2$ and $f(x_{k_1}{}^f) = f(x_{k_2}{}^f)$. Now let K_f be a clopen (in $\beta\mathbf{N} - \mathbf{N}$) subset of $C_f^{k_1} \cup C^{k_2}$ such that

$$K_f \cap C^{k_1} \neq \emptyset \neq K_f \cap C_f^{k_2}$$

and $f|K_f$ is constant (say $f|K_f = f(x_{k_1}{}^f)$). Let $\gamma\mathbf{N}$ be obtained from $\beta\mathbf{N}$ by identifying each K_f to a point and

$$\beta\mathbf{N} - \cup \{K_f: f \in C^*(N), |f(N)| < \aleph_0\}$$

to a point. Then $\gamma\mathbf{N}$ has the required properties. The proof is analogous to the proof of 5.3.

Remark. Proposition 5.5 is not explicitly stated in [**6**] but follows easily from Hechler's Theorem 10.2 or 3.2. A result which is stronger than 5.2 in which "$\subseteq^*$" is replaced by "$\subseteq$" is given in [**6**].

6. Non-pseudocompact spaces. We do not know of a large class of spaces X which admit compactifications Y such that $Y \cap \beta X = X$ multiply. However, we can find a large class (which includes all non-pseudocompact spaces) of spaces X which are contained as proper dense subsets of spaces Y with $Y \cap \beta X = X$ multiply.

6.1. PROPOSITION. *Suppose X contains a closed C^*-embedded copy of* $\mathbf{N}$. *Then X is a dense proper subset of a space Y such that $Y \cap \beta X = X$ multiply.*

Proof. Let $\mathbf{N}$ be a closed C^*-embedded copy of $\mathbf{N}$ in X. Let $W = \beta X - (\beta\mathbf{N} - \mathbf{N})$. Then $\beta W - W = \beta\mathbf{N} - \mathbf{N}$. Let $B\mathbf{N}$ be the compactification of $\mathbf{N}$ constructed in Example 4.2. Let $q:\beta\mathbf{N} - \mathbf{N}(= \beta W - W) \rightarrow B\mathbf{N} - \mathbf{N}$ be the natural quotient map, that is, q is the restriction to $\beta\mathbf{N} - \mathbf{N}$ of the Stone extension of the injection of $\mathbf{N}$ into $B\mathbf{N}$. By Magill's theorem, the quotient topology on $W \cup (B\mathbf{N} - \mathbf{N})$ induced by the map

$$\hat{q}:\beta W \rightarrow W \cup (B\mathbf{N} - \mathbf{N})$$

where

$$\hat{q}(x) = \begin{cases} x & \text{if } x \in W \\ q(x) & \text{if } x \in \beta W - W \end{cases}$$

is (compact) Hausdorff. Let $Y = X \cup (B\mathbf{N} - \mathbf{N})$. We claim $Y \cap \beta X = X$ multiply. We first show $Y \cap \beta X = X$. Suppose $p \in Y - X = B\mathbf{N} - \mathbf{N}$. Let $f \in C^*(\mathbf{N})$ be a function which does not extend to p. Since $\mathbf{N}$ is C^*-embedded in X, the function f extends to a function $\hat{f} \in C^*(X)$. Then $\hat{f}$ does not extend to p. We now show that it is not the case that $Y \cap \beta X = X$ singly. Suppose $f \in C^*(X)$. Then the restriction $f|\mathbf{N}$ extends to the point $k_{f|\mathbf{N}} \in B\mathbf{N} - \mathbf{N}$. Let DX be the quotient of βX which identifies $K_{f|\mathbf{N}}$ to a point and let $r:\beta X \rightarrow DX$ be the quotient map (so r is given by

$$r(x) = \begin{cases} x & \text{if } x \in \beta X - K_{f|\mathbf{N}} \\ k_{f|N} & \text{if } x \in K_{f|\mathbf{N}} \end{cases}.$$

Let $f^\beta:\beta X \rightarrow \mathbf{R}$ be the Stone extension of f. Define $g:DX \rightarrow \mathbf{R}$ by

$$g(x) = f^\beta(r^{-1}(x)).$$

Then g is well-defined and

$$g \circ r = f^\beta \in C^*(\beta X)$$

so (since r is a quotient map), $g \in C^*(DX)$. But $g|X = f$ and $g|X \cup \{k_{f|\mathbf{N}}\}$ is continuous so f extends to a point of $B\mathbf{N} - \mathbf{N} = Y - X$. Therefore $Y \cap \beta X = X$ multiply.

6.2. COROLLARY. *If X is not pseudocompact, there is a space Y which contains X as a proper dense subset such that $Y \cap \beta X = X$ multiply.*

Remark. We do not know if Y can be taken to be pseudocompact, or even compact, in 6.2, even in the case of $X = \mathbf{R}$. However, by Remark (iii) following 2.3, in the case of $\mathbf{R}$, no such Y can be locally connected.

