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ALGEBRAIC SURFACES WITH INFINITELY MANY TWISTOR LINES

A. ALTAVILLA✉ **and E. BALLICO**

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Abstract

We prove that a reduced and irreducible algebraic surface in $\mathbb{CP}^3$ containing infinitely many twistor lines cannot have odd degree. Then, exploiting the theory of quaternionic slice regularity and the normalisation map of a surface, we give constructive existence results for even degrees.

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1. Introduction and main results

In this paper we study integral (that is, reduced and irreducible) algebraic surfaces in $\mathbb{CP}^3$ containing infinitely many twistor lines. Let $\mathbb{HP}^1$ denote the *left* quaternionic projective line. This manifold is diffeomorphic to the 4-sphere $\mathbb{S}^4$. A twistor line is a fibre of the usual twistor fibration

$$\mathbb{CP}^1 \to \mathbb{CP}^3 \xrightarrow{\pi} \mathbb{HP}^1 (\simeq \mathbb{S}^4),$$

defined by

$$\pi[z_0, z_1, z_2, z_3] = [z_0 + z_1 j, z_2 + z_3 j],$$

where $j \in \mathbb{H}$ is such that $ij = k$ and (i, j, k) is the standard basis of imaginary units in $\mathbb{H}$. Motivation to study this fibration comes from its link with Riemannian and complex geometry (see, for example, [15]).

It is known (see, for example, [10]) that twistor lines can be identified with projective lines $\ell \subset \mathbb{CP}^3$ such that $j(\ell) = \ell$, where $j : \mathbb{CP}^3 \to \mathbb{CP}^3$ is the fixed-point-free antiholomorphic involution given by

$$j[z_0, z_1, z_2, z_3] \mapsto [-\bar{z}_1, \bar{z}_0, -\bar{z}_3, \bar{z}_2].$$

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Moreover, the map j induces (via Plücker embedding) a map (also called j) in the Grassmannian $\mathrm{Gr}(2,4) := \{t_1t_6 - t_2t_3 + t_4t_5 = 0\} \subset \mathbb{CP}^5$, defined by

$$j([t_1, t_2, t_3, t_4, t_5, t_6]) = [\overline{t_1}, \overline{t_5}, -\overline{t_4}, -\overline{t_3}, \overline{t_2}, \overline{t_6}], \tag{1.1}$$

(see, for example, [2, Section 3]). Twistor lines can be identified as points in $\mathrm{Gr}(2,4)$, which are fixed by this map j.

The study of algebraic surfaces from the twistor projection point of view is complete in the case of planes and quadrics [9, 16], but still partial in the case of cubics [5–7]. In a series of papers the authors have given general results on this topic by exploiting analytic [1, 4] (see also [10]) and algebraic [2, 3, 8] methods.

The goal of this paper is to use classical algebraic geometry and quaternionic slice regularity to show that there are no odd degree integral surfaces containing infinitely many twistor lines and that for each even degree there exists at least one. We will prove this last statement by giving two methods of construction. First of all, thanks to [16, Remark 14.5], an integral algebraic surface of degree d containing more than d^2 twistor lines must be j-invariant, and hence a surface containing infinitely many twistor lines is j-invariant. Surfaces with infinitely many lines are ruled and nonnormal and so we will deal with this class. However, we will show with a simple argument that cones are not allowed. Given a ruled surface Y we will recall its normalisation map $u : \mathbb{P}(\mathcal{E}) \to Y$, $\mathcal{E}$ being a rank-two vector bundle over a smooth curve C. Given such a vector bundle $\mathcal{E}$ and $L \subset \mathcal{E}$ a rank-one subsheaf of maximal degree, we say that $\mathcal{E}$ has the property £ if L is the unique rank-one subsheaf of maximal degree (see Definition 2.1). Afterwards, in Section 2, we will recall some known facts on the stability of rank-two vector bundles over a smooth curve and we will link them to the property £.

In Section 3, using the results proved in Section 2 and assuming that the surface Y is integral and ruled by twistor lines, we are able to prove that its normalisation $\mathbb{P}(\mathcal{E})$ is such that $\mathcal{E}$ does not have £ and, equivalently, that $\mathcal{E}$ is semistable (see Theorem 3.1). As a direct consequence we deduce that no odd degree integral rational surface with infinitely many twistor lines exists. More precisely, we prove the following result.

Proposition 1.1. *Let $Y \subset \mathbb{CP}^3$ be an integral rational surface containing infinitely many twistor lines. Then* $\deg(Y)$ *is even and* $\mathbb{CP}^1 \times \mathbb{CP}^1$ *is the normalisation of* Y.

