DUPLICATION OF ROOM SQUARES

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To Professor T. G. Room on his 70th birthday

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1. Room squares

A *Room square* $\mathscr{R}$ of order $2n$ is a way of arranging $2n$ objects (usually $1, 2, \cdots, 2n$) in a square array $\mathscr{R}$ of side $2n-1$ so that:

(i) every cell of the array is empty or contains two objects;
(ii) each unordered pair of objects occurs once in $\mathscr{R}$;
(iii) every row and column of $\mathscr{R}$ contains one copy of each object.

In his original note [6], Room showed that there are squares of orders 2 and 8, but not of orders 4 and 6. It is known that Room squares exist of every order 2^{2k+1}, k integral [1], of every order $2n = p^r + 1$ where p^r is a prime power other than a Fermat prime (of type $2^{2^s} + 1$) [4], and of every order from 10 to 48 [8]. Moreover, if there are squares of orders $2m$ and $2n$, then there is a square of order $(2m-1)(2n-1)+1$ [9].

In this paper we prove that

If $2n - 1 = \prod p_i^{r_i}$*, where each* p_i *is a prime congruent to* 3 *modulo* 4 *and no* $p_i^{r_i}$ *is* 3, 3^2, *or* 3^4, *then there is a Room square of order* $4n$.

This provides apparently new Room squares of orders 56, 88 and 96 (leaving six orders less than 100 undecided, namely 52, 58, 66, 76, 86 and 94) and seven new squares of orders between 100 and 500.

We consider two Room squares to be *isomorphic* if one can be obtained from the other by interchanging two rows, interchanging two columns, interchanging two objects, or any sequence of these operations. We say a square of order $2n$ is *standardized* if it has the pair $\{i, 2n\}$ in its ith diagonal position; obviously any Room square is isomorphic to a standardized one.

The *incidence matrix* of a Room square $\mathscr{R}$ of order $2n$ will mean the square matrix of size $2n-1$ which has 1 in its (i,j) position when $\mathscr{R}$ has an entry in that position and -1 otherwise. A standardized Room square will be called *skew-type* if its incidence matrix is $I + S$, where S is a skew-symmetric matrix; and an

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arbitrary Room square is skew-type if it is isomorphic to a skew-type standardized square.

2. Concerning Latin squares

A *Latin square* of order r is a square array of size r on r symbols (usually $1, 2, \cdots, r$) such that every symbol occurs once in each row and each column. Given arrays $L = (l_{ij})$ and $M = (m_{ij})$ of the same size, we will write (L, M) for the array whose (i, j) element is the ordered pair (l_{ij}, m_{ij}). In this notation two Latin squares L and M of the same order are called *orthogonal* if (L, M) has no two entries the same.

THEOREM 1. (i) *A Room square $\mathscr{R}$ of order $2n$ is equivalent to a pair of symmetric Latin squares R_1 and R_2 of order $2n$, each having constant diagonal* $(0, 0, \cdots, 0)$, *such that (R_1, R_2) has no two entries the same above the main diagonal.*

(ii) *If $\mathscr{R}$ is standardized then R_1 and R_2 each have last column* $(1, 2, \cdots, 2n-1, 0)$.

(iii) *If the standardized Room square is skew-type then (R_1, R_2) and (R_2, R_1) have no common entries above the main diagonal except for* $(1, 1)$, $(2, 2), \cdots (2n-1, 2n-1)$, *which appear in the last column of each.*

PROOF. (i) Given a Room square $\mathscr{R}$ on the objects $1, 2, \cdots 2n$, write r_{ij} for the number of the row and c_{ij} for the number of the column which contain the pair $\{i, j\}$. Write $r_{ii} = c_{ii} = 0$, and write $R_1 = (r_{ij})$, $R_2 = (c_{ij})$. It follows immediately from the definition of Room squares that R_1 and R_2 are symmetric Latin squares. They have the required diagonal. If the entry (a, b) occurs in the (i, j) position of (R_1, R_2) this means $\{i, j\}$ lies in row a and column b of $\mathscr{R}$ and consequently (a, b) cannot occur elsewhere in (R_1, R_2) except in the (j, i) position.

