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RATIONAL K-STABILITY OF CONTINUOUS $C(X)$-ALGEBRAS

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Abstract

We show that the properties of being rationally K-stable passes from the fibres of a continuous $C(X)$-algebra to the ambient algebra, under the assumption that the underlying space X is compact, metrizable, and of finite covering dimension. As an application, we show that a crossed product C*-algebra is (rationally) K-stable provided the underlying C*-algebra is (rationally) K-stable, and the action has finite Rokhlin dimension with commuting towers.

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1. Introduction

Given a compact Hausdorff space X, a continuous $C(X)$-algebra is the section algebra of a continuous field of C*-algebras over X. Such algebras form an important class of nonsimple C*-algebras, and it is often of interest to understand those properties of a C*-algebra which pass from the fibres to the ambient $C(X)$-algebra.

Given a unital C*-algebra A, we write $\mathcal{U}_n(A)$ for the group of $n \times n$ unitary matrices over A. This is a topological group, and its homotopy groups $\pi_j(\mathcal{U}_n(A))$ are termed the *nonstable K-theory* groups of A. These groups were first systematically studied by Rieffel [20] in the context of noncommutative tori. Thomsen [26] built on this work, and developed the notion of quasiunitaries, thus constructing a homology theory for (possibly nonunital) C*-algebras.

Unfortunately, the nonstable K-theory for a given C*-algebra is notoriously difficult to compute explicitly. Even for the algebra of complex numbers, these groups are naturally related to the homotopy groups of spheres $\pi_j(S^n)$, which are not known for

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many values of j and n. It is here that rational homotopy theory has proved to be useful to topologists and, in this paper, we employ this tool in the context of C*-algebras.

A C*-algebra A is said to be *K-stable* if the homotopy groups $\pi_j(\mathcal{U}_n(A))$ are naturally isomorphic to the K-theory groups $K_{j+1}(A)$, and *rationally* K-stable if the analogous statement holds for the rational homotopy groups (see Definition 2.3). In [23], we proved that, for a continuous $C(X)$-algebra, the properties of being K-stable passes from the fibres to the whole algebra, provided the underlying space X is metrizable and has finite covering dimension. The goal of this paper is to prove an analogous result for rational K-stability.

THEOREM A. *Let X be a compact metric space of finite covering dimension and let A be a continuous $C(X)$-algebra. If each fibre of A is rationally K-stable, then so is A.*

As an interesting application of these results, we consider crossed product C*-algebras where the action has finite Rokhlin dimension (with commuting towers). A theorem of Gardella *et al.* [11] states that such a crossed product C*-algebra can be locally approximated by a continuous $C(X)$-algebra (see Definition 4.3). This leads to the following result.

THEOREM B. *Let $\alpha : G \to Aut(A)$ be an action of a compact Lie group on a separable C*-algebra A such that α has finite Rokhlin dimension with commuting towers. If A is rationally K-stable (K-stable), then so is $A \rtimes_\alpha G$.*

The paper is organized as follows. In Section 2 we introduce the basic notions used throughout the paper: those of nonstable K-groups, $C(X)$-algebras, and the rationalization of H-spaces. In Section 3, we prove Theorem A along with some applications and examples. Finally, Section 4 is devoted to the proof of Theorem B.

2. Preliminaries

2.1. Nonstable *K*-theory. We begin by reviewing the work of Thomsen in constructing the nonstable K-groups associated to a C*-algebra. For the proofs of the results mentioned in this section, the reader is referred to [26].

Let A be a C*-algebra (not necessarily unital). Define an associative composition $\cdot$ on A by

$$a \cdot b = a + b - ab. \tag{2-1}$$

An element $u \in A$ is said to be a quasiunitary if

$$u \cdot u^* = u^* \cdot u = 0.$$

We write $\widehat{\mathcal{U}}(A)$ for the set of all quasiunitary elements in A. For elements $u, v \in \widehat{\mathcal{U}}(A)$, we write $u \sim v$ if there is a continuous function $f : [0, 1] \to \widehat{\mathcal{U}}(A)$ such that $f(0) = u$ and $f(1) = v$. We write $\widehat{\mathcal{U}}_0(A)$ for the set of $u \in \widehat{\mathcal{U}}(A)$ such that $u \sim 0$. Note that $\widehat{\mathcal{U}}_0(A)$ is a closed, normal subgroup of $\widehat{\mathcal{U}}(A)$. We now define the two functors we are interested in.

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DEFINITION 2.1. Let A be a C^*-algebra, and $k \geq 0$ and $m \geq 1$ be integers. Define

$$G_k(A) := \pi_k(\widehat{\mathcal{U}}(A)) \quad \text{and} \quad F_m(A) := \pi_m(\widehat{\mathcal{U}}_0(A)) \otimes \mathbb{Q} \cong G_m(A) \otimes \mathbb{Q}.$$

Recall [21] that a homology theory on the category of C^*-algebras is a sequence $\{h_n\}$ of covariant, homotopy-invariant functors from the category of C^*-algebras to the category of abelian groups such that, if $0 \to J \xrightarrow{\iota} B \xrightarrow{p} A \to 0$ is a short exact sequence of C^*-algebras, then for each $n \in \mathbb{N}$, there exists a connecting map $\partial : h_n(A) \to h_{n-1}(J)$, making the sequence

$$\cdots \xrightarrow{\partial} h_n(J) \xrightarrow{h_n(\iota)} h_n(B) \xrightarrow{h_n(p)} h_n(A) \xrightarrow{\partial} h_{n-1}(J) \to \cdots$$

exact, and furthermore, ∂ is natural with respect to morphisms of short exact sequences. Furthermore, we say that a homology theory $\{h_n\}$ is continuous if, whenever $A = \lim A_i$ is an inductive limit in the category of C^*-algebras, then $h_n(A) = \lim h_n(A_i)$ in the category of abelian groups. The next proposition is a consequence of [26, Proposition 2.1] and [9, Theorem 4.4].

PROPOSITION 2.2. *For each $m \geq 1$, G_m and F_m are continuous homology theories.*

The notion of K-stability given below is due to Thomsen [26, Definition 3.1], and that of rational K-stability has been studied by Farjoun and Schochet [5, Definition 1.2], where it was termed rational Bott-stability.

DEFINITION 2.3. Let A be a C^*-algebra and $j \geq 2$. Define $\iota_j : M_{j-1}(A) \to M_j(A)$ to be the natural inclusion map

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

A is said to be *K-stable* if $G_k(\iota_j) : G_k(M_{j-1}(A)) \to G_k(M_j(A))$ is an isomorphism for all $k \geq 0$ and all $j \geq 2$. Furthermore, A is said to be *rationally K-stable* if the induced map $F_m(\iota_j) : F_m(M_{j-1}(A)) \to F_m(M_j(A))$ is an isomorphism for all $m \geq 1$ and all $j \geq 2$.

Note that, for a K-stable C^*-algebra, $G_k(A) \cong K_{k+1}(A)$, and for a rationally K-stable C^*-algebra, $F_m(A) \cong K_{m+1}(A) \otimes \mathbb{Q}$. A variety of interesting C*-algebras are known to be K-stable (see [23, Remark 1.5]). Clearly, K-stability implies rational K-stability. By [22, Theorem B], the converse is true for approximately finite-dimensional (AF) algebras. However, as Example 3.1 shows, the converse is not true in general.

2.2. $C(X)$-algebras. Let A be a C^*-algebra, and X a compact Hausdorff space. We say that A is a $C(X)$-algebra [13, Definition 1.5] if there is a unital $*$-homomorphism $\theta : C(X) \to \mathcal{Z}(M(A))$, where $\mathcal{Z}(M(A))$ denotes the center of the multiplier algebra of A. For simplicity of notation, if $f \in C(X)$ and $a \in A$, we write $fa := \theta(f)(a)$.

If $Y \subset X$ is closed, the set $C_0(X, Y)$ of functions in $C(X)$ that vanish on Y is a closed ideal of $C(X)$. Hence, $C_0(X, Y)A$ is a closed, two-sided ideal of A. The quotient of A by this ideal is denoted by $A(Y)$, and we write $\pi_Y : A \to A(Y)$ for the quotient map (also referred to as the restriction map). If $Z \subset Y$ is a closed subset of Y, we write

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$\pi_Z^Y : A(Y) \to A(Z)$ for the natural restriction map, so that $\pi_Z = \pi_Z^Y \circ \pi_Y$. If $Y = \{x\}$ is a singleton, we write $A(x)$ for $A(\{x\})$ and π_x for $\pi_{\{x\}}$. The algebra $A(x)$ is called the fibre of A at x. For $a \in A$, write $a(x)$ for $\pi_x(a)$. For each $a \in A$, there is a map

$$\Gamma_a : X \to \mathbb{R} \quad \text{given by } x \mapsto \|a(x)\|.$$

This map is, in general, upper semicontinuous [14, Lemma 2.3]. We say that A is a *continuous* $C(X)$*-algebra* if Γ_a is continuous for each $a \in A$.

If A is a $C(X)$-algebra, we often have reason to consider other $C(X)$-algebras obtained from A. For this purpose, the following result of Kirchberg and Wasserman is useful.

THEOREM 2.4 [14, Remark 2.6]. *Let X be a compact Hausdorff space, and let A be a continuous $C(X)$-algebra. If B is a nuclear C^*-algebra, then $A \otimes B$ is a continuous $C(X)$-algebra whose fibre at a point $x \in X$ is $A(x) \otimes B$.*

In particular, if A is a continuous $C(X)$-algebra, then so is $M_2(A)$. If $Y \subset X$ is a closed set, we denote the restriction map by $\eta_Y : M_2(A) \to M_2(A(Y))$, and we write $\iota_Y : A(Y) \to M_2(A(Y))$ for the natural inclusion map. If $Y = X$, we simply write ι (or ι^A) for ι_X. Note that $\eta_Y \circ \iota = \iota_Y \circ \pi_Y$. Once again, if $Y = \{x\}$, we simply write ι_x for $\iota_{\{x\}}$.

Finally, the notion of a pullback is important for our investigation. Let B, C, and D be C^*-algebras, and $\delta : B \to D$ and $\gamma : C \to D$ be $*$-homomorphisms. We define the pullback of this system to be

$$A = B \oplus_D C := \{(b, c) \in B \oplus C : \delta(b) = \gamma(c)\}.$$

This is described by a diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\psi"'] & B \arrow[d, "\delta"] \\ C \arrow[r, "\gamma"] & D \end{tikzcd} \tag{2-2}$$

where $\phi(b, c) = b$ and $\psi(b, c) = c$. The next lemma allows us to inductively put together a $C(X)$-algebra from its natural quotients.

LEMMA 2.5 [2, Lemma 2.4]. *Let X be a compact Hausdorff space and Y and Z be two closed subsets of X such that $X = Y \cup Z$. If A is a $C(X)$-algebra, then A is isomorphic to the pullback*

$$\begin{tikzcd} A \arrow[r, "\pi_Y"] \arrow[d, "\pi_Z"'] & A(Y) \arrow[d, "\pi^Y_{Y\cap Z}"] \\ A(Z) \arrow[r, "\pi^Z_{Y\cap Z}"] & A(Y \cap Z) \end{tikzcd}$$

2.3. Rational homotopy theory. We now discuss some basic facts about the rationalization of groups and spaces as developed in [10].

