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ON THE MACKEY BOREL STRUCTURE

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Let A be a C^*-algebra and $\mathfrak{H}$ a Hilbert space which is infinite dimensional and of Hilbert dimension $\geqq \dim \pi$ for all $\pi \in \hat{A}$. Suppose that the set $\mathrm{Irr}(A, \mathfrak{H})$ of all non-null *-representations π of A on $\mathfrak{H}$, irreducible on the essential space $\mathfrak{H}(\pi)$, is given the relative strong topology as a subspace of $\mathrm{Rep}(A, \mathfrak{H})$ [**2**; **4**; **6**]. That is, the topology is that of simple convergence in $\mathscr{L}(\mathfrak{H})$ with the strong topology. Finally, let $\sim$ denote equivalence of representations in $\mathrm{Irr}(A, \mathfrak{H})$ implemented by partial isometries in $\mathscr{L}(\mathfrak{H})$: $\pi_1 \sim \pi_2$ if and only if there exists a partial isometry $v \in \mathscr{L}(\mathfrak{H})$ with $vv^* \supset H(\pi_1)$ and $v^*v \supset H(\pi_2)$ satisfying $\pi_2(a) = v^*\pi_1(a)v$ for all $a \in A$.

In [**6**], we showed that the quotient topology on $\hat{A} = \mathrm{Irr}(A, \mathfrak{H})/\sim$ agrees with the hull-kernel topology. In this note, we prove that if A and $\mathfrak{H}$ are separable, the quotient Borel structure and the Mackey Borel structure agree on $\hat{A}$.

1. We use the terminology and some results of [**5**].

Let $\mathscr{L}_1(\mathfrak{H})$ be the closed unit ball in $\mathscr{L}(\mathfrak{H})$, let $\mathscr{J}$ be the semigroup of all isometries on the separable Hilbert space $\mathfrak{H}$ and let $\mathscr{U}$ be the full unitary subgroup of $\mathscr{J}$. Give to each the *-strong topology. Then $\mathscr{L}_1(\mathfrak{H})$ and $\mathscr{J}$ are topological semigroups and $\mathscr{U}$ is a topological subgroup of $\mathscr{J}$.

Make $(\mathscr{U}, \mathscr{J})$ a transformation group by

$$\begin{aligned} \mathscr{U} \times \mathscr{J} &\to \mathscr{J} \\ U, J &\mapsto JU^*. \end{aligned}$$

(We will use juxtaposition for operator products only.)

LEMMA 1. *$(\mathscr{U}, \mathscr{J})$ is a polonais transformation group; $\mathscr{J}/\mathscr{U}$ has a Borel transversal.*

Proof. That $\mathscr{U}$ is a polonais topological group is shown in [**3**, Lemma 4]. The proof adapts immediately to $\mathscr{J}$, as well. In fact, the equations:

$$F_{ik} = \left\{T \in \mathscr{L}_1(\mathfrak{H}) : \sup_j |\langle Tx_i, x_j\rangle| > 1 - \frac{1}{k}\right\}, \quad i, k = 1, 2, \ldots$$

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where (x_i) is a dense sequence in the unit sphere of $\mathfrak{H}$;

$$\mathcal{J} = \bigcap_{ik} F_{ik};$$

and

$$\mathcal{U} = \mathcal{J} \cap \mathcal{J}^*$$

show that each is a G_δ in the polonais space $\mathcal{L}_1(\mathfrak{H})$ with its weak topology. But $\mathcal{L}_1(\mathfrak{H})$ is polonais in the *-strong topology, as well, and the F_{ik} (respectively F_{ik}^*) are *-strongly open. Thus each of $\mathcal{U}, \mathcal{J}$ is a polonais space. Of course, on $\mathcal{U}$ the weak, strong, and *-strong topologies coincide.

The continuity of the mapping $U, J \mapsto JU^*$ follows from that of * on $\mathcal{U}$ and from the strong joint continuity of multiplication on bounded subsets of $\mathcal{L}(\mathfrak{H})$. Thus $(\mathcal{U}, \mathcal{J})$ is a polonais transformation group.