Given two Latin squares of the specified type, this construction can be reversed to give the Room square.

(ii) is obvious.

(iii) follows because entry (i, j) occurs above the diagonal in (R_1, R_2) if and only if the incidence matrix of $\mathscr{R}$ has 1 in its (i, j) position, and because Latin squares R_2 and R_1 correspond via part (i) to the transpose of $\mathscr{R}$.

We will call the R_1 and R_2 which we have constructed the row and column Latin squares of $\mathscr{R}$ respectively.

Part (i) of Theorem 1 could be deduced from the facts that both Latin squares and Room squares are related to loops [3, Chapters 6, 10].

3. Duplication of skew-type Room squares

In this section we give a construction for a Room square twice the order of a given skew-type square.

Suppose $\mathscr{R}$ is a skew-type Room square of order $2n$. We may assume $\mathscr{R}$ to be standardized. Write R_1 and R_2 for the row and column Latin squares of $\mathscr{R}$. Suppose further that there are orthogonal Latin squares L_1 and L_2 of order $2n$, each having last column $(1, 2, \cdots, 2n)$. Write M_α to mean the square L_α with each entry increased by $2n-1$. Now write

$$Q = \boxed{\begin{matrix} R_1 & M_1 \\ M_1^T & R_2 \end{matrix}} \qquad S_2 = \boxed{\begin{matrix} R_2 & M_2 \\ M_2^T & R_1 \end{matrix}}$$

and write S_1 for the array obtained from Q by interchanging rows $2n$ and $4n$ and interchanging columns $2n$ and $4n$. Then (S_1, S_2) is as shown, where the notations are that m_i^α is the $(2n, i)$ element of M_α and that $\hat{X}$ means the array X with the last row and column deleted.

S_1 and S_2 are symmetric Latin squares of order $4n$ with constant diagonal $(0, 0, \cdots, 0)$. In the event of no duplications above the diagonal in (S_1, S_2) they will imply the existence of a Room square of order $4n$, by Theorem 1.

The entries above the diagonal in (S_1, S_2) are of three types:

(a) *both elements less than* $2n$: we have all the entries above the diagonal in $(\hat{R}_1, \hat{R}_2)$ and $(\hat{R}_2, \hat{R}_1)$ once each. By Theorem 2(iii) these include no duplications.

(b) *both elements at least* $2n$: we have certain of the entries from (M_1, M_2) (not including the entry $(4n-1, 4n-1)$ which was in the $2n$th row and column of (M_1, M_2)) and also $(4n-1, 4n-1)$. As M_1 and M_2 are orthogonal, these include no duplications.

(c) *one element less than* $2n$, *and one not*: we have all the $(i, 2n+i-1)$, (i, m_i^2), $(2n+i-1, i)$ and (m_i^1, i) for $i = 1, 2, \cdots, 2n-1$. These will include duplications if and only if $m_i^1 = 2n+i-1$ or $m_i^2 = 2n+i-1$ for some $i \leqq 2n-1$; in terms of the original squares L_1 and L_2, if and only if the ith element of the last row of L_1 or L_2 is i for some $i \neq 2n$.

Clearly, no duplications occur between the different types.

From [2] there are two orthogonal Latin squares of order $2n$ provided $2n \neq 2$ or 6. Suppose we have two such Latin squares. By proper labelling of the entries and ordering of the columns, we can assure that both squares have last column $(1, 2, \cdots, 2n)$, and that the last row is $(1, 2, \cdots, 2n)$ in one and $(a_1, a_2, \cdots, a_{2n-1}, 2n)$ in the other where $a_i \neq i$. By Theorem 2.2 of Chapter 6 of [7] we can find $b_1, b_2, \cdots, b_{2n-1}$ such that $i \neq b_i \neq a_i$. Re-order the columns of the squares so that column i becomes column b_i. If we take these squares as L_1 and L_2, there will be no duplications in the elements of type (c).

Summarising, we have the following result.