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A connected CW complex Y is said to be nilpotent if $\pi_1(Y)$ is a nilpotent group and $\pi_1(Y)$ acts nilpotently on $\pi_j(Y)$ for all $j \geq 2$. A nilpotent space Y is a rational space if, for each $j \geq 1$, the homotopy group $\pi_j(Y)$ is a $\mathbb{Q}$-vector space. A continuous map $r : Y \to Z$ is said to be a rationalization of Y if Z is a rational space and

$$r_* \otimes \mathrm{id} : \pi_*(Y) \otimes \mathbb{Q} \to \pi_*(Z) \otimes \mathbb{Q} \cong \pi_*(Z)$$

is an isomorphism. The next theorem (see [10, Theorem II.3A]) is fundamental to the theory.

THEOREM 2.6 (Hilton, Mislin, and Roitberg). *Every nilpotent CW complex Y has a rationalization $r : Y \to Y_{\mathbb{Q}}$, where $Y_{\mathbb{Q}}$ is a CW complex. The space $Y_{\mathbb{Q}}$ is uniquely determined up to homotopy equivalence.*

We now specialize to the situation of our interest. Recall that an H-space is a pointed space (Y, e) endowed with a 'multiplication' map $\mu : Y \times Y \to Y$ such that e is a homotopy unit, that is, the maps $\lambda, \rho : Y \to Y$ given by $\lambda(y) := \mu(e, y)$ and $\rho(y) := \mu(y, e)$ are both homotopic to id_Y. We denote this H-space by the triple (Y, e, μ). We say that (Y, e, μ) is homotopy-associative if the maps

$$\mu \circ (\mu \times \mathrm{id}_Y) \quad \text{and} \quad \mu \circ (\mathrm{id}_Y \times \mu) : Y \times Y \times Y \to Y$$

are homotopic. In what follows, we implicitly assume that the H-spaces under consideration are all homotopy-associative.

Now suppose (Y, e, μ) is an H-space, where the space Y is a connected CW complex. Since Y is nilpotent, it has a rationalization $r : Y \to Y_{\mathbb{Q}}$ by Theorem 2.6. Now, by [16, Theorem 6.2.3], $r \times r : Y \times Y \to Y_{\mathbb{Q}} \times Y_{\mathbb{Q}}$ is a rationalization. By the universal properties of the rationalization, there is a unique map $\rho : Y_{\mathbb{Q}} \times Y_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ such that the following diagram commutes up to homotopy:

$$\begin{tikzcd} Y \times Y \arrow[r, "\mu"] \arrow[d, "r\times r"'] & Y \arrow[d, "r"] \\ Y_{\mathbb{Q}} \times Y_{\mathbb{Q}} \arrow[r, "\rho"] & Y_{\mathbb{Q}} \end{tikzcd} \tag{2-3}$$

By the mapping cylinder construction, we may assume that r is a cofibration. Then $r \times r$ is also a cofibration as it is the composition of two cofibrations

$$Y \times Y \to Y \times Y_{\mathbb{Q}} \to Y_{\mathbb{Q}} \times Y_{\mathbb{Q}}.$$

Hence, by [24, Problem 5.3], we may assume that the above diagram commutes strictly. If we set $e_{\mathbb{Q}} := r(e)$, then it follows from [16, Proposition 6.6.2] that the triple $(Y_{\mathbb{Q}}, e_{\mathbb{Q}}, \rho)$ is an H-space. Furthermore, by universality, we may also ensure that the triple $(Y, e_{\mathbb{Q}}, \rho)$ is homotopy-associative. We summarize this result below.

PROPOSITION 2.7. *If (Y, e, μ) is a homotopy-associative H-space, where Y is a connected CW complex, then there is a homotopy-associative H-space $(Y_{\mathbb{Q}}, e_{\mathbb{Q}}, \rho)$ and a map $r : Y \to Y_{\mathbb{Q}}$ such that r is a rationalization, and Diagram (2-3) commutes strictly.*

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If A is a C*-algebra, then $\widehat{\mathcal{U}}(A)$ has the homotopy type of a CW complex [26, Corollary 1.6]. Therefore, $\widehat{\mathcal{U}}_0(A)$ may be regarded as a connected CW complex. Since $\widehat{\mathcal{U}}_0(A)$ is a topological group (and hence a connected H-space), it has a rationalization $r : \widehat{\mathcal{U}}_0(A) \to \widehat{\mathcal{U}}_0(A)_{\mathbb{Q}}$. By Proposition 2.7, $\widehat{\mathcal{U}}_0(A)_{\mathbb{Q}}$ has the structure of an H-space, which we write as $(\widehat{\mathcal{U}}_0(A)_{\mathbb{Q}}, e_{\mathbb{Q}}, \rho)$, where $e_{\mathbb{Q}} = r(0)$. Finally, observe that the commutativity of Diagram (2-3) implies that $\rho(e_{\mathbb{Q}}, e_{\mathbb{Q}}) = e_{\mathbb{Q}}$.

2.4. Notational conventions. If A and B are two C^*-algebras, the symbol $A \otimes B$ will always denote the minimal tensor product. If $B = C_0(X)$ is commutative, we identify $C_0(X) \otimes A$ with $C_0(X, A)$, the space of continuous A-valued functions on X that vanish at infinity.

Suppose f and g are two continuous paths in a topological space Y. If $f(1) = g(0)$, we write $f \bullet g$ for the concatenation of the two paths. If f and g agree at end-points, we write $f \sim_h g$ if there is a path homotopy between them. Furthermore, we write $\overline{f}$ for the path $\overline{f}(t) := f(1-t)$ and the constant path at a point $*$ as e_*.

If X and Y are two pointed spaces, we write $C_*(X, Y)$ for the space of base-point-preserving continuous functions from X to Y. Note that if A is a C*-algebra, and Y is either A or $\widehat{\mathcal{U}}_0(A)$, then we always take 0 to be the base point. In that case, $C_*(X, A)$ is a C*-algebra, and, for any path-connected space X, there is a natural isomorphism

$$\widehat{\mathcal{U}}(C_*(X, A)) \cong C_*(X, \widehat{\mathcal{U}}_0(A)).$$

Henceforth, we identify these two spaces without further comment.

If (Y, e, μ) is an H-space and $a \in Y$, we may define nonnegative powers of a inductively by $\mu_0(a) := e$ and $\mu_n(a) := \mu(\mu_{n-1}(a), a)$. Similarly, if $f : X \to Y$ is any function, we define nonnegative powers of f pointwise, that is, $\mu_n(f)(x) := \mu_n(f(x))$ for all $n \geq 0$. Note that, if $f \in C_*(S^j, Y)$, then $[\mu_n(f)] = n[f]$ in $\pi_j(Y)$ by [28, Theorem 4.7]. Throughout the rest of the paper, for any C^*-algebra B, we write μ^B for the multiplication in $\widehat{\mathcal{U}}_0(B)$ given by Equation (2-1), and ρ^B for the multiplication in $\widehat{\mathcal{U}}_0(B)_{\mathbb{Q}}$ given by Proposition 2.7.

3. Main results

The goal of this section is to provide a proof for Theorem A. To put things in perspective, we begin by constructing an example of a C*-algebra that is rationally K-stable, but not K-stable.

EXAMPLE 3.1. Let X be a connected, finite CW complex such that $H^i(X; \mathbb{Z})$ is a finite group for all $i \geq 1$ (for instance, we may take X to be the real projective space $\mathbb{RP}^2$), and set

$$A := C_*(X, \mathbb{C}).$$

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Note that, for all $n, m \geq 1$,

$$F_n(M_m(A)) = \pi_n(C_*(X; \mathcal{U}_m)) \otimes \mathbb{Q} = \bigoplus_{l \geq n} \tilde{H}^{l-n}(X; \pi_l(\mathcal{U}_m) \otimes \mathbb{Q}) = 0$$

by [19, Theorem 4.20]. Hence, A is rationally K-stable.

Now suppose that A is K-stable. We fix a path connected H-space Y, and consider the following fibration sequence (see the proof of [19, Proposition 4.9]):

$$C_*(X, Y) \to C(X, Y) \to Y.$$

This fibration has a section, hence the long exact homotopy sequence breaks into split short exact sequences

$$0 \to \pi_n(C_*(X, Y)) \to \pi_n(C(X, Y)) \to \pi_n(Y) \to 0 \tag{3-1}$$

for all $n \in \mathbb{N}$. By a result of Thom [25, Theorem 2], if $Y = S^1 = K(\mathbb{Z}, 1)$, then $\pi_n(C(X, S^1)) \cong H^{1-n}(X; \mathbb{Z})$. It follows that

$$G_n(A) = \pi_n(\widehat{\mathcal{U}}(C_*(X, \mathbb{C}))) \cong \pi_n(C_*(X, S^1)) = 0$$

for all $n \geq 1$. If A were K-stable, it would follow that

$$\pi_n(C_*(X, \mathcal{U}_m)) \cong G_n(M_m(A)) \cong G_n(A) = 0$$

for all $n, m \geq 1$. Hence, $\pi_n(C_*(X, \widehat{\mathcal{U}}(\mathcal{K}))) \cong G_n(A \otimes \mathcal{K}) = 0$ for all $n \geq 1$. Taking $Y = \widehat{\mathcal{U}}(K)$ in Equation (3-1), we conclude that

$$\pi_n(C(X, \widehat{\mathcal{U}}(\mathcal{K}))) \cong \pi_n(\widehat{\mathcal{U}}(\mathcal{K})) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{3-2}$$

Thus, in order to show that A is not K-stable, it suffices to show that Equation (3-2) cannot hold. To do this, we consider the work of Federer [6], who constructed a spectral sequence converging to these homotopy groups (note that X is a finite CW complex, and $\widehat{\mathcal{U}}(K)$ is a simple space, so the results of [6] do apply). The first page of this spectral sequence, which converges to $\pi_p(C(X, \widehat{\mathcal{U}}(K)))$, is of the form

$$C^{(1)}_{p,q} \cong H^q(X; \pi_{p+q}(\widehat{\mathcal{U}}(K)))$$

with differential $d : C^{(1)}_{p,q} \to C^{(1)}_{p-1,q+2}$. Therefore, for $C^{(1)}_{p,q}$ to be nonzero, $p + q$ must be odd. But in that case, $C^{(1)}_{p-1,q+2}$ is zero. Hence, the spectral sequence collapses at the very first page, so $C^{(1)}_{p,q} = C^{(\infty)}_{p,q}$. Therefore,

$$\pi_n(C(X, \widehat{\mathcal{U}}(K))) = \bigoplus_{q \geq 0} H^q(X; \pi_{n+q}(\widehat{\mathcal{U}}(K)))$$

for all $n \geq 1$. This is a finite sum of finite groups (by our choice of X), contradicting Equation (3-2). Thus, A is not K-stable.

We now turn to the proof of Theorem A, and begin with some lemmas that will be useful to us. The first lemma, which we use repeatedly throughout the paper, follows from [26, Theorem 1.9] and [3, Theorem 4.8].