With a bit more effort we are able to remove the rationality hypothesis and to prove the following result.

Theorem 1.2. *Let $Y \subset \mathbb{CP}^3$ be an integral surface containing infinitely many twistor lines. Then* $\deg(Y)$ *is even.*

After that we give two existence results for even degrees by constructive methods. In the first result, using the theory of slice regularity (see [11] for an overview in the quaternionic setting) and the results contained in [1, 4, 10], we solve the problem with the hypothesis of rationality. A slice regular function f defined on $\Omega \subset \mathbb{H}$ is a quaternionic function of a quaternionic variable $f : \Omega \to \mathbb{H}$ whose restrictions to any complex plane $\mathbb{C}_p = \mathrm{span}_{\mathbb{R}}\langle 1, p\rangle \subset \mathbb{H}$, such that $p^2 = -1$, are holomorphic functions. For any fixed orthonormal basis $\{1, i, j, k\} \subset \mathbb{H}$, it is possible to split a slice regular function f as $f = g + hj$, where $g|_{\mathbb{C}_i}, h|_{\mathbb{C}_i}$ are complex holomorphic functions of a $\mathbb{C}_i$-variable. The key result is that any slice regular function f can be *lifted* (via π^{-1}) to a holomorphic function $\tilde{f} : Q \subset \mathbb{CP}^3 \to \mathbb{CP}^3$, where $Q \cong \mathbb{CP}^1 \times \mathbb{CP}^1$, and the expression of $\tilde{f}$ is explicitly given in terms of the splitting $f = g + hj$. With this tool, the precise result that we are able to prove is as follows.

PROPOSITION 1.3. *For each even integer $d \geq 2$ there is a rational degree d ruled surface $Y \subset \mathbb{CP}^3$ containing infinitely many twistor lines.*

The last (constructive) result of the paper states that it is possible to select a smooth curve C to construct an integral surface Y with infinitely many twistor lines, such that $u : \mathbb{P}(\mathcal{E}) \to Y$ is its normalisation and $\mathcal{E}$ is a rank-two vector bundle over a smooth curve C. The precise statement is as follows.

THEOREM 1.4. *Let C be a smooth connected complex projective curve defined over $\mathbb{R}$ with $C(\mathbb{R}) \neq \emptyset$. Fix an integer d_0. Then there is an integer $d \geq d_0$ and a degree d integral surface $Y \subset \mathbb{CP}^3$ such that Y contains infinitely many twistor lines and the normalisation of Y is a $\mathbb{CP}^1$-bundle over C.*

The proof of this last theorem is constructive and it is exploited in the last example to generate a class of integral ruled surfaces of even degree each containing infinitely many twistor lines.

2. Preliminary results

The main reference for this section is [13, Ch. V]. Let C be a smooth connected complex projective curve of genus $g \geq 0$, $\mathcal{E}$ a rank-two holomorphic vector bundle on C and $L \subset \mathcal{E}$ a rank-one subsheaf of $\mathcal{E}$ with maximal degree. As in [13, V.2] or [14], but with opposite sign, set $s(\mathcal{E}) := 2\deg(L) - \deg(\mathcal{E})$. Note that for any line bundle R on C, the line subbundle $L \otimes R$ of $\mathcal{E} \otimes R$ is a rank-one subsheaf of $\mathcal{E} \otimes R$ with maximal degree and hence $s(\mathcal{E}) = s(\mathcal{E} \otimes R)$. Thus the integer $s(\mathcal{E})$ depends only on the isomorphism classes of the $\mathbb{CP}^1$-bundle $\mathbb{P}(\mathcal{E})$. Since $s(\mathcal{E}) \equiv \deg(\mathcal{E}) \pmod 2$, the parity classes of the integer $\deg(\mathcal{E})$ and $s(\mathcal{E})$ are constant in connected families of rank-two vector bundles on C and the parity class of $s(\mathcal{E})$ is a deformation invariant for the smooth surface $\mathbb{P}(\mathcal{E})$.

DEFINITION 2.1. Let C be a smooth connected complex projective curve of genus $g \geq 0$, $\mathcal{E}$ a rank-two holomorphic vector bundle on C and $L \subset \mathcal{E}$ a rank-one subsheaf of $\mathcal{E}$ with maximal degree. We say that $\mathcal{E}$ *has (the property)* £ if L is the unique rank-one subsheaf of $\mathcal{E}$ with maximal degree.

Thanks to the previous considerations, for any line bundle R on C the vector bundle $\mathcal{E}$ has £ if and only if $\mathcal{E} \otimes R$ has £. Hence it makes sense to say that a $\mathbb{CP}^1$-bundle $\mathbb{P}(\mathcal{E})$ has £ or does not have £.