We note that the isotropy subgroup of J

$$\mathcal{U}_J = \{U \in \mathcal{U}: JU^* = J\}$$

is trivial for each $J \in \mathcal{J}$ and that the map $U \mapsto JU^*$ carries $\mathcal{U} = \mathcal{U}/\mathcal{U}_J$ homeomorphically onto the orbit $J\mathcal{U}$.

We next prove that $(\mathcal{U}, \mathcal{J})$ satisfies [**5**, Condition D]; i.e. that, given a neighborhood $\mathcal{N}$ of I in $\mathcal{U}$ and a decreasing basis Q_m of open sets about $J_0 \in \mathcal{J}$, there exists a neighborhood $\mathcal{M}$ of I in $\mathcal{U}$ such that

$$\bigcap_m \mathrm{Cl}[Q_m\mathcal{M}^*] \subset J_0\mathcal{N}^*.$$

Clearly, we may take $\mathcal{N}$ to be of the form $\mathcal{N} = \mathcal{N}_{\xi,\epsilon} = \mathcal{V} \cap \mathcal{V}^*$, with

$$\mathcal{V} = \{U \in \mathcal{U}: ||(U - I)\xi_i|| < \epsilon, 1 \leqq i \leqq n\},$$

where the $\xi_i \in \mathfrak{H}$, $1 \leqq i \leqq n$, and $\epsilon > 0$.

Let $0 < \delta < \epsilon$, and take $\mathcal{M}$ to be $\mathcal{N}_{\xi,\delta}$. Then $\mathrm{Cl}\,\mathcal{M} \subset \mathcal{N}$, so it suffices to prove that

$$\bigcap_m \mathrm{Cl}[Q_m\mathcal{M}^*] \subset J_0\,\mathrm{Cl}\,\mathcal{M}^*.$$

Let $J \in \bigcap_m \mathrm{Cl}[Q_m\mathcal{M}^*]$, and choose sequences $R_m \in Q_m$ and $S_m \in \mathcal{M}$ so that $R_mS_m^*$ *-strongly converges to J:

$$R_mS_m^* \xrightarrow{*} J.$$

Then

$$R_mS_m^*S_mR_m^* = R_mR_m^* \xrightarrow{*} JJ^*,$$

while since $R_m \in Q_m$,

$$R_mR_m^* \xrightarrow{*} J_0J_0^*,$$

so J and J_0 have the same final space. Especially, $J = J_0J_0^*J$.

Now

$$S_m{}^* = (R_m{}^*R_m)S_m{}^* = R_m{}^*(R_mS_m{}^*) \xrightarrow{\ *\ } J_0{}^*J,$$

yielding $J_0{}^*J \in \mathrm{Cl}\,\mathscr{M}^*$, whence $J = J_0J_0{}^*J \in J_0\,\mathrm{Cl}\,\mathscr{M}^*$, as desired†.

We have shown that $(\mathscr{U}, \mathscr{J})$ satisfies [**5**, Theorem 2.1, condition (1), and also condition D]. By [**5**, Theorem 2.9], $\mathscr{J}/\mathscr{U}$ has a Borel transversal. The Lemma is proved.

In the commutative diagram, $\mathrm{Irr}_\infty(A, \mathfrak{H}_\infty) = \{\pi \in \mathrm{Irr}(A, \mathfrak{H})\colon \mathfrak{H}(\pi) = \mathfrak{H}\}$, $\mathrm{Irr}_\infty(A, \mathfrak{H}) = \{\pi \in \mathrm{Irr}(A, \mathfrak{H})\colon \dim \pi = \infty\}$, j is the natural injection and q and r are quotient maps.

$$\begin{array}{ccc} \mathrm{Irr}_\infty(A, \mathfrak{H}_\infty) = X & \xrightarrow{\quad j \quad} & Y = \mathrm{Irr}_\infty(A, \mathfrak{H}) \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle r} \\ X/\sim\; = Q & \xrightarrow{\quad \tilde{j} \quad} & R = Y/\sim \end{array}$$

If X and Y are given their strong topological Borel structures and Q and R the corresponding quotient Borel structures, then all the maps are Borel. (j is continuous.) Q carries by definition the Mackey Borel structure.