LEMMA 3.2. *Let $\varphi : A \to B$ be a surjective $*$-homomorphism between two C*-algebras. Then the induced maps $\varphi : \widehat{\mathcal{U}}(A) \to \varphi(\widehat{\mathcal{U}}(A))$ and $\varphi : \widehat{\mathcal{U}}_0(A) \to \widehat{\mathcal{U}}_0(B)$ are both Serre fibrations.*

LEMMA 3.3 [23, Lemma 2.2]. *Let $a, b \in \widehat{\mathcal{U}}(A)$ such that $\|a - b\| < 2$. Then $a \sim b$ in $\widehat{\mathcal{U}}(A)$.*

Note that, for any element a in a C*-algebra A (not necessarily a quasiunitary), we write $\mu_N^A(a)$ for $a \cdot a \cdots a$ (N times). The next lemma is a variation of [23, Lemma 2.3] that we need for our purposes.

LEMMA 3.4. *For any $\epsilon > 0$ and any $N \in \mathbb{N}$, there exists $\delta > 0$ satisfying the following condition. For any C*-algebra A, and any element $a \in A$ such that $\|a\| \leq 2$, $\|a \cdot a^*\| < \delta$, and $\|a^* \cdot a\| < \delta$, there exists a quasiunitary $u \in \widehat{\mathcal{U}}(A)$ such that*

$$\|\mu_N^A(u) - \mu_N^A(a)\| < \epsilon.$$

PROOF. Note that the function $d \mapsto \mu_N^A(d)$ is a polynomial in d (that is independent of A). Thus, for any $\epsilon > 0$, there exists $\eta > 0$ satisfying the following condition. For any C*-algebra A and any $c, d \in A$ with $\|c\|, \|d\| \leq 2$ such that $\|c - d\| < \eta$, we have $\|\mu_N^A(c) - \mu_N^A(d)\| < \epsilon$.

We choose $\delta > 0$ satisfying the conditions of [23, Lemma 2.3] with $\epsilon = \eta$. Then there exists $u \in \widehat{\mathcal{U}}(A)$ such that $\|u - a\| < \eta$, so that $\|\mu_N^A(u) - \mu_N^A(a)\| < \epsilon$. □

Our proof of Theorem A is by induction on the covering dimension of the underlying space. The next theorem is the base case, and it holds even if the space is not metrizable. In what follows we repeatedly use the fact that, for any abelian group A, any element in $A \otimes \mathbb{Q}$ can be represented as an elementary tensor of the form $u \otimes 1/m$ for some $u \in A$ and $m \in \mathbb{Z}$.

THEOREM 3.5. *Let X be a compact Hausdorff space of zero covering dimension, and let A be a continuous $C(X)$-algebra. If each fibre of A is rationally K-stable, then so is A.*

PROOF. We show that the map

$$\iota_* \otimes \mathrm{id} : \pi_j(\widehat{\mathcal{U}}_0(A)) \otimes \mathbb{Q} \to \pi_j(\widehat{\mathcal{U}}_0(M_n(A))) \otimes \mathbb{Q}$$

is an isomorphism for each $n \geq 2$ and $j \geq 1$. For simplicity of notation, we fix $n = 2$.

We first consider injectivity. Suppose $[f] \in \pi_j(\widehat{\mathcal{U}}_0(A))$ and $q \in \mathbb{Q}$ are such that$[\iota \circ f] \otimes q = 0$ in $\pi_j(\widehat{\mathcal{U}}_0(M_2(A)) \otimes \mathbb{Q}$. Then, by elementary group theory, $[\iota \circ f]$ has finite order in $\pi_j(\widehat{\mathcal{U}}_0(M_2(A)))$. Thus, for $x \in X$, $[\iota_x \circ \pi_x \circ f]$ has finite order in $\pi_j(\widehat{\mathcal{U}}_0(M_2(A(x)))$. Since $A(x)$ is rationally K-stable, $[\pi_x \circ f]$ has finite order in

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$\pi_j(\widehat{\mathcal{U}}_0(A(x)))$. Hence, there exist $N_x \in \mathbb{N}$ and a path $F : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(A(x)))$ such that

$$F(0) = 0 \quad \text{and} \quad F(1) = \mu_{N_x}^{A(x)}(\pi_x \circ f).$$

Note that by [23, Lemma 2.4], there is a closed neighbourhood Y_x of x such that $\mu_{N_x}^{A(Y_x)}(\pi_{Y_x} \circ f) \sim 0$ in $C_*(S^j, \widehat{\mathcal{U}}_0(A(Y_x)))$. Since X is zero-dimensional, we may assume that the sets $\{Y_x : x \in X\}$ are clopen and disjoint. Since X is compact, we may obtain a finite subcover $\{Y_{x_1}, Y_{x_2}, \dots, Y_{x_n}\}$. By Lemma 2.5,

$$A \cong A(Y_{x_1}) \oplus A(Y_{x_2}) \oplus \cdots \oplus A(Y_{x_n})$$

via the map $b \mapsto (\pi_{Y_{x_1}}(b), \pi_{Y_{x_2}}(b), \dots, \pi_{Y_{x_n}}(b))$. If $N := \mathrm{lcm}_{1 \le i \le n}(N_{x_i})$, then we must have $\mu_N^{A(Y_{x_i})}(\pi_{Y_{x_i}} \circ f) \sim 0$ in $C_*(S^j, \widehat{\mathcal{U}}_0(A(Y_{x_i})))$, for each $1 \le i \le n$. Thus, $\mu_N(f) \sim 0$ in $\pi_j(\widehat{\mathcal{U}}_0(A))$. Hence, $[f]$ has finite order in $\pi_j(\widehat{\mathcal{U}}_0(A))$, so $[f] \otimes q = 0$ in $\pi_j(\widehat{\mathcal{U}}_0(A)) \otimes \mathbb{Q}$. Thus, $\iota_* \otimes \mathrm{id}$ is injective.

For surjectivity, choose $[u] \in \pi_j(\widehat{\mathcal{U}}_0(M_2(A)))$ and $m \in \mathbb{Z}$ nonzero. We wish to construct an element $[\omega] \in \pi_j(\widehat{\mathcal{U}}_0(A))$ and $q \in \mathbb{Q}$ such that

$$\iota_* \otimes \mathrm{id}([\omega] \otimes q) = [u] \otimes \frac{1}{m}.$$

To this end, fix $x \in X$. Since $A(x)$ is rationally K-stable, there exist $[f_x] \in \pi_j(\widehat{\mathcal{U}}_0(A(x)))$ and $q_x \in \mathbb{Q}$ such that

$$(\iota_x)_* \otimes \mathrm{id}([f_x] \otimes q_x) = [\eta_x \circ u] \otimes \frac{1}{m}.$$

Replacing f_x by a multiple of itself if need be, we obtain integers $L_x, N_x \in \mathbb{N}$ such that

$$N_x[\iota_x \circ f_x] = L_x[\eta_x \circ u]$$

in $\pi_j(\widehat{\mathcal{U}}_0(M_2(A(x))))$. Hence, there is a path $g_x : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(M_2(A(x))))$ such that $g_x(0) = \mu_{L_x}^{M_2(A(x))}(\eta_x \circ u)$ and $g_x(1) = \mu_{N_x}^{M_2(A(x))}(\iota_x \circ f_x)$. Choose $e_x \in C_*(S^j, A)$ such that $\pi_x \circ e_x = f_x$. Note that e_x may not be a quasiunitary, but we may ensure that $\|e_x\| = \|f_x\| \le 2$. Since the map

$$\eta_x : C_*(S^j, \widehat{\mathcal{U}}_0(M_2(A))) \to \eta_x(C_*(S^j, \widehat{\mathcal{U}}_0(M_2(A))))$$

is a fibration, g_x lifts to a path $G_x : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(M_2(A)))$ such that $G_x(0) = \mu_{L_x}^{M_2(A)}(u)$. Let $b_x := G_x(1)$, so that $\eta_x \circ b_x = \mu_{N_x}^{M_2(A(x))}(\iota_x \circ \pi_x \circ e_x)$. Choose $\delta > 0$ so that the conclusion of Lemma 3.4 holds for $\epsilon = 1$ and $N = N_x$. Since A is a continuous $C(X)$-algebra, there is a closed neighbourhood Y_x of x such that

$$\|\pi_{Y_x} \circ (e_x^* \cdot e_x)\| < \delta, \quad \|\pi_{Y_x} \circ (e_x \cdot e_x^*)\| < \delta,$$

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and $\|\eta_{Y_x} \circ b_x - \mu_{N_x}^{M_2(A(Y_x))}(\eta_{Y_x} \circ \iota \circ e_x)\| < 1$. By Lemma 3.4, there is a quasiunitary $d_x \in C_*(S^j, \widehat{\mathcal{U}}_0(A(Y_x)))$ such that $\|\mu_{N_x}^{A(Y_x)}(d_x) - \mu_{N_x}^{A(Y_x)}(\pi_{Y_x} \circ e_x)\| < 1$, so that

$$\|\mu_{N_x}^{M_2(A(Y_x))}(\iota_{Y_x} \circ d_x) - \eta_{Y_x} \circ b_x\| < 2.$$

By Lemma 3.3, $\mu_{N_x}^{M_2(A(Y_x))}(\iota_{Y_x} \circ d_x) \sim \eta_{Y_x} \circ b_x$ in $C_*(S^j, \widehat{\mathcal{U}}_0(M_2(A(Y_x)))$. Hence, we have $\iota_{Y_x} \circ \mu_{N_x}^{A(Y_x)}(d_x) \sim \mu_{L_x}^{M_2(A(Y_x))}(\eta_{Y_x} \circ u)$. As before, since X is compact and zero-dimensional, we may choose a finite refinement of $\{Y_x : x \in X\}$ consisting of disjoint clopen sets, which we denote by $\{Y_{x_1}, Y_{x_2}, \ldots, Y_{x_n}\}$. Then, by Lemma 2.5,

$$A \cong A(Y_{x_1}) \oplus A(Y_{x_2}) \oplus \cdots \oplus A(Y_{x_n})$$

via the map $a \mapsto (\pi_{Y_{x_1}}(a), \pi_{Y_{x_2}}(a), \ldots, \pi_{Y_{x_n}}(a))$. Similarly,

$$M_2(A) \cong M_2(A(Y_{x_1})) \oplus M_2(A(Y_{x_2})) \oplus \cdots \oplus M_2(A(Y_{x_n}))$$

via the map $b \mapsto (\eta_{Y_{x_1}}(b), \eta_{Y_{x_2}}(b), \ldots, \eta_{Y_{x_n}}(b))$. Define $L := \mathrm{lcm}_{1 \le i \le n}(L_{x_i})$, so that

$$\iota_{Y_{x_i}} \circ c_{x_i} \sim \mu_L^{M_2(A(Y_{x_i}))}(\eta_{Y_{x_i}} \circ u)$$

in $C_*(S^j, \widehat{\mathcal{U}}_0(M_2(A(Y_{x_i}))))$, where $c_{x_i} \in C_*(S^j, \widehat{\mathcal{U}}_0(A(Y_{x_i})))$ is an appropriate power of d_{x_i}. Choose $\omega \in C_*(S^j, \widehat{\mathcal{U}}_0(A))$ such that $\pi_{Y_{x_i}} \circ \omega = c_{x_i}$ for all $1 \le i \le n$. Furthermore, for each $1 \le i \le n$,

$$\eta_{Y_{x_i}} \circ \iota \circ \omega = \iota_{Y_{x_i}} \circ c_{x_i} \sim \mu_L^{M_2(A(Y_{x_i}))}(\eta_{Y_{x_i}} \circ u)$$

in $C_*(S^j, \widehat{\mathcal{U}}_0(M_2(A(Y_{x_i}))))$, so that $\iota \circ \omega \sim \mu_L^{M_2(A)}(u)$ in $C_*(S^j, \widehat{\mathcal{U}}_0(M_2(A)))$. Thus,

$$\iota_* \otimes \mathrm{id}\left([\omega] \otimes \frac{1}{Lm}\right) = [u] \otimes \frac{1}{m}.$$

This proves the surjectivity of $\iota_* \otimes \mathrm{id}$. □

The next few lemmas allow us to extend this argument to higher-dimensional spaces.