7. Unbounded functions. In this section we briefly discuss the υX analogue of the material of the earlier sections. If X is a proper dense subset of Y, we write $Y \cap \upsilon X = X$ if for each $p \in Y - X$, there is a function $f_p \in C(X)$ such that f does not extend to p. If a single function serves as every f_p, we say $Y \cap \upsilon X = X$ singly. If $Y \cap \upsilon X = X$ but it is not the case that $Y \cap \upsilon X = X$ singly, then we say $Y \cap \upsilon X = X$ multiply. We will be concerned here with the case where $Y = \beta X$. We first note an easy fact.

7.1. PROPOSITION. *The following statements are equivalent for a space X:* (i) $\beta X \cap \upsilon X = X$. (ii) $\upsilon X \cap \beta X = X$. (iii) *$X$ is realcompact.*

Proof. The first statement says that for each $p \in \beta X - X$, there is a function $f_p \in C(X)$ which does not extend to p, that is, X is realcompact. The second statement says that for each $p \in \upsilon X - X$, there is a function $f_p \in C^*(X)$ such that f_p does not extend to p. This can happen if and only if $\upsilon X - X = \emptyset$.

7.2. LEMMA. *Suppose X is normal and $f \in C(X)$. Then f extends to no point of $\beta X - X$ if and only if $f^{-1}([-n, n])$ is compact for each $n \in \mathbf{N}$.*

Proof. Suppose first that for some $n_0 \in \mathbf{N}$, $f^{-1}([-n_0, n_0])$ is not compact. Let $Z_1 = f^{-1}([-n_0, n_0])$ and let $Z_2 = f^{-1}([-n_0 - 1, n_0 + 1])$. Then Z_2 is not compact. Choose $p \in (\mathrm{Cl}_{\beta X} Z_1) - X$. We claim f extends to p. Since Z_2 is C^*-embedded in X, the restriction $f|Z_2$ extends to a function $g \in C^*(Z_2 \cup \{p\})$. Define $\hat{f}: X \cup \{p\} \to \mathbf{R}$ by

$$\hat{f}|X - Z_1 = f \quad \text{and} \quad \hat{f}|Z_2 \cup \{p\} = g.$$

Then $\hat{f}$ is a well-defined extension of f. We claim $\hat{f}$ is continuous. $\hat{f}$ is continuous on $X - Z_1$. If

$$U = [(|f| \wedge (n+1))^\beta]^{-1}(-1, n_0 + \tfrac{1}{2}), \text{ and}$$
$$V = U \cap (X \cup \{p\}),$$

then V is an $X \cup \{p\}$ neighborhood of p such that

$$Z_1 \cup \{p\} \subseteq V \subseteq Z_2 \cup \{p\}.$$

Then $\hat{f}|V = g|V$ which is continuous. $X \cup \{p\} = V \cup (X - Z_1)$. Therefore, since each of the sets V and $X - Z$ is open in $X \cup \{p\}$, $\hat{f}$ is continuous. For the converse, suppose $f^{-1}([-n, n])$ were compact for each $n \in N$ and suppose there were a $p \in \beta X - X$ and an $\hat{f} \in C(X \cup \{p\})$ such that $\hat{f}|X = f$. Suppose $\hat{f}(p) \in [-n_0, n_0]$ where $n_0 \in \mathbf{N}$. Let

$$K = f^{-1}([-n_0 - 1, n_0 + 1]).$$

By assumption K is compact. But if U is any $X \cup \{p\}$ neighborhood of p, then $U - K$ is an $X \cup \{p\}$ neighborhood of p each of whose elements $x \neq p$ satisfies $|\hat{f}(x) - \hat{f}(p)| \geqq 1$. This contradicts the continuity of $\hat{f}$.

7.3. COROLLARY. *Suppose X is normal, realcompact, but not σ-compact. Then $\beta X \cap \upsilon X = X$ multiply.*

Proof. By 7.1, $\beta X \cap \upsilon X = X$. If $f \in C(X)$, there is an $n \in \mathbf{N}$ such that $f^{-1}([-n, n])$ is not compact; otherwise X would be σ-compact. By 7.2, f extends to a point of $\beta X - X$.

7.4. COROLLARY. *Suppose D is an infinite discrete space. Then $\beta D \cap \upsilon D = D$ singly if and only if D is countable.*

Proof. If D is countable, then $D = \mathbf{N}$ and the injection $\iota: N \rightarrow \mathbf{R}$ extends to no point of $\beta D - D$. For the converse, suppose $\beta D \cap \upsilon D = D$ singly. By 7.1, D has non-measurable cardinal. By 7.3, since an uncountable discrete space is not σ-compact, D must be countable.

Added in proof. Eric van Douwen has proved that there is a compactification Y of $\mathbf{R}$ such that $Y \cap \beta\mathbf{R} = \mathbf{R}$ multiply. He has also proved that for metric X and a dense subset G of X, G is a G_δ in X if and only if $X \cap \beta G = G$ singly. This extends 2.4.

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