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REMARK 2.2. If the genus g of C is equal to zero, then $s(\mathcal{E}) = s$ if and only if $\mathbb{P}(\mathcal{E}) \cong F_{-s} := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(s))$, the Hirzebruch surface with invariant $-s$ (see [13, V, 2.13, 2.14]).

By the definition of stability and semistability for rank-two vector bundles we see that $s(\mathcal{E}) < 0$ (respectively $s(\mathcal{E}) \leq 0$) if and only if $\mathcal{E}$ is stable (respectively semistable). Moreover, $s(\mathcal{E}) = 0$ if and only if $\mathcal{E}$ is strictly semistable, that is, it is semistable but not stable (see [13, V, Exercise 2.8] and also [14]).

LEMMA 2.3. *Let C be a smooth curve and $\mathcal{E}$ a rank-two vector bundle on C without £. Then $\mathcal{E}$ is semistable.*

PROOF. Let $\mathcal{G}$ be a rank-two vector bundle on C, which is not semistable. It is sufficient to prove that $\mathcal{G}$ has £. Let $L \subset \mathcal{G}$ be a maximal degree rank-one subsheaf. Since L has maximal degree, $\mathcal{G}/L$ has no torsion. Since C is a smooth curve, this means that $\mathcal{G}/L$ is a line bundle. We have the exact sequence

$$0 \to L \to \mathcal{G} \xrightarrow{v} \mathcal{G}/L \to 0.$$

Let M be any maximal degree rank-one subsheaf of $\mathcal{G}$. We have $\deg(M) = \deg(L)$. Since $\mathcal{G}$ is not semistable, $\deg(L) > \deg(\mathcal{G}/L)$ and so $\mathrm{Hom}(M, \mathcal{G}/L) = 0$. Thus $v_{|M} \equiv 0$, that is, $M \subseteq L$. Since $\deg(M) = \deg(L)$, it follows that $M = L$, that is $\mathcal{G}$ has £. □

Let C be any smooth connected projective curve of genus $g \geq 2$. See [14] for a large number of examples of stable rank-two vector bundles on C with or without £.

It is proved in [14] that if $\mathcal{E}$ is a general rank-two stable vector bundle on C with degree d, then the integer $s(\mathcal{E})$ is the only integer in $\{-g, 1-g\}$, which is $\equiv d \pmod 2$. If $s(\mathcal{E}) = -g$, then $\mathcal{E}$ has ∞^1 maximal degree rank-one subbundles and hence it does not have £. If $s(\mathcal{E}) = 1 - g$ and $\mathcal{E}$ is general, then $\mathcal{E}$ has exactly 2^g maximal degree-one subbundles (a result discovered by C. Segre in 1889).

We recall the following well-known observation, which characterises the property £ in the case of strict semistability.

LEMMA 2.4. *Assume $s(\mathcal{E}) = 0$. Then $\mathcal{E}$ has £ if and only if $\mathcal{E}$ is indecomposable. If $\mathcal{E}$ is decomposable, then either $\mathcal{E} \cong L^{\oplus 2}$ for some line bundle L (and in this case $\mathcal{E}$ has ∞^1 rank-one subsheaves with maximal degree) or $\mathcal{E} \cong L \oplus M$ with L, M line bundles, $\deg(L) = \deg(M)$ and $L \not\cong M$ (and in this case L and M are the only line subbundles of $\mathcal{E}$ with maximal degree).*

PROOF. Assume $\mathcal{E}$ is decomposable, say $\mathcal{E} \cong L_1 \oplus L_2$ with L_1 and L_2 line bundles on C with $\deg(L_2) \geq \deg(L_1)$. Since $s(\mathcal{E}) = 0$, in particular, $s(\mathcal{E}) \leq 0$ and $\deg(L_2) = \deg(L_1)$. Let $\pi_i : \mathcal{E} \to L_i$ denote the projections. Let L be a maximal degree rank-one subsheaf $L \subset L_1 \oplus L_2$. Then there is $i \in \{1, 2\}$ with $\pi_{i|L} \neq 0$. Hence $\deg(L) \leq \deg(L_i)$ and equality holds if and only if π_i induces an isomorphism $L \to L_i$. The maximality property of $\deg(L)$ implies $\deg(L) = \deg(L_i)$. This gives the second assertion of the lemma and the 'only if' part of the first assertion.