LEMMA 3.6. *Let (Y, e, μ) be an H-space, where Y is a connected CW complex. Let $r : (Y, e, \mu) \to (Y_{\mathbb{Q}}, e_{\mathbb{Q}}, \rho)$ be the rationalization map from Proposition 2.7, and let $j \ge 1$ be a fixed integer.*

(1) *Let $[f] \in \pi_j(Y)$ and $n \in \mathbb{N}$, and suppose there is a path $H : [0,1] \to C_*(S^j, Y)$ such that $H(0) = e$ and $H(1) = \mu_n(f)$. Then there exists a path $G : [0,1] \to C_*(S^j, Y_{\mathbb{Q}})$ with $G(0) = e_{\mathbb{Q}}$ and $G(1) = r \circ f$, such that*

$$r \circ H \sim_h \rho_n(G)$$

in $C_*(S^j, Y_{\mathbb{Q}})$.

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(2) *Let* $[f] \in \pi_j(Y)$, *and suppose there is a path* $G' : [0,1] \to C_*(S^j, Y_{\mathbb{Q}})$ *such that* $G'(0) = e_{\mathbb{Q}}$ *and* $G'(1) = r \circ f$. *Then there exists a natural number* $N \in \mathbb{N}$ *and a path* $H' : [0,1] \to C_*(S^j, Y)$ *with* $H'(0) = e$ *and* $H'(1) = \mu_N(f)$, *such that*

$$r \circ H' \sim_h \rho_N(G')$$

in $C_*(S^j, Y_{\mathbb{Q}})$.

PROOF. (1) Since $Y_{\mathbb{Q}}$ is a rational space, $[r \circ f] = 0$ in $\pi_j(Y_{\mathbb{Q}})$. Then, there is a path $L : [0,1] \to C_*(S^j, Y_{\mathbb{Q}})$ such that $L(0) = e_{\mathbb{Q}}$ and $L(1) = r \circ f$. Thus, $\rho_n(L) : [0,1] \to C_*(S^j, Y_{\mathbb{Q}})$ is a path that satisfies $\rho_n(L)(0) = e_{\mathbb{Q}}$ and $\rho_n(L)(1) = \rho_n(r \circ f)$. Note that $\pi_1(C_*(S^j, Y_{\mathbb{Q}}))$ is itself a $\mathbb{Q}$-vector space [10, Theorem II.3.11] and $(r \circ H) \bullet \overline{\rho_n(L)}$ is a loop in $C_*(S^j, Y_{\mathbb{Q}})$. Thus, there exists $[T] \in \pi_1(C_*(S^j, Y_{\mathbb{Q}}))$ such that

$$[(r \circ H) \bullet \overline{\rho_n(L)}] = n[T] = [\rho_n(T)].$$

Hence, $G := T \bullet L$ is the required homotopy (since the operation ρ_n respects concatenation).

(2) Since $[r \circ f] = 0$, in $\pi_j(Y_{\mathbb{Q}})$, under $r_* \otimes \mathbb{Q} : \pi_j(Y) \otimes \mathbb{Q} \to \pi_j(Y_{\mathbb{Q}})$ it follows that $[r \circ f] \cong [f] \otimes 1 = 0$ in $\pi_j(Y) \otimes \mathbb{Q}$. Hence, by elementary group theory, this implies that $[f]$ has finite order in $\pi_j(Y)$. Thus, there exists $n \in \mathbb{N}$ such that $n[f] = 0$ in $\pi_j(Y)$, say, by homotopy $K : [0,1] \to C_*(S^j, Y)$ such that

$$K(0) = e, \quad K(1) = \mu_n(f).$$

Now, by a similar argument to that of part (1), $(r \circ K) \bullet \overline{\rho_n(G')}$ is a loop in $C_*(S^j, Y_{\mathbb{Q}})$, which is a rational space. Hence, there exists $[T] \in \pi_1(C_*(S^j, Y_{\mathbb{Q}}))$ satisfying

$$n[T] = [\rho_n(T)] = [r \circ K \bullet \overline{\rho_n(G')}].$$

Now $n[T] \in \pi_1(C_*(S^j, Y_{\mathbb{Q}})) \cong \pi_1(C_*(S^j, Y)) \otimes \mathbb{Q}$, so there exist $[h] \in \pi_1(C_*(S^j, Y))$ and $m \in \mathbb{Z}$ such that

$$n[T] = \frac{[r \circ h]}{m}.$$

Thus, by the fact that ρ is homotopy-associative,

$$m[r \circ K \bullet \overline{\rho_n(G')}] = [\rho_m(r \circ K) \bullet \overline{\rho_{mn}(G')}] = mn[T] = [r \circ h].$$

Thus, if $H' := \overline{h} \bullet \mu_m(K)$ and $N := mn$, then $H'(0) = e, H'(1) = \mu_N(f)$, and $r \circ H'$ is path homotopic to $\rho_N(G')$. □

The next result will be useful to us in the following context. Suppose that B, C, and D are C^*-algebras, and $\delta : B \to D$ and $\gamma : C \to D$ are $*$-homomorphisms. Let $A = B \oplus_D C$ be the pullback as in Equation (2-2). Then $\widehat{\mathcal{U}}(A)$ may be described as

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a pullback (in the category of pointed topological spaces) by the induced diagram

$$\begin{tikzcd} \widehat{\mathcal{U}}(A) \arrow[r, "\phi"] \arrow[d, "\psi"'] & \widehat{\mathcal{U}}(B) \arrow[d, "\delta"] \\ \widehat{\mathcal{U}}(C) \arrow[r, "\gamma"'] & \widehat{\mathcal{U}}(D) \end{tikzcd}$$

In other words, a pair $(b, c) \in A$ is in $\widehat{\mathcal{U}}(A)$ if and only if $b \in \widehat{\mathcal{U}}(B)$ and $c \in \widehat{\mathcal{U}}(C)$. We now introduce some notation for later use. Given a path $G : [0, 1] \to Y$ in a topological space Y, $\widetilde{G}$ is a path given by

$$\widetilde{G}(s) = \begin{cases} e_{G(0)}(3s) & \text{if } 0 \le x \le \frac{1}{3} \\ G(3s - 1) & \text{if } \frac{1}{3} \le x \le \frac{2}{3} \\ e_{G(1)}(3s - 2) & \text{if } \frac{2}{3} \le x \le 1. \end{cases} \tag{3-3}$$

LEMMA 3.7. *Consider a pullback diagram of pointed topological spaces given by*

$$\begin{tikzcd} P \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & X \arrow[d, "\pi_1"] \\ Y \arrow[r, "\pi_2"'] & Z \end{tikzcd}$$

such that one of the maps π_1 or π_2 is a Serre fibration. Let $p = (x, y)$, $p' = (x', y')$ be in P, such that there exist paths

$$G_1 : [0, 1] \to X, \quad G_2 : [0, 1] \to Y$$

with the properties that $G_1(0) = x$, $G_1(1) = x'$, $G_2(0) = y$, $G_2(1) = y'$ and $\pi_1 \circ G_1 \sim_h \pi_2 \circ G_2$ in Z. Then there is a path $H : [0, 1] \to P$ such that $H(0) = p$ and $H(1) = p'$.

PROOF. Assume without loss of generality that π_1 is a Serre fibration. Then since $\pi_1 \circ G_1 \sim_h \pi_2 \circ G_2$, there is a homotopy $F : [0, 1] \times [0, 1] \to D$ such that

$$F(s, 0) = \pi_1 \circ G_1(s), \quad F(s, 1) = \pi_2 \circ G_2(s),$$
$$F(0, t) = \pi_1(x) = \pi_2(y), \quad F(1, t) = \pi_1(x') = \pi_2(y').$$

Then F lifts to a homotopy $F' : [0, 1] \times [0, 1] \to X$, such that

$$F'(s, 0) = G_1, \quad \pi_1 \circ F' = F, \quad \pi_1 \circ F'(t, 1) = \pi_2 \circ G_2(t).$$

So if we define

$$G_X(s) = \begin{cases} F'(0, 3s) & \text{if } 0 \le s \le \frac{1}{3} \\ F'(3s - 1, 1) & \text{if } \frac{1}{3} \le s \le \frac{2}{3} \\ F'(1, 3 - 3s) & \text{if } \frac{2}{3} \le s \le 1 \end{cases}$$

then $\pi_1 \circ G_X = \pi_2 \circ \widetilde{G_2}$. Therefore, the pair $(G_X, \widetilde{G_2})$ defines a path in P from p to p'. □

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LEMMA 3.8. *Let X and Y be two connected topological spaces, and $i : X \to Y$ and $q : Y \to X$ be homotopy inverses of each other. For $x \in X$, let $H : [0,1] \to Y$ be a path in Y, such that*

$$H(0) = i(x), \quad H(1) = i \circ q \circ i(x).$$

Then there exists a path $T : [0,1] \to X$ such that

$$T(0) = x, \quad T(1) = q \circ i(x)$$

and $i \circ T$ is path homotopic to H in Y.

PROOF. Since $q \circ i \sim_h \mathrm{id}_X$, there is a path $S : [0,1] \to X$ such that $S(0) = q \circ i(x)$, and $S(1) = x$. Thus, $H \bullet (i \circ S)$ is a loop in Y based at $i(x)$. Since $\pi_1(Y) = i_*(\pi_1(X))$, there exists a loop L based at x in X such that

$$i_*[L] = [H \bullet (i \circ S)].$$

Then $T := L \bullet \overline{S}$ is the required path. □

Note that, if B is a C*-algebra, then the rationalization $\widehat{\mathcal{U}}_0(B)_{\mathbb{Q}}$ of $\widehat{\mathcal{U}}_0(B)$ carries an H-space structure by Proposition 2.7. We use ρ^B to denote this multiplication map. Furthermore, we write $e_{\mathbb{Q}}$ and $e^2_{\mathbb{Q}}$ for the units of $\widehat{\mathcal{U}}_0(B)_{\mathbb{Q}}$ and $\widehat{\mathcal{U}}_0(M_2(B))_{\mathbb{Q}}$, respectively.