THEOREM 2. *If there is a skew-type Room square of order* $2n > 2$, *then there is a Room square of order* $4n$.

$(\hat{R}_1, \hat{R}_2)$	$(2n, 1)$ $(2n+1, 2)$ $\vdots$ $(4n-2, 2n-1)$	$(\hat{M}_1, \hat{M}_2)$	$(1, 2n)$ $(2, 2n+1)$ $\vdots$ $(2n-1, 4n-2)$
$(2n, 1)(2n+1, 2)\cdots(4n-2, 2n-1)$	$(0, 0)$	$(1, m_1^2)(2, m_2^2)\cdots(2n-1, m_{2n-1}^2)$	$(4n-1, 4n-1)$
$(\hat{M}_1^T, \hat{M}_2^T)$	$(1, m_1^2)$ $(2, m_2^2)$ $\vdots$ $(2n-1, m_{2n-1}^2)$	$(\hat{R}_2, \hat{R}_1)$	$(m_1^1, 1)$ $(m_2^1, 2)$ $\vdots$ $(m_{2n-1}^1, 2n-1)$
$(1, 2n)(2, 2n+1)\cdots(2n-1, 4n-2)$	$(4n-1, 4n-1)$	$(m_1^1, 1)(m_2^1, 2)\cdots(m_{2n-1}^1, 2n-1)$	$(0, 0)$

(S_1, S_2)

4. The Mullin-Nemeth squares

Suppose n is even and $2n-1$ is a prime power greater than 3. Choose a generator x of the multiplicative group of non-zero elements in the Galois field $GF(2n-1)$, and write the elements of $GF(2n-1)$ as $g_1, g_2, \cdots, g_{2n-1}$, where $g_1 = 0$ and $g_i = x^{i-2}$ otherwise. A Room square of order $2n$ may be formed in the following way:

(i) the (i,i) cell contains $\{i, 2n\}$;

(ii) the $(1,k)$ cell contains $\{2p, 2p+1\}$ when $g_k = g_{2p} + g_{2p+1}$, $p = 1, 2, \cdots n-1$; if g_k cannot be expressed in this way, then cell $(1,k)$ is empty;

(iii) the (i,j) cell depends on the $(1,k)$ cell where $g_k = g_j - g_i$. If the $(1,k)$ cell is empty then the (i,j) cell is empty; if the $(1,k)$ cell contains $\{2p, 2p+1\}$ then the (i,j) cell contains $\{q, r\}$, defined by

$$g_q = g_{2p} + g_i,\ q_r = g_{2p+1} + g_i.$$

This is the construction given by Mullin and Nemeth in [4]. (For proof of the Room property see [4] and [5].) Its incidence matrix A has diagonal elements $a_{ii} = 1$; if $i \neq j$, $a_{ij} = 1$ if and only if $g_j - g_i$ is $(x+1)$ times an even power of x. If $a_{ij} = a_{ji} = 1$ then both $(x+1)^{-1}(g_j - g_i)$ and $(x+1)^{-1}(g_i - g_j)$ are quadratic elements in $GF(2n-1)$, which is impossible as $2n-1 \equiv 3 (\text{mod } 4)$. So the Mullin-Nemeth construction gives a skew-type Room square.

COROLLARY 3. *There is a skew-type Room square of order* $2n = p^r + 1 > 4$ *whenever* p^r *is a prime power congruent to* 3 *modulo* 4.

5. A multiplication theorem

THEOREM 4. *If there are Room squares* $\mathscr{M}$ *and* $\mathscr{N}$ *of orders* $2m$ *and* $2n$, *then there is a Room square* $\mathscr{R}$ *of order* $(2m-1)(2n-1)+1$. *If* $\mathscr{M}$ *and* $\mathscr{N}$ *are skew-type, then so is* $\mathscr{R}$.