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Now assume that £ fails and take rank-one subsheaves L, M of $\mathcal{E}$ with maximal degree. M and L may be isomorphic as abstract line bundles, but they are supposed to be different subsheaves of $\mathcal{E}$. Since $\deg(L) = \deg(\mathcal{E})/2 = \deg(M)$, we have $L \nsubseteq M$ and $M \nsubseteq L$. Hence the map $f : L \oplus M \to \mathcal{E}$ induced by the inclusions $L \hookrightarrow \mathcal{E}$ and $R \hookrightarrow \mathcal{E}$ has generic rank two. Since f has generic rank two and $L \oplus M$ is a rank-two vector bundle, f is injective. Since $s(\mathcal{E}) = 0$, we have $\deg(L \oplus M) = \deg(\mathcal{E})$. Thus f is an isomorphism, concluding the proof of the lemma. □

3. Surfaces with infinitely many twistor lines

Suppose that $Y \subset \mathbb{CP}^3$ is an integral ruled projective surface of degree > 1 and let $u : X \to Y$ denote the normalisation map. Assume that Y is not a cone. Then X is a $\mathbb{CP}^1$-bundle on a smooth curve C, that is, there is a rank-two vector bundle $\mathcal{E}$ on C such that $X \cong \mathbb{P}(\mathcal{E})$. Let $v : \mathbb{P}(\mathcal{E}) \to C$ denote the map with $\mathbb{CP}^1$ as fibres. In particular, u sends each fibre of the ruling $v : \mathbb{P}(\mathcal{E}) \to C$ to a line of $\mathbb{CP}^3$. The map v is a locally trivial fibration (both in the Zariski and the Euclidean topology) and the curve C may be obtained in the following way. Fix a general hyperplane $H \subset \mathbb{CP}^3$. Since H is general, $H \cap Y$ is an integral plane curve. The curve C is the normalisation of the curve $H \cap Y$.

In this section we discuss the existence and nonexistence of integral surfaces $Y \subset \mathbb{CP}^3$ of degree d with $d \geq 3$, which contain infinitely many twistor lines. As stated in the introduction, any integral surface $Y \subset \mathbb{CP}^3$ of degree d containing at least $d^2 + 1$ twistor lines is j-invariant. Hence, if Y contains infinitely many twistor lines, then $j(Y) = Y$. Since any two twistor lines are disjoint and a cone has only finitely many curves not through the vertex, we may exclude cones. Moreover, no smooth surface of degree $d > 2$ contains infinitely many lines and therefore we need to allow singular surfaces. Hence, our situation accords with the construction described at the beginning of the section.

Our first result concerns the property £ and semistability.

Theorem 3.1. *Let $Y \subset \mathbb{CP}^3$ be an integral surface containing infinitely many twistor lines. Let $\mathcal{E}$ be a rank-two vector bundle on a smooth curve C and $\mathbb{P}(\mathcal{E})$ the normalisation of Y. Then $\mathcal{E}$ does not have £ and in particular, by Lemma 2.3, it is semistable.*

Proof. We know that $j(Y) = Y$, that Y is not a cone and that Y contains infinitely many twistor lines appearing as lines of the ruling. Let $u : \mathbb{P}(\mathcal{E}) \to Y$ denote the normalisation map. Assume that $\mathcal{E}$ has £ and let $T \subset \mathbb{P}(\mathcal{E})$ be the section of the ruling $v : \mathbb{P}(\mathcal{E}) \to C$ associated to the unique rank-one line subbundle L of $\mathcal{E}$ with maximal degree. The maximality of the integer $\deg(L)$ implies that L is a rank-one subbundle of $\mathcal{E}$, that is, $\mathcal{E}/L$ is a line bundle on C, and $\deg(\mathcal{E}/L)$ is the minimal degree of a line bundle M such that there is a surjective map $f : \mathcal{E} \to M$. Surjective maps $f : \mathcal{E} \to M$ (or, equivalently, embeddings of rank-one subbundles $R \hookrightarrow \mathcal{E}$) correspond to sections of the ruling v [13, Proposition V.2.6]. Since u is the normalisation map, u is finite. Thus

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$L \subset \mathbb{P}(\mathcal{E})$ corresponds to a minimal degree curve $D \subset Y$, which is the image of T, that is, $D = u(T)$. Since T intersects each fibre of Y, D intersects each line of the ruling of Y. Each section of v different from T has as image in $\mathbb{CP}^3$ a curve of degree greater than $\deg(D)$. Since $\deg(j(D)) = \deg(D)$, the section giving $j(D)$ is equal to the one giving D and hence $j(D) = D$. Fix a twistor line ℓ of the ruling and take $z \in D \cap \ell$ (z exists, because each fibre of v meets T). Then $j(z) \in \ell$ because ℓ is a twistor line, and $j(z) \in D$ because $j(D) = D$. Since $j : \mathbb{CP}^3 \to \mathbb{CP}^3$ has no fixed point, ℓ contains at least two different points of D. Hence the fibre of v over $v(u^{-1}(\ell))$ meets T in at least two different points, a contradiction. □

Proposition 1.1 now follows as an easy corollary of the previous result.