If $E \subset \mathrm{Irr}(A, \mathfrak{H})$, $E^\sim$ denotes the saturation of E by $\sim$.

LEMMA 2. *$\tilde{j}$ is a Borel isomorphism.*

Proof. Since $\tilde{j}$ is a Borel bijection, it remains only to show that if $B \subset Q$ is Borel, $\tilde{j}(B)$ is Borel.

Now $q^{-1}(B) = D$ is Borel and unitarily saturated in X, hence in Y. Since $\tilde{j}(B) = r(D^\sim)$, our task is to prove that $D^\sim$ is Borel. Let τ be a Borel transversal of $\mathscr{J}/\mathscr{U}$ (Lemma 1). We note that the strong and the *-strong topologies on $\mathscr{L}(\mathfrak{H})$ define the same simple-convergence topologies on $\mathrm{Irr}(A, \mathfrak{H})$ (and, for that matter, the same Borel structures on $\mathscr{J}$). In this connection, see also [**2**].

Now it is easy to see that the mapping $f\colon D \times \tau \to Y$ is a Borel bijection of the standard Borel space $D \times \tau$ onto $D^\sim$, if $f(\pi, J) = J\pi J^*$, where $J\pi J^*(a) = J\pi(a)J^*$ for $a \in A$.

In fact it is shown in [**4**, 3.7.4] that X is standard, so its Borel subspace D is standard. The same argument, with Lemma 1, shows that τ is standard. So $D \times \tau$ is standard.

Let us show that f is injective. If $\pi = J_1\pi_1J_1{}^* = J_2\pi_2J_2{}^*$, $\pi_i \in D$, $J_i \in \tau$, then $H(\pi) = J_1J_1{}^* = J_2J_2{}^*$, so $U = J_1{}^*J_2$ is unitary and $J_1U = J_1J_1{}^*J_2 = J_2J_2{}^*J_2 = J_2$. Since τ is a transversal and $J_2 = J_1U$, $J_1 = J_2$ and $\pi_1 = J_1{}^*J_2\pi_2J_2{}^*J_1 = \pi_2$.

f is continuous, so it is Borel. Finally, if $\pi \in D^\sim$, let $\pi = J\pi_1J^*$, $\pi_1 \in D$, $J \in \mathscr{J}$. Then $J = J_0U^*$ for some $J_0 \in \tau$, $U \in \mathscr{U}$, and $\pi = J_0(U^*\pi_1U)J_0{}^*$. But $U^*\pi_1U = \pi_0 \in D$, since D is unitarily saturated, so $\pi = f(\pi_0, J_0)$, showing that f is surjective.

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†I am grateful to the referee for a remark which helped simplify this portion of the proof.

Now an application of [**1**, Proposition 2.5] shows that $f(D \times \tau) = D^{\sim}$ is Borel.

2. For $n = 1, 2, \ldots$, $\mathrm{Irr}_n(A, \mathfrak{H}_n)$ denotes the set of all irreducible representations of A on the standard n-dimensional Hilbert space $\mathfrak{H}_n =$ linear span $\{e_1, \ldots, e_n\}$, where $(e_i)_{1 \leqq i < \infty}$ is a fixed orthonormal basis for the infinite dimensional, separable Hilbert space $\mathfrak{H}$. $\mathrm{Irr}_n(A, \mathfrak{H})$ denotes the set of all irreducible representations of A on n-dimensional subspaces of $\mathfrak{H}$.

We have natural maps

$$\begin{array}{ccc} \mathrm{Irr}_n(A, \mathfrak{H}_n) = R & \xrightarrow{j} & P = \mathrm{Irr}_n(A, \mathfrak{H}) \\ \downarrow s & & \downarrow q \\ \mathrm{Irr}_n(A, \mathfrak{H}_n)/\sim\; = S & \xrightarrow{\tilde{j}} & Q = \mathrm{Irr}_n(A, \mathfrak{H})/\sim \end{array}$$

in the categories of sets, topological spaces and Borel spaces, if the quotient sets are equipped with the quotient structures. By [**4**, 3.5.8], $\tilde{j}$ is a homeomorphism. We claim that it is a Borel isomorphism.