PROPOSITION 3.9. *Let B be a rationally K-stable C^*-algebra, $[f] \in \pi_j(\widehat{\mathcal{U}}_0(B))$ and $n \in \mathbb{N}$ such that $[\iota \circ f]$ is an element of order n in $\pi_j(\widehat{\mathcal{U}}_0(M_2(B))$. Consider a path $H : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(M_2(B)))$ satisfying*

$$H(0) = 0 \quad \textit{and} \quad H(1) = \mu_n^{M_2(B)}(\iota \circ f) = \iota \circ \mu_n^B(f).$$

Then there exist a natural number $N \in \mathbb{N}$ and a path $H' : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(B))$ such that

$$\mu_N^{M_2(B)}(H) \sim_h \iota \circ H'$$

in $C_*(S^j, \widehat{\mathcal{U}}_0(M_2(B)))$.

PROOF. Since B is rationally K-stable, there are maps $\widehat{\mathcal{U}}_0(B)_{\mathbb{Q}} \to \widehat{\mathcal{U}}_0(M_2(B))_{\mathbb{Q}}$ and $\widehat{\mathcal{U}}_0(M_2(B))_{\mathbb{Q}} \to \widehat{\mathcal{U}}_0(B)_{\mathbb{Q}}$ which are homotopy inverses of each other. Therefore, we get a commuting diagram

$$\begin{tikzcd} C_*(S^j, \widehat{\mathcal{U}}_0(B)) \arrow[r, "\iota"] \arrow[d, "r"'] & C_*(S^j, \widehat{\mathcal{U}}_0(M_2(B))) \arrow[d, "R"] \\ C_*(S^j, \widehat{\mathcal{U}}_0(B)_{\mathbb{Q}}) \arrow[r, "i"] & C_*(S^j, \widehat{\mathcal{U}}_0(M_2(B))_{\mathbb{Q}}) \end{tikzcd}$$

where r and R represent the rationalization maps. Furthermore, i has a homotopy inverse $q : C_*(S^j, \widehat{\mathcal{U}}_0(M_2(B))_{\mathbb{Q}}) \to C_*(S^j, \widehat{\mathcal{U}}_0(B)_{\mathbb{Q}})$. Let $H : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(M_2(B)))$

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as above. Since R is a rationalization map, applying Lemma 3.6, we get a homotopy $G : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(M_2(B))_{\mathbb{Q}})$ such that

$$G(0) = e_{\mathbb{Q}}^2, \quad \text{and} \quad G(1) = R \circ \iota \circ f = \iota \circ r \circ f.$$

Furthermore, $\rho_n^{M_2(B)}(G)$ is path homotopic to $R \circ H$ in $C_*(S^j, \widehat{\mathcal{U}}_0(M_2(B))_{\mathbb{Q}})$. Now, $q \circ G : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(B)_{\mathbb{Q}})$ is such that

$$q \circ G(0) = e_{\mathbb{Q}} \quad \text{and} \quad q \circ G(1) = q \circ i \circ r \circ f.$$

Note that i and q are homotopy equivalences, hence G and $i \circ q \circ G$ are homotopic in $C_*(S^j, \widehat{\mathcal{U}}_0(M_2(B))_{\mathbb{Q}})$, say, by $K : [0,1] \times [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(M_2(B))_{\mathbb{Q}})$ satisfying

$$K(s,0) = G(s), \quad K(s,1) = i \circ q \circ G(s), \quad K(0,t) = e_{\mathbb{Q}}^2.$$

Define $T : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(M_2(B))_{\mathbb{Q}})$ as $T(t) = K(1, 1-t)$. Then

$$T(0) = i \circ q \circ i \circ r \circ f, \quad T(1) = i \circ r \circ f.$$

Thus, by Lemma 3.8, there is a homotopy $S : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(B)_{\mathbb{Q}})$ such that

$$S(0) = q \circ i \circ r \circ f, \quad S(1) = r \circ f$$

and $i \circ S$ is path homotopic to T in $C_*(S^j, \widehat{\mathcal{U}}_0(M_2(B))_{\mathbb{Q}})$. Since $(i \circ q \circ G) \bullet T$ is path homotopic to G, this implies $(i \circ q \circ G) \bullet (i \circ S)$ is path homotopic to G in $C_*(S^j, \widehat{\mathcal{U}}_0(M_2(B))_{\mathbb{Q}})$. Thus, we get a path $(q \circ G) \bullet S : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(B)_{\mathbb{Q}})$ so that

$$(q \circ G) \bullet S(0) = e_{\mathbb{Q}}, \quad q \circ G \bullet S(1) = r \circ f, \quad i \circ (q \circ G \bullet S) \sim_h G.$$

Again, since r is a rationalization map, by Lemma 3.6, there exist a natural number $m \in \mathbb{N}$ and a path $H' : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(B))$ such that

$$H'(0) = 0, \quad H'(1) = \mu_m^B(f), \quad r \circ H' \sim_h \rho_m^B((q \circ G) \bullet S).$$

Take $k = \operatorname{lcm}\{n, m\}$, and write $k = n\ell_1 = m\ell_2$ for some $\ell_1, \ell_2 \in \mathbb{N}$. Then the path $\mu_{\ell_2}^B(H') : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(B))$ is such that

$$\mu_{\ell_2}^B(H')(0) = 0, \mu_{\ell_2}^B(H')(1) = \mu_k^B(f) \quad \text{and} \quad r \circ \mu_{\ell_2}^B(H') \sim_h \rho_k^B((q \circ G) \bullet S).$$

Also $\mu_{\ell_1}^{M_2(B)}(H) : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(M_2(B)))$ is such that

$$\mu_{\ell_1}^{M_2(B)}(H)(0) = 0, \mu_{\ell_1}^{M_2(B)}(H)(1) = \iota \circ \mu_k^B(f) \quad \text{and} \quad R \circ \mu_{\ell_1}^{M_2(B)}(H) \sim_h \rho_k^{M_2(B)}(G).$$

Then, from the earlier arguments, we have the relations

$$\begin{aligned} \rho_k^{M_2(B)}(G) &\sim_h R \circ \mu_{\ell_1}^{M_2(B)}(H), \\ i \circ ((q \circ G) \bullet S) &\sim_h G, \\ r \circ \mu_{\ell_2}^B(H') &\sim_h \rho_k^B((q \circ G) \bullet S). \end{aligned}$$

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Also $i \circ r \circ H' = R \circ \iota \circ H'$. Hence,

$$R \circ \iota \circ \mu^B_{\ell_2}(H') = i \circ r \circ \mu^B_{\ell_2}(H') \sim_h \rho_k^{M_2(B)}(i \circ ((q \circ G) \bullet S)) \\ \sim_h \rho_k^{M_2(B)}(G) \sim_h R \circ \mu_{\ell_1}^{M_2(B)}(H).$$

Thus

$$[R \circ (\iota \circ \mu^B_{\ell_2}(H') \bullet \overline{\mu_{\ell_1}^{M_2(B)}(H)})] = 0$$

in $\pi_1(C_*(S^j, \widehat{\mathcal{U}}_0(M_2(B))_{\mathbb{Q}}))$. Then, by Lemma 3.6, there exists a natural number $P \in \mathbb{N}$ such that

$$\iota \circ \mu^B_{P\ell_2}(H') = \mu_P^{M_2(B)}(\iota \circ \mu^B_{\ell_2}(H')) \sim_h \mu_P^{M_2(B)}(\mu_{\ell_1}^{M_2(B)}(H)) = \mu_{P\ell_1}^{M_2(B)}(H)$$

in $C_*(S^1, \widehat{\mathcal{U}}_0(M_2(B)))$. Thus, replacing H' by $\mu^B_{P\ell_2}(H')$ and taking $N := P\ell_1$, we have

$$\iota \circ H' \sim_h \mu_N^{M_2(B)}(H),$$

proving the result. □

The next lemma is an analogue of [23, Lemma 2.7], and is a consequence of that result and Proposition 3.9.

Lemma 3.10. *Let X be a compact Hausdorff space, A a continuous $C(X)$-algebra, and $x \in X$ such that $A(x)$ is rationally K-stable. For $[f] \in \pi_j(\widehat{\mathcal{U}}_0(A))$, let $F : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(M_2(A)))$ be a path and $n \in \mathbb{N}$ such that*

$$F(0) = 0 \quad \text{and} \quad F(1) = \mu_n^{M_2(A)}(\iota \circ f).$$

Then there is a closed neighbourhood Y of x, a natural number $N_x \in \mathbb{N}$, and a path $L_Y : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(A(Y)))$ such that $L_Y(0) = 0, L_Y(1) = \mu_{N_x n}^{A(Y)}(\pi_Y \circ f)$, and

$$\iota_Y \circ L_Y \sim_h \mu_{N_x}^{M_2(A(Y))}(\eta_Y \circ F)$$

in $C_*(S^j, \widehat{\mathcal{U}}_0(M_2(A(Y))))$.

Remark 3.11. We are now in a position to prove Theorem A, but first, we need one important fact, which allows us to use induction. If X is a finite-dimensional compact metric space, then the covering dimension agrees with the small inductive dimension [4, Theorem 1.7.7]. Therefore, by [4, Theorem 1.1.6], X has an open cover $\mathcal{B}$ such that, for each $U \in \mathcal{B}$,

$$\dim(\partial U) \leq \dim(X) - 1.$$

Now suppose $\{U_1, U_2, \dots, U_m\}$ is an open cover of X such that $\dim(\partial U_i) \leq \dim(X) - 1$ for $1 \leq i \leq m$. We define sets $\{V_i : 1 \leq i \leq m\}$ inductively by

$$V_1 := \overline{U_1}, \quad \text{and} \quad V_k := \overline{U_k \setminus \Big(\bigcup_{i<k} U_i\Big)} \text{ for } k > 1,$$

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and subsets $\{W_j : 1 \leq j \leq m-1\}$ by

$$W_j := \Big(\bigcup_{i=1}^{j} V_i\Big) \cap V_{j+1}.$$

It is easy to see that $W_j \subset \bigcup_{i=1}^{j} \partial U_i$, so by [4, Theorem 1.5.3], $\dim(W_j) \leq \dim(X) - 1$ for all $1 \leq j \leq m-1$.

Proof of Theorem A. Let A be a continuous $C(X)$-algebra such that each fibre of A is rationally K-stable. By Theorem 3.5, we assume that $\dim(X) \geq 1$, and we assume that $A(Y)$ is rationally K-stable for any closed subset $Y \subset X$ with $\dim(Y) \leq \dim(X) - 1$. We now show that the map

$$\iota_* \otimes \mathrm{id} : \pi_j(\widehat{\mathcal{U}}_0(M_n(A))) \otimes \mathbb{Q} \to \pi_j(\widehat{\mathcal{U}}_0(M_{n+1}(A))) \otimes \mathbb{Q}$$

is an isomorphism for $j \geq 1$, $n \geq 1$. For simplicity of notation, we assume that $n = 1$.