Proof of Proposition 1.1. Set $d := \deg(Y)$. By Theorem 3.1 the normalisation of Y is associated to a degree d rank-two vector bundle on C. Since Y is rational, the genus g of C is not positive and therefore $C \cong \mathbb{CP}^1$. But then the normalisation of Y is the Hirzebruch surface with invariant $s = s(\mathcal{E}) = 0$ (see Remark 2.2). Thanks to Theorem 3.1 and Lemma 2.4, two line bundles on $\mathbb{CP}^1$ of the same degree are isomorphic. It follows that $\mathcal{E} \cong L^{\oplus 2}$ with $L \cong \mathcal{O}_{\mathbb{CP}^1}(d/2)$. Therefore, d is even and $\mathbb{P}(L^{\otimes 2}) \cong \mathbb{P}(\mathcal{O}_{\mathbb{CP}^1}^{\oplus 2}) \cong \mathbb{CP}^1 \times \mathbb{CP}^1$. □

To remove the hypothesis of rationality in Proposition 1.1, we need the following Lemma whose proof is in the same spirit as that of Theorem 3.1.

Lemma 3.2. *Let $Y \subset \mathbb{CP}^3$ be an integral surface containing infinitely many twistor lines. Let $\mathcal{E}$ be a rank-two vector bundle on a smooth curve C and $\mathbb{P}(\mathcal{E})$ the normalisation of Y. Let $L \subset \mathcal{E}$ be a line bundle with maximal degree and $D \subset Y$ be a curve of minimal degree, which is the image of a section T of v corresponding to L. Then $j(D) \neq D$.*

Proof. Assume $j(D) = D$. Thus j induces an antiholomorphic involution of D. Fix a twistor line ℓ of the ruling and take $z \in D \cap \ell$. Then $j(z) \in \ell$ because ℓ is a twistor line and $j(z) \in D$ because $j(D) = D$. Since $j : \mathbb{CP}^3 \to \mathbb{CP}^3$ has no fixed point, ℓ contains at least two different points of D. Hence the fibre v over $v(u^{-1}(\ell))$ meets D in at least two different points, a contradiction. □

Proof of Theorem 1.2. Since any plane contains exactly one twistor line, we may assume $d \geq 3$. Let $u : X = \mathbb{P}(\mathcal{E}) \to Y$ denote the normalisation map. We recall that there is a smooth connected curve C and a rank-two vector bundle $\mathcal{E}$ on C such that $X = \mathbb{P}(\mathcal{E})$ and u sends each fibre of the ruling $v : \mathbb{P}(\mathcal{E}) \to C$ to a line of $\mathbb{CP}^3$.

Let $L \subset \mathcal{E}$ be a line bundle with maximal degree. As explained in the proof of Theorem 3.1, $L \subset \mathbb{P}(\mathcal{E})$ corresponds to a curve $D \subset Y$ of minimal degree, which is the image of a section of v. Since $j(Y) = Y$, $\deg(j(D)) = \deg(D)$ and $j(D) \neq D$ (by Lemma 3.2), and $j(D)$ corresponds to a maximal degree line subbundle $R \subset \mathcal{E}$ with $\deg(R) = \deg(L)$ and $R \neq L$.

Since $j(D) \neq D$ and D is an integral curve, the set $S := j(D) \cap D$ is finite. Note that $j(S) = S$. Since the antiholomorphic involution j has no base points, $b := |S|$ is even. Moreover, $j_{|D} : D \to j(D)$ is a bijection and hence j induces an antiholomorphic involution $\hat{\mathcal{J}} : X \to X$.