In fact, if $\mathscr{U}_n$ is the unitary group on $\mathfrak{H}_n$, $(\mathscr{U}_n, R)$ is a polonais transformation group, in which the orbits are the equivalence classes modulo $\sim$ in R. These orbits are compact since $\mathscr{U}_n$ is compact, and hence closed since R is Hausdorff. In addition, the saturation of an open set is open in R. Hence by [**3**, Lemma 2], there is a Borel transversal in R of the equivalence classes modulo $\sim$. Let E be such a transversal. If B is a Borel subset of S, let $F = s^{-1}(B) \cap E$. Then F is Borel in R and $j(F)^{\sim} = F^{\sim} = (j(s^{-1}(B)))^{\sim}$ in P.

Since $n < \infty$, equivalence is unitary equivalence in P (via unitaries in $\mathscr{L}(\mathfrak{H})$). So $F^{\sim}$ is the image of $\mathscr{U} \times F$ under the mapping

$$f\colon U, \pi \mapsto U\pi U^*.$$

Since R is polonais and $\mathscr{U}$ is standard, $\mathscr{U} \times F$ is standard, while f is continuous, hence Borel. But f is clearly injective, so as before, f has a Borel image $F^{\sim}$. Since $q(F^{\sim}) = \tilde{j}(B)$, we have proved our claim.

Now the quotient Borel structure on $S = \hat{A}_n$ is the Mackey Borel structure.

We have proved

LEMMA 3. *For $n = \infty, 1, 2, \ldots$, the quotient Borel structure on* $\mathrm{Irr}_n(A, \mathfrak{H})/\sim$ *agrees with the Mackey Borel structure.*

3. Now consider the commutative diagram:

$$\begin{array}{ccc} P = \mathrm{Irr}_n(A, \mathfrak{H}) & \xrightarrow{j_n} & \mathrm{Irr}(A, \mathfrak{H}) = L \\ \downarrow q & & \downarrow t \\ Q & \xrightarrow{\tilde{j}_n} & T \end{array},$$

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where once again the vertical mappings are quotients and j_n is natural injection, t is open [**6**] and P is saturated.

Now all we need, since P is saturated, is that for each n, $n = \infty, 1, 2, \ldots$, $\mathrm{Irr}_n(A, \mathfrak{H})$ is a Borel subset of $\mathrm{Irr}(A, \mathfrak{H})$. [**4**, Proposition 3.6.3] says, in part, that for $n < \infty$, ${}_n\hat{A}$, the set of all $\pi \in \hat{A}$ with $\dim \pi \leqq n$ is closed in $\hat{A}$, so by [**6**] $t^{-1}({}_n\hat{A}) = \bigcup_{k=1}^{n} \mathrm{Irr}_k(A, \mathfrak{H})$ is closed in $\mathrm{Irr}(A, \mathfrak{H})$ for $n < \infty$, and it is clear how to generate the $\mathrm{Irr}_n(A, \mathfrak{H})$, $n = \infty, 1, 2, \ldots$, by elementary Borel operations. So $\tilde{\jmath}_n$ is a Borel isomorphism. Especially, $\tilde{\jmath}_n(\hat{A}_n) = T_n$ is a Borel subset of the quotient structure T for $n = \infty, 1, 2, \ldots$. But this implies that T is the Borel direct sum of its subspaces T_n and that T_n is Borel isomorphic to $\hat{A}_n$ for $n = \infty, 1, 2, \ldots$.

We have proved

THEOREM. *Let A be a separable C^*-algebra and $\mathfrak{H}$ an infinite-dimensional separable Hilbert space. Let* $\mathrm{Irr}(A, \mathfrak{H})$ *be given the Borel structure generated by its strong topology. Then the quotient Borel structure and the Mackey Borel structure agree on* $\hat{A} = \mathrm{Irr}(A, \mathfrak{H})/\sim$.

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