We first prove injectivity. Fix $[f] \in \pi_j(\widehat{\mathcal{U}}_0(A))$ such that $[\iota \circ f]$ has order n in $\pi_j(\widehat{\mathcal{U}}_0(M_2(A)))$. Then we wish to prove that $[f]$ has finite order in $\pi_j(\widehat{\mathcal{U}}_0(A))$. For this purpose consider $F : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(M_2(A)))$ such that

$$F(0) = 0, \quad F(1) = \mu_n^{M_2(A)}(\iota \circ f).$$

For $x \in X$, by Lemma 3.10, there is a closed neighbourhood Y_x of x, $N_x \in \mathbb{N}$, and a path $L_{Y_x} : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(A(Y_x)))$ such that

$$L_{Y_x}(0) = 0, \quad L_{Y_x}(1) = \mu_{N_x n}^{A(Y_x)}(\pi_{Y_x} \circ f)$$

and $\iota_{Y_x} \circ L_{Y_x} \sim_h \mu_{N_x}^{M_2(A(Y_x))}(\eta_{Y_x} \circ F)$ in $C_*(S^j, \widehat{\mathcal{U}}_0(M_2(A(Y_x))))$. We may choose Y_x to be the closure of a basic open set U_x such that $\dim(\partial U_x) \leq \dim(X) - 1$. Since X is compact, we may choose a finite subcover $\{U_1, U_2, \ldots, U_m\}$. Now define $\{V_1, V_2, \ldots, V_m\}$ and $\{W_1, W_2, \ldots, W_{m-1}\}$ as in Remark 3.11. We observe that each V_i is a closed set such that $\mu_{N_i n}^{A(V_i)}(\pi_{V_i} \circ f) \sim 0$ in $C_*(S^j, \widehat{\mathcal{U}}_0(A(V_i)))$ since $V_i \subset \overline{U_i}$ for all $1 \leq i \leq m$.

Note that $W_1 = V_1 \cap V_2$, and $\dim(W_1) \leq \dim(X) - 1$. By the induction hypothesis, $A(W_1)$ is rationally K-stable. Let $H_i : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(A(V_i)))$, $i = 1, 2$, be paths such that $H_i(0) = 0, H_i(1) = \mu_{N_i n}^{A(V_i)}(\pi_{V_i} \circ f)$, and

$$\iota_{V_i} \circ H_i \sim_h \mu_{N_i}^{M_2(A(V_i))}(\eta_{V_i} \circ F).$$

Setting $M := \mathrm{lcm}(N_1, N_2)$, we may assume that $H_i : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(A(V_i)))$, $i = 1, 2$, are paths such that $H_i(0) = 0, H_i(1) = \mu_{Mn}^{A(V_i)}(\pi_{V_i} \circ f)$, and

$$\iota_{V_i} \circ H_i \sim_h \mu_M^{M_2(A(V_i))}(\eta_{V_i} \circ F).$$

Let $S : [0,1] \to C_*(S^j, \widehat{\mathcal{U}}_0(A(W_1)))$ be the path

$$S := (\pi_{W_1}^{V_1} \circ H_1) \bullet \overline{(\pi_{W_1}^{V_2} \circ H_2)}.$$

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Note that $S(0) = S(1) = 0$, so S is a loop in $C_*(S^j, \widehat{\mathcal{U}}_0(A(W_1)))$, and

$$\begin{aligned}\iota_{W_1} \circ S &= (\eta_{W_1}^{V_1} \circ \iota_{V_1} \circ H_1) \bullet (\eta_{W_1}^{V_2} \circ \iota_{V_2} \circ \overline{H_2})\\ &\sim_h \mu_M^{M_2(A(W_1))}(\eta_{W_1} \circ F \bullet \overline{(\eta_{W_1} \circ F)}) \sim_h 0.\end{aligned}$$

Also, since $A(W_1)$ is rationally K-stable,

$$\iota_{W_1}^* \otimes \mathrm{id} : \pi_1(C_*(S^j, \widehat{\mathcal{U}}_0(A(W_1)))) \otimes \mathbb{Q} \to \pi_1(C_*(S^j, \widehat{\mathcal{U}}_0(M_2(A(W_1))))) \otimes \mathbb{Q}$$

is an isomorphism. Hence, there is $m \in \mathbb{N}$ such that $m[S] = 0$ in $\pi_1(C_*(S^j, \widehat{\mathcal{U}}_0(A(W_1))))$. Thus

$$\pi_{W_1}^{V_1} \circ \mu_m^{A(V_1)}(H_1) = \mu_m^{A(W_1)}(\pi_{W_1}^{V_1} \circ H_1) \sim_h \mu_m^{A(W_1)}(\pi_{W_1}^{V_2} \circ H_2) = \pi_{W_1}^{V_2} \circ \mu_m^{A(V_2)}(H_2)$$

in $C_*(S^j, \widehat{\mathcal{U}}_0(A(W_1)))$. Now, by Lemma 2.5, and [18, Theorem 3.9], we have a pullback diagram

$$\begin{array}{ccc} C_*(S^j, A(V_1 \cup V_2)) & \xrightarrow{\pi_{V_1}^{V_1 \cup V_2}} & C_*(S^j, A(V_1)) \\ \Big\downarrow{\scriptstyle \pi_{V_2}^{V_1 \cup V_2}} & & \Big\downarrow{\scriptstyle \pi_{W_1}^{V_1}} \\ C_*(S^j, A(V_2)) & \xrightarrow{\pi_{W_1}^{V_2}} & C_*(S^j, A(W_1)) \end{array}$$

As mentioned before, this induces a pullback diagram of groups of quasiunitaries. Furthermore, the map $\pi_{W_1}^{V_1} : \widehat{\mathcal{U}}_0(A(V_1)) \to \widehat{\mathcal{U}}_0(A(W_1))$ is a Serre fibration. Thus, by Lemma 3.7,

$$\mu_{mMn}^{A(V_1 \cup V_2)}(\pi_{V_1 \cup V_2} \circ f) \sim_h 0$$

in $C_*(S^j, \widehat{\mathcal{U}}_0(A(V_1 \cup V_2)))$. Thus, $mMn[\pi_{V_1 \cup V_2} \circ f] = 0$, so that $[\pi_{V_1 \cup V_2} \circ f]$ has finite order in $\pi_j(\widehat{\mathcal{U}}_0(A(V_1 \cup V_2)))$.

Now observe that $W_2 = (V_1 \cup V_2) \cap V_3$, and $\dim(W_2) \leq \dim(X) - 1$. Replacing V_1 by $V_1 \cup V_2$ and V_2 by V_3 in the above argument, we may repeat the earlier procedure. By induction on the number of elements in the finite subcover, we conclude that $[f]$ has finite order in $\pi_j(\widehat{\mathcal{U}}_0(A))$, as required.

We now prove surjectivity of $\iota_* \otimes \mathrm{id}$. Choose $[u] \in \pi_j(\widehat{\mathcal{U}}_0(M_2(A)))$ and $m \in \mathbb{Z}$ nonzero. We wish to construct an element $[\omega] \in \pi_j(\widehat{\mathcal{U}}_0(A))$ and $q \in \mathbb{Q}$ such that

$$\iota_* \otimes \mathrm{id}([\omega] \otimes q) = [u] \otimes \frac{1}{m}.$$

So, fix $x \in X$. Then, by rationally K-stability of $A(x)$ (as in the proof of Theorem 3.5), there is a closed neighbourhood Y_x of x, a natural number $L_x \in \mathbb{N}$, and a quasiunitary $c_x \in C_*(S^j, \widehat{\mathcal{U}}_0(A(Y_x)))$ such that

$$\mu_{L_x}^{M_2(A(Y_x))}(\eta_{Y_x} \circ u) \sim_h \iota_{Y_x} \circ c_x.$$

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As in the first part of the proof, we may reduce to the case where $X = V_1 \cup V_2$, and there are quasiunitaries $c_{V_1} \in C_*(S^j, \widehat{\mathcal{U}}_0(A(V_1))), c_{V_2} \in C_*(S^j, \widehat{\mathcal{U}}_0(A(V_2)))$ such that

$$\mu_{L_i}^{M_2(A(V_i))}(\eta_{V_i} \circ u) \sim \iota_{V_i} \circ c_{V_i} \quad \text{in } C_*(S^j, \widehat{\mathcal{U}}_0(M_2(A(V_i)))), \quad i = 1, 2,$$

and if $W := V_1 \cap V_2$, then $\dim(W) \leq \dim(X) - 1$. Furthermore, by replacing the $\{L_i\}$ by their least common multiple, we may assume that $L_1 = L_2 =: L$. Now, fix paths $H_i : [0, 1] \to C_*(S^j, \widehat{\mathcal{U}}_0(M_2(A(V_i))))$ such that

$$H_1(0) = \iota_{V_1} \circ c_{V_1}, \quad H_1(1) = \mu_L^{M_2(A(V_1))}(\eta_{V_1} \circ u)$$
$$H_2(0) = \mu_L^{M_2(A(V_2))}(\eta_{V_2} \circ u), \quad H_2(1) = \iota_{V_2} \circ c_{V_2}.$$

Consider the path $F : [0, 1] \to C_*(S^j, \widehat{\mathcal{U}}_0(M_2(A(W))))$ given by

$$F := (\eta_W^{V_1} \circ H_1) \bullet (\eta_W^{V_2} \circ H_2).$$

Then $F(0) = \iota_W \circ \pi_W^{V_1} \circ c_{V_1}$ and $F(1) = \iota_W \circ \pi_W^{V_2} \circ c_{V_2}$. Then since $A(W)$ is rationally K-stable, by Proposition 3.9, there exist a path $F' : [0, 1] \to C_*(S^j, \widehat{\mathcal{U}}_0(A(W)))$ and a natural number $N \in \mathbb{N}$ such that

$$F'(0) = \mu_N^{A(W)}(\pi_W^{V_1} \circ c_{V_1}), \quad F'(1) = \mu_N^{A(W)}(\pi_W^{V_2} \circ c_{V_2})$$

and $\iota_W \circ F'$ is path homotopic to $\mu_N^{M_2(A(W))}(F)$ in $C_*(S^j, \widehat{\mathcal{U}}_0(M_2(A(W))))$. The map $\pi_W^{V_2} : C_*(S^j, \widehat{\mathcal{U}}_0(A(V_2))) \to \pi_W^{V_2}(C_*(S^j, \widehat{\mathcal{U}}_0(A(V_2))))$ is a fibration, so there is a path $F'' : [0, 1] \to C_*(S^j, \widehat{\mathcal{U}}_0(A(V_2)))$ such that

$$F''(1) = \mu_N^{A(V_2)}(c_{V_2}), \quad \text{and} \quad \pi_W^{V_2} \circ F'' = F'.$$

Define $e_{V_2} := F''(0)$ so that

$$\pi_W^{V_2} \circ e_{V_2} = \mu_N^{A(W)}(\pi_W^{V_1} \circ c_{V_1}).$$

Recall that, given a path G in a topological space, the path $\widetilde{G}$ is defined by Equation (3-3). Define $H_3 : [0, 1] \to C_*(S^j, \widehat{\mathcal{U}}_0(M_2(V_2)))$ as

$$H_3 := \mu_N^{M_2(A(V_2))}(H_2) \bullet (\widetilde{\iota_{V_2} \circ F''}).$$

Then $H_3(0) = \mu_{NL}^{M_2(A(V_2))}(\eta_{V_2} \circ u), H_3(1) = \iota_{V_2} \circ e_{V_2}$, and

$$\eta_W^{V_2} \circ H_3 = \eta_W^{V_2} \circ (\mu_N^{M_2(A(V_2))}(H_2)) \bullet (\widetilde{\iota_W \circ F'}).$$