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Now we prove that d is even. Let $T_1 \subset X$ and $T_2 \subset X$ be the sections of v such that $u(T_1) = D$ and $u(T_2) = j(D)$. Since $u_{|T_1} : T_1 \to D$ and $u_{|T_2} : T_2 \to j(D)$ are bijections, $\hat{\mathcal{J}}$ induces a bijection $T_1 \to T_2$ and $S' := T_1 \cap T_2$ has cardinality b. For any divisors A, B on X let $A \cdot B$ denote the intersection product in the Chow ring of X, or equivalently, the cup product $H^2(X, \mathbb{C}) \times H^2(X, \mathbb{C}) \to H^4(X, \mathbb{C}) \cong \mathbb{C}$. We have $A \cdot B \in \mathbb{Z}$. For $p \in S'$ (if any) let c_p be the degree of the connected component containing p of the zero-dimensional scheme $T_1 \cap T_2$ (scheme-theoretic intersection). Since $\hat{\mathcal{J}}$ is an antiholomorphic isomorphism, we have $c_{j(p)} = c_p$ for all $p \in S'$. Since $T_1 \cap T_2$ is finite, we have $T_1 \cdot T_2 = \sum_{p \in S'} c_p$. Decomposing S' into the disjoint union of pairs $\{o, \hat{\mathcal{J}}(o)\}$, we see that $T_1 \cdot T_2$ is an even nonnegative integer. Since Y is not a cone, there is an ample and base point free line bundle $\mathcal{O}_X(1)$ on X such that u is induced by a four-dimensional linear subspace of $H^0(\mathcal{O}_X(1))$ and $d = \mathcal{O}_X(1) \cdot \mathcal{O}_X(1)$. We have $\mathrm{Pic}(X) \cong v^*(\mathrm{Pic}(C)) \oplus \mathbb{Z}T_1$ [13, Proposition V.2.3]. Write $\sim$ for the numerical equivalence of divisors and line bundles on X: by definition, two fibres F and F' are equivalent if $A \cdot F = A \cdot F'$ for each divisor A. Let F denote the numerical equivalence class of a fibre of v. Since two different fibres of v are disjoint, we have $F \cdot F = 0$. For any degree x line bundle A on C we have $v^*(A) \sim xF$. Since Y is ruled by lines, $\mathcal{O}_X(1) \cdot F = 1$. Thus there is an integer x such that $\mathcal{O}_X(1) \sim T_1 + xF$. Since T_2 is a section of v, there is an integer y such that $T_2 \sim T_1 + yF$. We have $\deg(D) = \mathcal{O}_X(1) \cdot T_1 = (T_1 + xF) \cdot T_1 = T_1 \cdot T_1 + x$ and $\deg(j(D)) = \mathcal{O}_X(1) \cdot T_2 = (T_1 + xF) \cdot (T_1 + yF) = T_1 \cdot T_1 + x + y$. Since $\deg(j(D)) = \deg(D)$, we have $y = 0$. Thus $T_1 \cdot T_1 = T_1 \cdot T_2$. Hence $T_1 \cdot T_1$ is an even nonnegative integer. Moreover, $d = \mathcal{O}_X(1) \cdot \mathcal{O}_X(1) = (T_1 + xF) \cdot (T_1 + xF) = T_1 \cdot T_1 + 2x$. Since $T_1 \cdot T_1$ is even, then d is even. □

We now give two different methods to construct integral surfaces with infinitely many twistor lines. We begin with Proposition 1.3. As stated in the introduction, the theory of quaternionic slice regularity can be exploited in this case. In fact, as explained in [1, 4, 10], any slice regular function $f : \Omega \to \mathbb{H}$ where $\Omega \subset \mathbb{H}$ can be lifted, with an explicit parametrisation, to a holomorphic curve $\tilde{f} : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^3$. This geometric construction is the core of the proof.

Proof of Proposition 1.3. For any even degree d it is possible to construct a rational ruled surface $Y \subset \mathbb{CP}^3$ parameterised by the twistor lift $\tilde{f}$ of a slice regular function f [1, 4, 10], that is,

$$\tilde{f} : \mathbb{CP}^1 \times \mathbb{CP}^1 \to Y, \quad ([s, u], [1, v]) \mapsto [s, u, sg(v) - u\hat{h}(v), sh(v) + u\hat{g}(v)],$$

where $g, \hat{g}, h$ and $\hat{h}$ are holomorphic functions defined on $\mathbb{C}$. From [4, Remark 4.9], if $\hat{g}(v) = \overline{g(\overline{v})}$ and $\hat{h}(v) = \overline{h(\overline{v})}$, then $\deg(Y)$ is even and Y contains infinitely many twistor fibres (namely the fibres over $f(\mathbb{R})$). By suitably choosing these functions, it is possible to construct a birational morphism between $\mathbb{CP}^1 \times \mathbb{CP}^1$ and Y. □

Finally, we prove Theorem 1.4. Note that the map $j : \mathbb{CP}^3 \to \mathbb{CP}^3$ can be decomposed as $j = {}^{-} \circ \sigma = \sigma \circ {}^{-}$, where ${}^{-}$ denotes the usual conjugation and $\sigma[z_0, z_1, z_2, z_3] = [-z_1, z_0, -z_3, z_2]$. Analogously, the map j defined in Equation (1.1)