PROOF. If $n = 1$ there is nothing to prove. So we assume $n > 1$, whence there exist a pair of orthogonal Latin squares of order $2n-1$; call them L_1 and L_2. Assume that $\mathscr{M}$ and $\mathscr{N}$ are standardized; write the row and column Latin squares of $\mathscr{M}$ as $M_1 = (a_{ij}^1)$ and $M_2 = (a_{ij}^2)$, and those of $\mathscr{N}$ as N_1 and N_2. Again $\hat{N}_\alpha$ is N_α with the last row and column deleted.

For convenience we will write the numbers from 1 to $(2m-1)(2n-1)$ in the form x_y, where

$$x_y = x + (y-1)(2n-1),\ 0 < x < 2n,\ 0 < y < 2m.$$

(Define 0_y as 0.) If A is any of the matrices $\hat{N}_1$, $\hat{N}_2$, L_1, L_2, we will write $A(y)$ to mean the matrix A with each entry x replaced by x_y.

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$\mathscr{R}$ is defined by its row and column squares R_1 and R_2. Each $\hat{R}_\alpha$ is a $(2m-1)\times(2m-1)$ array whose entries are $(2n-1)\times(2n-1)$ blocks:

$$\text{the } (i,i) \text{ block of } \hat{R}_\alpha \text{ is } N_\alpha(i),$$

$$\text{the } (i,j) \text{ block of } \hat{R}_\alpha \text{ is } L_\alpha(a^\alpha_{ij}) \text{ if } i<j,$$

$$\text{the } (i,j) \text{ block of } \hat{R}_\alpha \text{ is } L_\alpha(a^\alpha_{ij})^T \text{ if } i>j.$$

The last row and column of R_α are $(1,2,\cdots,(2m-1)(2n-1),0)$. One sees immediately that the R_α are symmetric Latin squares with constant diagonal 0.

Assume that k and l are non-zero. If the entry (k_i, l_i) occurs in $(\hat{R}_1, \hat{R}_2)$ then it occurs in the block $(\hat{N}_1(i), \hat{N}_2(i))$ at the place where (k,l) occurs in $(\hat{N}_1, \hat{N}_2)$; since $\mathscr{N}$ is a Room square, (k,l) can arise at most once above the diagonal, and if also $\mathscr{N}$ is skew-type, (l,k) cannot also arise above the diagonal in $(\hat{N}_1, \hat{N}_2)$. So (k_i, l_i) can occur at most once above the diagonal in $(\hat{R}_1, \hat{R}_2)$; if $\mathscr{N}$ is skew-type, (l_i, k_i) and (k_i, l_i) cannot both occur. (We need not distinguish the case $k_i = l_i$, since this pair does not arise in $(\hat{R}_1, \hat{R}_2)$.) If the entry (k_i, l_j) occurs in $(\hat{R}_1, \hat{R}_2)$, $i \neq j$, it occurs above the diagonal in the block $(L_1(i), L_2(j))$, which appears at most once since $\mathscr{M}$ is a Room square; (k_i, l_j) occurs only once in $(L_1(i), L_2(j))$ since L_1 and L_2 are orthogonal. If further $\mathscr{M}$ is skew-type then $(L_1(i), L_2(j))$ and $(L_1(j), L_2(i))$ cannot both be in $(\hat{R}_1, \hat{R}_2)$, so (k_i, l_j) and (l_j, k_i) cannot both occur. The last column of (R_1, R_2) contains the pairs (i,i) once each. So, from Theorem 1, R_1 and R_2 define a (standardized) Room square $\mathscr{R}$ of the required order. If $\mathscr{M}$ and $\mathscr{N}$ are skew-type then $\mathscr{R}$ is skew-type.

COROLLARY 5. *If* $2n-1 = \prod p_i^{r_i}$, *where each* p_i *is a prime congruent to* 3 *modulo* 4 *and no* $p_i^{r_i}$ *is* 3, 3^2, or 3^4, *then there is a skew-type Room square of order* $2n$ *and consequently a Room squre of order* $4n$.

This Corollary simply combines Theorems 2 and 4 and Corollary 3. It gives ten new Room square orders less than 500, including 56, 88 and 96. If there is a skew-type Room square of order 10, then we need only bar the case $p_i^{r_i} = 3$ in Corollary 5.

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