Also $\eta_W^{V_1} : C_*(S^j, \widehat{\mathcal{U}}_0(M_2(A(V_1)))) \to \eta_W^{V_1}(C_*(S^j, \widehat{\mathcal{U}}_0 M_2((A(V_1)))))$ is a fibration, thus $\eta_W^{V_2} \circ (\mu_N^{M_2(A(V_2))}(H_2))$ has a lift, denoted by $T : [0, 1] \to C_*(S^j, \widehat{\mathcal{U}}_0(M_2(A(V_1))))$ so that

$$T(0) = \mu_{NL}^{M_2(A(V_1))}(\eta_{V_1} \circ u).$$

Then letting $G := \mu_N^{M_2(A(V_1))}(H_1) \bullet T$ gives $\eta_W^{V_1} \circ G = \mu_N^{M_2(A(W))}(F)$. Again by the above fibration map, since $\eta_W^{V_1} \circ G = \mu_N^{M_2(A(W))}(F) \sim_h \iota_W \circ F'$, by the calculation done in

Lemma 3.7, $\widetilde{\iota_W \circ F'}$ has a lift in $C_*(S^j, \widehat{\mathcal{U}}_0(M_2(V_1)))$, denoted by T'. Then

$$\eta_W^{V_1} \circ (T \bullet \overline{T'}) = \eta_W^{V_2} \circ (\mu_N^{M_2(A(V_2))}(H_2)) \bullet (\widetilde{\iota_W \circ \overline{F'}}).$$

As before, $C_*(S^j, A)$ is a pullback

$$\begin{tikzcd}
C_*(S^j, A) \arrow[r, "\pi_{V_1}"] \arrow[d, "\pi_{V_2}"'] & C_*(S^j, A(V_1)) \arrow[d, "\pi_W^{V_1}"] \\
C_*(S^j, A(V_2)) \arrow[r, "\pi_W^{V_2}"] & C_*(S^j, A(W))
\end{tikzcd}$$

so that $\omega := (\mu_N^{A(V_1)}(c_{V_1}), e_{V_2})$ defines a quasiunitary in $C_*(S^j, A)$, and $\iota \circ \omega \sim \mu_{NL}^{M_2(A)}(u)$ in $C_*(S^j, \widehat{\mathcal{U}}_0(M_2(A)))$, where the path is given by the pair $(H_3, T \bullet \overline{T'})$. Hence, for $q := 1/(mNL)$, we have

$$\iota_* \otimes \mathrm{id}([\omega] \otimes q) = [u] \otimes \frac{1}{m}$$

as required. □

We conclude this section with a discussion on the extent to which the converse of Theorem A holds. □

Proposition 3.12. *Let X be a locally compact, Hausdorff space, and A be a C^*-algebra. If A is rationally K-stable, then so is $C_0(X) \otimes A$. The converse is true if X is a finite CW complex.*

Proof. If A is rationally K-stable, we wish to show that $C_0(X) \otimes A$ is rationally K-stable. By appealing to the five lemma (as in [23, Lemma 2.1]), we may assume that X is compact. Now, X is an inverse limit of compact metric spaces (X_i) by [15], so that $C(X) \otimes A \cong \lim C(X_i) \otimes A$. Since the functors F_j are continuous (Proposition 2.2), we may assume that X itself is a compact metric space. Any metric space can, in turn, be written as an inverse limit of finite CW complexes [7]. Therefore, we may further assume that X is a finite CW complex. In that case, by [19, Theorem 4.20], one has

$$F_j(C(X, A)) \cong \bigoplus_{n \geq j} H^{n-j}(X; F_j(A)) \tag{3-4}$$

where the isomorphism is natural. Since the map $\iota_* : F_j(M_{n-1}(A)) \to F_j(M_n(A))$ is an isomorphism, it follows that $\iota_* : F_j(C(X, M_{n-1}(A))) \to F_j(C(X, M_n(A)))$ is an isomorphism as well. Hence, $C(X) \otimes A$ is rationally K-stable.

Now suppose X is a finite CW complex and $C(X) \otimes A$ is K-stable. Then

$$F_j(C(X, M_{n-1}(A))) \cong F_j(C(X, M_n(A)))$$

and the isomorphism of Equation (3-4) is componentwise. This implies that

$$H^{n-j}(X; F_j(M_{n-1}(A))) \cong H^{n-j}(X; F_j(M_n(A)))$$

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for all $n \geq j$. For any connected H-space Y, as in Example 3.1, there is a fibration sequence $C_*(X, Y) \to C(X, Y) \to Y$, which induces a short exact sequence of rational homotopy groups

$$0 \to F_j(C_*(X, Y)) \to F_j(C(X, Y)) \to F_j(Y) \to 0. \tag{3-5}$$

Now, we take $Y = \widehat{\mathcal{U}}_0(M_k(A))$ and apply [19, Theorem 4.20] to get

$$F_j(C_*(X, M_k(A))) \cong \bigoplus_{n \geq j} \widetilde{H}^{n-j}(X; F_j(M_k(A)))$$

and the isomorphism is natural. Hence, we conclude that

$$F_j(C_*(X, M_{n-1}(A))) \cong F_j(C_*(X, M_n(A)))$$

as well. By Equation (3-5) and the five lemma, we conclude that A is rationally K-stable. □

In [22, Theorem B], we proved that, for an AF algebra, rational K-stability is equivalent to K-stability. Combining this fact with Proposition 3.12, and [23, Theorem A], we have the following result.

COROLLARY 3.13. *Let X be a finite CW complex, and A be an AF-algebra. Then $C(X) \otimes A$ is K-stable if and only if A is K-stable.*

The next example shows that the converse of Theorem A need not hold for arbitrary continuous $C(X)$-algebras.

EXAMPLE 3.14. Let $D_1 := M_{2^\infty}$ denote the UHF algebra of type 2^∞, and let $D_2 := D_1 \oplus M_2(\mathbb{C})$. Consider the $C[0, 1]$-algebra

$$A := \{(f, g) \in C[0, 1/2] \otimes D_1 \oplus C[1/2, 1] \otimes D_2 : \Phi(f) = g(1/2)\}$$

where $\Phi : C[0, 1/2] \otimes D_1 \to D_2$ is given by $\Phi(f) = (f(1/2), 0)$. Since Φ is injective, it follows that A is a continuous $C[0, 1]$-algebra. Note that A may be described as a pullback

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C[\tfrac{1}{2}, 1] \otimes D_2 \arrow[d, "\mathrm{ev}"] \\ C[0, \tfrac{1}{2}] \otimes D_1 \arrow[r, "\Phi"'] & D_2 \end{tikzcd}$$

where ev is evaluation at $1/2$. The Mayer–Vietoris theorem [21, Theorem 4.5] for the functor F_m gives a long exact sequence

$$\cdots \to F_m(A) \to F_m(D_2) \oplus F_m(D_1) \xrightarrow{\mathrm{ev}_* - \Phi_*} F_m(D_2) \to \cdots$$

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where $\mathrm{ev}_* : F_m(D_2) \to F_m(D_2)$ is the identity map and $\Phi_* : F_m(D_1) \to F_m(D_2)$ is given as $\Phi_*(r) = (r, 0)$, thus $(\mathrm{ev}_* - \Phi_*) : F_m(D_2) \oplus F_m(D_1) \to F_m(D_2)$ is given by

$$(\mathrm{ev}_* - \Phi_*)((a, b), c) = (a - c, b).$$

Consider the case where m is odd. By [22, Lemma 3.2], $F_{m-1}(D_i) = F_{m+1}(D_i) = 0$ for $i = 1, 2$. Hence, the above long exact sequence boils down to

$$0 \to F_m(A) \to F_m(D_2) \oplus F_m(D_1) \xrightarrow{\mathrm{ev}_* - \Phi_*} F_m(D_2) \to \cdots .$$

Thus, there is a natural isomorphism

$$F_m(A) = \ker(\mathrm{ev}_* - \Phi_*) \cong F_m(D_1).$$

Similarly, $F_m(M_2(A)) \cong F_m(M_2(D_1))$ and the following diagram commutes:

$$\begin{tikzcd} F_m(A) \arrow[r, "\cong"] \arrow[d, "\iota^A"'] & F_m(D_1) \arrow[d, "\iota^{D_1}"'] \\ F_m(M_2(A)) \arrow[r, "\cong"] & F_m(M_2(D_1)) \end{tikzcd}$$

Since D_1 is rationally K-stable by [22, Theorem B], it follows that ι^A is an isomorphism. Doing the same for the inclusion map $M_n(A) \hookrightarrow M_{n+1}(A)$, we conclude that the map $F_m(M_n(A)) \to F_m(M_{n+1}(A))$ is an isomorphism if m is odd.

Now suppose m is even, The above long exact sequence reduces to

$$F_{m-1}(A) \to F_{m-1}(D_2) \oplus F_{m-1}(D_1) \xrightarrow{\mathrm{ev}_* - \Phi_*} F_{m-1}(D_2) \to F_m(A) \to 0$$

so that $F_m(A) \cong \mathrm{coker}(\mathrm{ev}_* - \Phi_*)$. Now, by [22, Theorem A], it follows that $F_{m-1}(D_1) \cong \mathbb{Q}$ for all even m, and

$$F_{m-1}(D_2) \cong \begin{cases} \mathbb{Q} \oplus \mathbb{Q} & \text{if } m = 2, 4 \\ \mathbb{Q} & m > 4 \text{ and even.} \end{cases}$$

Thus, elementary linear algebra proves that $\mathrm{ev}_* - \Phi_*$ is surjective, so that $F_m(A) = 0$. Similarly, $F_m(M_n(A)) = 0$ for all $n \geq 2$ as well (if m is even).

Thus, we conclude that A is rationally K-stable. However, one of its fibres (namely D_2) is not rationally K-stable because it has a nonzero finite-dimensional representation [22, Theorem B].

4. An application to crossed product C*-algebras

As an application of our earlier results, we wish to show that the class of (rationally) K-stable C*-algebras is closed under the formation of certain crossed products. To begin with, we fix some conventions. In what follows, G will denote a compact, second countable group, and A will denote a separable C*-algebra. By an action of G on A, we mean a continuous group homomorphism $\alpha : G \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is equipped

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with the point-norm topology. We write $\sigma : G \to \mathrm{Aut}(C(G))$ for the left action of G on $C(G)$, given by $\sigma_s(f)(t) := f(s^{-1}t)$.

The notion of Rokhlin dimension was invented by Hirshberg *et al.* [12] for actions of finite groups (and the integers). The definition for compact, second countable groups is due to Gardella [8]. The 'local' definition we give below is different from the original, but is equivalent due to [8, Lemma 3.7] (see also [27, Lemma 1.5]).

DEFINITION 4.1. Let G be a compact, second countable group, and let A be a separable C*-algebra. We say that an action $\alpha : G \to \mathrm{Aut}(A)$ has Rokhlin dimension d (with commuting towers) if d is the least integer such that, for any pair of finite sets $F \subset A, K \subset C(G)$, and any $\epsilon > 0$, there exist $(d + 1)$ contractive, completely positive maps

$$\psi_0, \psi_1, \ldots, \psi_d : C(G) \to A$$

satisfying the following conditions.