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can be decomposed as $j = \bar{\ } \circ \sigma = \sigma \circ \bar{\ }$, where $\bar{\ }$ is again the usual conjugation in $\mathbb{CP}^5$, and $\sigma : \mathbb{CP}^5 \to \mathbb{CP}^5$ is defined by

$$\sigma([t_1 : t_2 : t_3 : t_4 : t_5 : t_6]) = [t_1 : t_5 : -t_4 : -t_3 : t_2 : t_6]. \tag{3.1}$$

The compatibility of these two maps is given by the following observation. Using standard Plücker coordinates $t_1 = z_0 \wedge z_1$, $t_2 = z_0 \wedge z_2$, $t_3 = z_0 \wedge z_3$, $t_4 = z_1 \wedge z_2$, $t_5 = z_1 \wedge z_3$ and $t_6 = z_2 \wedge z_3$,

$$\begin{aligned}
\sigma(t_1) &= \sigma(z_0) \wedge \sigma(z_1) = (-z_1) \wedge (z_0) = z_0 \wedge z_1 = t_1\\
\sigma(t_2) &= \sigma(z_0) \wedge \sigma(z_2) = (-z_1) \wedge (-z_3) = z_1 \wedge z_3 = t_5\\
\sigma(t_3) &= \sigma(z_0) \wedge \sigma(z_3) = (-z_1) \wedge z_2 = -t_4\\
\sigma(t_4) &= \sigma(z_1) \wedge \sigma(z_2) = z_0 \wedge (-z_3) = -t_3\\
\sigma(t_5) &= \sigma(z_1) \wedge \sigma(z_3) = z_0 \wedge z_2 = t_2\\
\sigma(t_6) &= \sigma(z_2) \wedge \sigma(z_3) = (-z_3) \wedge z_2 = z_2 \wedge z_3 = t_6.
\end{aligned}$$

We recall that $\mathbb{CP}^1$ is defined over $\mathbb{R}$ with $\mathbb{RP}^1$ as its real points. Moreover, if $g > 0$ there are infinitely many nonisomorphic smooth and connected complex projective curves C of genus g defined over $\mathbb{R}$ and with $C(\mathbb{R}) \neq \emptyset$ [12, 17].

Proof of Theorem 1.4. Set $F := \{t_2 - t_5 = t_3 + t_4 = 0\} \subset \mathbb{CP}^5$ and $E := \mathrm{Gr}(2, 4) \cap F$. Note that $\sigma_{|F}$ is the identity map. Take homogeneous coordinates t_1, t_5, t_4, t_6 on $F \cong \mathbb{CP}^3$. Note that E is the smooth quadric surface of F with $t_1t_6 - t_5^2 - t_4^2 = 0$ as its equation and, over $\mathbb{R}$, the quadric E has signature $(1, 3)$. So E has many real points, but it is not projectively isomorphic to $\mathbb{RP}^1 \times \mathbb{RP}^1$. Set $F' := F \setminus \{t_6 = 0\} \cong \mathbb{C}^3$.

Let C be a smooth and geometrically connected projective curve defined over $\mathbb{R}$ with $C(\mathbb{R}) \neq \emptyset$. Then $C(\mathbb{R})$ is topologically isomorphic to the disjoint union of k circles with $1 \leq k \leq g + 1$ (because $C(\mathbb{R})$ is infinite and hence it is dense in $C(\mathbb{C})$ for the Zariski topology). Fix $p \in C(\mathbb{R})$ and set $C' := C \setminus \{p\}$ so that C' is an affine and connected rational curve defined on $\mathbb{R}$. Thus there are nonconstant algebraic maps $f_4 : C' \to \mathbb{C}$ and $f_5 : C' \to \mathbb{C}$ defined over $\mathbb{R}$. Set $f_1 := f_4^2 + f_5^2$.

The map $(f_1, f_4, f_5) : C' \to \mathbb{C}^3$ is defined over $\mathbb{R}$ and it maps C' into $F' \cap E$. Since C is a smooth projective curve, (f_1, f_4, f_5) extends in a unique way to a regular map $\psi : C \to F$. As (f_1, f_4, f_5) is defined over $\mathbb{R}$, the uniqueness of the extension ψ means that ψ commutes with the complex conjugation. The image of C' is contained in $F' \cap E$, and therefore $D := \psi(C) \subset E \subset \mathrm{Gr}(2, 4)$ and, because f_5 is not constant, D is an integral projective curve. Since C and ψ are defined over $\mathbb{R}$, then D is defined over $\mathbb{R}$. We now state three claims, which lead us to the desired conclusion.

Claim 1. We can find f_4 and f_5 such that ψ is birational onto D (that is, C is the normalisation of D).

Since C is compact, ψ is a proper map. Moreover, since f_5 is a nonconstant algebraic map, f_5^2 is a proper map deleting finitely many points of $\mathbb{C}$, that is, there is a finite set $S \subset \mathbb{C}$ such that, taking $C'' := \{(f_5^2)^{-1}(\mathbb{C} \setminus S)\}$, f_5^2 induces a proper nonconstant map $u' : C'' \to \mathbb{C} \setminus S$. Since C'' is an irreducible (affine) curve, the differential of this map vanishes only at finitely many points. Increasing if necessary the finite set S, we may assume that u' has everywhere nonzero differential. Fix any $a \in \mathbb{C} \setminus S$ and set $S' := f_5^{-1}(a)$. Then S' is a nonempty finite set. To prove Claim 1 it is sufficient to take f_5 such that $f_5^2(b) \neq f_5^2(b')$ for all $b, b' \in S'$ such that $b \neq b'$ and to repeat the same argument for f_4. It is possible to find f_4 and f_5 satisfying the previous condition for the following reason. Since S' is a finite set, there are infinitely many algebraic maps $h : C' \to \mathbb{C}$ such that $h(p) \notin h(S' \setminus \{p\}) \cup -h(S')$ for all $p \in S'$ (and hence such that $h^2(p) \neq h^2(p')$ if $p \neq p'$ are element of S'). This proves Claim 1.

Claim 2. $j(D) = D$.

Since C and ψ are defined over $\mathbb{R}$, D is defined over $\mathbb{R}$, that is, complex conjugation induces a real analytic isomorphism between D and itself. In particular, complex conjugation induces a bijection of D. Thus to prove Claim 2 it is sufficient to prove that $\sigma(D) = D$. This follows since $\sigma_{|F}$ is the identity map. This proves Claim 2.

Claim 3. If Y is the integral surface in $\mathbb{CP}^3$ defined by $Y = \cup_{p \in D} \ell_p$, where ℓ_p is the line in $\mathbb{CP}^3$ corresponding to $p \in \mathrm{Gr}(2,4)$, then Y contains infinitely many twistor lines.

Fix $a \in D(\mathbb{R}) \subset E = \mathrm{Gr}(2,4) \cap F$ and let $\ell_a \subset Y$ be the line associated to a. We prove that $j(\ell_a) = \ell_a$ for any $a \in D(\mathbb{R})$. Since $a \in D(\mathbb{R})$, then ℓ_a is defined over $\mathbb{R}$ and hence the complex conjugation sends ℓ_a into itself. Moreover, as $D(\mathbb{R}) \subset F$ and $\sigma|_F$ is the identity map, we have $\sigma(a) = a$. Thus, Claim 3 follows from the decomposition $j = {}^- \circ \sigma = \sigma \circ {}^-$. This proves Claim 3.

Given these three claims, to conclude the proof of Theorem 1.4 we only need to observe that we may take f_5 such that the map $f_5^2 : C' \to \mathbb{C}$ has degree $\geq d_0$. □

The proof of Theorem 1.4 allows us to give several examples, all with Y having even degree, as explained in the following example.

Example 3.3. In the set-up of Theorem 1.4, take $F = \mathbb{CP}^3$ with homogeneous coordinates t_1, t_4, t_5, t_6. Let $E \subset F$ be the smooth quadric surface with equation $t_1 t_6 = t_5^2 + t_4^2$. All integral projective curves $D \subset E$ defined over $\mathbb{R}$ give examples of surfaces $Y \subset \mathbb{P}^3$ with infinitely many twistor lines. Taking D to be a complete intersection of E with a smooth quadric surface defined over $\mathbb{R}$ gives a degree-four elliptic ruled surface $Y \subset \mathbb{CP}^3$ with infinitely many twistor lines. Taking general intersections of E with a degree $t \geq 3$ hypersurface of F defined over $\mathbb{R}$ gives a smooth curve C with degree $2t$ and genus $t^2 - 2t + 1$ (adjunction formula). This construction only gives ruled surfaces Y of even degree (even allowing singular curves D) because E contains only even degree curves defined over $\mathbb{R}$.

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A. ALTAVILLA, Dipartimento di Ingegneria Industriale e Scienze Matematiche,
Università Politecnica delle Marche,
Via Brecce Bianche, 60131,
Ancona, Italy
e-mail: altavilla@dipmat.univpm.it

E. BALLICO, Dipartimento Di Matematica,
Università di Trento, Via Sommarive 14, 38123,
Povo, Trento, Italy
e-mail: edoardo.ballico@unitn.it

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