(1) For $f_1, f_2 \in K$ such that $f_1 \perp f_2$, $\|\psi_j(f_1)\psi_j(f_2)\| < \epsilon$ for all $0 \leq j \leq d$.
(2) For any $a \in F$ and $f \in K$, $\|[\psi_j(f), a]\| < \epsilon$ for all $0 \leq j \leq d$.
(3) For any $f \in K$ and $s \in G$, $\|\alpha_s(\psi_j(f)) - \psi_j(\sigma_s(f))\| < \epsilon$ for all $0 \leq j \leq d$.
(4) For any $a \in F$, $\| \sum_{j=0}^{d} \psi_j(1_{C(G)})a - a\| < \epsilon$.
(5) For any $f_1, f_2 \in K$, $\|[\psi_j(f_1), \psi_k(f_2)]\| < \epsilon$ for all $0 \leq j, k \leq d$.

We denote the Rokhlin dimension (with commuting towers) of α by $\dim_{\mathrm{Rok}}^{c}(\alpha)$. If no such integer exists, we say that α has infinite Rokhlin dimension (with commuting towers), and write $\dim_{\mathrm{Rok}}^{c}(\alpha) = +\infty$.

We now describe the local approximation theorem due to Gardella *et al.* [11] that will help prove the permanence result we are interested in.

PROPOSITION 4.2. [11, Corollary 4.9] *Let G be a compact, second countable group, X be a compact Hausdorff space, and A be a separable C*-algebra. Let $G \curvearrowright X$ be a continuous, free action of G on X, and $\alpha : G \to Aut(A)$ be an action of G on A. Equip the C*-algebra $C(X, A)$ with the diagonal action of G, denoted by γ. Then the crossed product C*-algebra $C(X, A) \rtimes_\gamma G$ is a continuous $C(X/G)$-algebra, each of whose fibres are isomorphic to $A \otimes \mathcal{K}(L^2(G))$.*

In the context of Proposition 4.2, the natural inclusion map $\rho : A \to C(X, A)$ is a G-equivariant $*$-homomorphism. Hence, it induces a map $\rho : A \rtimes_\alpha G \to C(X, A) \rtimes_\gamma G$. To describe the nature of this map, we need the next definition, which is due to Barlak and Szabó [1]. Once again, we choose to work with the local definition as it is more convenient for our purpose.

DEFINITION 4.3. Let A and B be separable C*-algebras. A $*$-homomorphism $\varphi : A \to B$ is said to be *sequentially split* if, for every compact set $F \subset A$, and for every $\epsilon > 0$, there exists a $*$-homomorphism $\psi = \psi_{F,\epsilon} : B \to A$ such that

$$\|\psi \circ \phi(a) - a\| < \epsilon$$

for all $a \in F$.

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The next theorem, due to Gardella *et al.* [11, Proposition 4.11] is an important structure theorem that allows one to prove permanence results concerning crossed products with finite Rokhlin dimension (with commuting towers).

THEOREM 4.4. *Let $\alpha : G \to Aut(A)$ be an action of a compact, second countable group on a separable C*-algebra such that $\dim_{\mathrm{Rok}}^c(\alpha) < \infty$. Then there exist a compact metric space X and a free action $G \curvearrowright X$ such that the canonical embedding*

$$\rho : A \rtimes_\alpha G \to C(X, A) \rtimes_\gamma G$$

is sequentially split. Furthermore, if G finite-dimensional, then X may be chosen to be finite-dimensional as well.

In light of Theorem 4.4, we now show that the properties of being rationally K-stable (K-stable) passes from the target algebra B to the domain algebra A, in the presence of a sequentially split $*$-homomorphism. To this end, we fix the following notation. Given $*$-homomorphism $\varphi : A \to B$, $\varphi_n : M_n(A) \to M_n(B)$ represents the inflation of φ, given by $\varphi_n((a_{i,j})) = (\varphi(a_{i,j}))$. Furthermore, $\iota^B : B \to M_2(B)$ represents the canonical inclusion.

PROPOSITION 4.5. *Let A and B be separable C*-algebras, and $\varphi : A \to B$ be a sequentially split $*$-homomorphism. If B is rationally K-stable (K-stable), then so is A.*

PROOF. Since the proofs of both cases are entirely similar, we only prove that rational K-stability passes from B to A. As before, we need to show that the map

$$(\iota^A)_* \otimes \mathrm{id} : \pi_j(\widehat{\mathcal{U}}_0(M_n(A))) \otimes \mathbb{Q} \to \pi_j(\widehat{\mathcal{U}}_0(M_{n+1}(A))) \otimes \mathbb{Q}$$

is an isomorphism for all $j \geq 1$, and $n \geq 1$. If $\varphi : A \to B$ is sequentially split, then so is $\varphi_n : M_n(A) \to M_n(B)$, so we may assume without loss of generality that $n = 1$.

We first show that $(\iota^A)_* \otimes \mathrm{id}$ is injective. So suppose $[f] \otimes q \in \pi_j(\widehat{\mathcal{U}}_0(A)) \otimes \mathbb{Q}$ is such that $[\iota^A \circ f] \otimes q = 0$ in $\pi_j(\widehat{\mathcal{U}}_0(M_2(A)))$. Then $[\iota^A \circ f]$ has finite order in $\pi_j(\widehat{\mathcal{U}}_0(M_2(A)))$, which implies $[\varphi_2 \circ \iota^A \circ f] = [\iota^B \circ \varphi \circ f]$ has finite order in $\pi_j(\widehat{\mathcal{U}}_0(M_2(B)))$. Since B is rationally K-stable, $[\varphi \circ f]$ also has finite order in $\pi_j(\widehat{\mathcal{U}}_0(B))$. Let $F := \{f(x) : x \in S^j\}$, which is a compact set in A, so there exists a $*$-homomorphism $\psi = \psi_{F,1} : B \to A$ such that $\|\psi \circ \varphi(a) - a\| < 1$ for all $a \in F$. Hence,

$$\|\psi \circ \varphi \circ f - f\| < 1$$

in $\widehat{\mathcal{U}}(C_*(S^j, A))$. Thus, by Lemma 3.3, we conclude that

$$[\psi \circ \varphi \circ f] = [f]$$

in $\pi_j(\widehat{\mathcal{U}}_0(A))$. However, since $[\varphi \circ f]$ has finite order in $\pi_j(\widehat{\mathcal{U}}_0(B))$, $[\psi \circ \varphi \circ f] = [f]$ has finite order in $\pi_j(\widehat{\mathcal{U}}_0(A))$. Hence, $[f] \otimes q = 0$, proving that $(\iota^A)_* \otimes \mathrm{id}$ is injective.

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For surjectivity, fix an element $[u] \in \pi_j(\widehat{\mathcal{U}}_0(M_2(A)))$ and $m \in \mathbb{Z}$. We wish to construct elements $[\omega] \in \pi_j(\widehat{\mathcal{U}}_0(A))$ and $q \in \mathbb{Q}$ such that

$$((\iota^A)_* \otimes \mathrm{id})([\omega] \otimes q) = [u] \otimes \frac{1}{m}.$$

Now, $[\varphi_2 \circ u] \otimes (1/m) \in \pi_j(\widehat{\mathcal{U}}_0(M_2(B))) \otimes \mathbb{Q}$. Since B is rationally K-stable, there exist $[g] \in \pi_j(\widehat{\mathcal{U}}_0(B))$ and $n \in \mathbb{Z}$ such that

$$((\iota^B)_* \otimes \mathrm{id})\Big([g] \otimes \frac{1}{n}\Big) = [\varphi_2 \circ u] \otimes \frac{1}{m}.$$

Again, as in previous calculations, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$N_1[\iota^B \circ g] = N_2[\varphi_2 \circ u] \tag{4-1}$$

in $\pi_j(\widehat{\mathcal{U}}_0(M_2(B)))$. Now, fix $F := \{u(x) : x \in S^j\}$, so we get a $*$-homomorphism $\psi_F : M_2(B) \to M_2(A)$ such that $\|\psi_F \circ \varphi_2 \circ u(x) - u(x)\| < \frac{1}{2}$ for all $x \in S^j$. Hence,

$$[\psi_F \circ \phi_2 \circ u] = [u] \tag{4-2}$$

in $\pi_j(\widehat{\mathcal{U}}_0(M_2(A)))$. Now, we write $u = (u_{i,j})_{1 \le i,j \le 2}$, and take

$$K = \{u_{i,j}(x) : 1 \le i,j \le 2, x \in S^j\} \subset A.$$

Then K is compact, so we get a $*$-homomorphism $\psi_K : B \to A$ such that

$$\|\psi_K \circ \varphi \circ u_{i,j}(x) - u_{i,j}(x)\| < \tfrac{1}{8}$$

for all $x \in S^j$ and $1 \le i,j \le 2$. Thus, $\|(\psi_K)_2 \circ \varphi_2 \circ u(x) - u(x)\| < \frac{1}{2}$ for all $x \in S^j$. Therefore, $\|\psi_F \circ \varphi_2 \circ u(x) - (\psi_K)_2 \circ \phi_2 \circ u(x)\| < 1$ for all $x \in S^j$, so that we have $[\psi_F \circ \varphi_2 \circ u] = [(\psi_K)_2 \circ \varphi_2 \circ u]$ in $\pi_j(\widehat{\mathcal{U}}_0(M_2(A)))$. Now, from Equations (4-1) and (4-2),

$$N_1[(\psi_K)_2 \circ \iota^B \circ g] = N_2[(\psi_K)_2 \circ \phi_2 \circ u] = N_2[\psi_F \circ \phi_2 \circ u] = N_2[u].$$

Since $(\psi_K)_2 \circ \iota^B \circ g = \iota^A \circ \psi_K \circ g$, we have

$$N_1[\iota^A \circ \psi_K \circ g] = N_2[u].$$

Therefore, if $\omega := \psi_K \circ g$ and $q := N_1/N_2 m$, then

$$(\iota^A)_* \otimes \mathrm{id}([\omega] \otimes q) = [u] \otimes \frac{1}{m}$$

proving that $(\iota^A)_* \otimes \mathrm{id}$ is surjective. □

We are now in a position to complete the proof of Theorem B, restated as follows.

COROLLARY 4.6. *Let $\alpha : G \to Aut(A)$ be an action of a compact Lie group on a separable C^*-algebra A such that $\dim_{\mathrm{Rok}}^c(\alpha) < \infty$. If A is rationally K-stable (K-stable), then so is $A \rtimes_\alpha G$.*

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PROOF. We first discuss the case of K-stability. Let X be the (finite-dimensional) metric space obtained from Theorem 4.4. By Proposition 4.2, $C(X, A) \rtimes_\gamma G$ is a continuous $C(X/G)$-algebra, each of whose fibres is isomorphic to $A \otimes \mathcal{K}(L^2(G))$, and is hence K-stable. Since X is compact and metrizable, so is X/G. Furthermore, since G is a compact Lie group, it follows that

$$\dim(X/G) \leq \dim(X) < \infty$$

by [17, Corollary 1.7.32]. By [23, Theorem A], we conclude that $C(X, A) \rtimes_\gamma G$ is K-stable, and hence $A \rtimes_\alpha G$ is K-stable by Proposition 4.5.

The argument for rational K-stability is entirely similar, except that we apply Theorem A instead of [23, Theorem A]. □

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