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RESEARCH ARTICLE

The Brouwer invariance theorems in reverse mathematics

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Abstract
In [12], John Stillwell wrote, 'finding the exact strength of the Brouwer invariance theorems seems to me one of the most interesting open problems in reverse mathematics.' In this article, we solve Stillwell's problem by showing that (some forms of) the Brouwer invariance theorems are equivalent to the weak König's lemma over the base system RCA_0. In particular, there exists an explicit algorithm which, whenever the weak König's lemma is false, constructs a topological embedding of $\mathbb{R}^4$ into $\mathbb{R}^3$.

Contents

1. Introduction

How different are $\mathbb{R}^m$ and $\mathbb{R}^n$? It is intuitively obvious that $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic whenever $m \neq n$. However, this is not as easy as it appears. Quite a few prominent mathematicians have tried to solve this *invariance of dimension* problem, and nobody before Brouwer succeeded in providing a correct rigorous proof (see [14, Section 5.1] for the history of the invariance of dimension problem).

In the early days of topology, Brouwer proved three important theorems: the *Brouwer fixed point theorem*, the *invariance of dimension theorem*, and the *invariance of domain theorem*. Modern proofs of these theorems make use of singular homology theory [3] or its relative of the same nature, but even today, no direct proofs (only using elementary topology) have been found.

Brouwer's intuitionist standpoint eventually led him to refuse his theorems and even propose a counterexample to his fixed point theorem. As an alternative, he introduced an approximate version of the fixed point theorem (which follows from Sperner's lemma); however, it does not provide us an approximation of an actual fixed point, as Brouwer himself already pointed out (cf. [14, p. 503]). Indeed, there is *no* computable algorithm which, given a sequence $(x_n)_{n\in\mathbb{N}}$ of points such that x_n looks like a fixed point within precision 2^{-n}, produces an approximation of an actual fixed point. How nonconstructive, then, are Brouwer's original theorems?

We examine this problem from the perspective of reverse mathematics. Reverse mathematics is a program to determine the exact (set-existence) axioms needed to prove theorems of ordinary mathematics. We employ a subsystem RCA_0 of second-order arithmetic as our base system, which consists of Robinson arithmetic (or the theory of the nonnegative parts of discretely ordered rings), Σ^0_1-induction schema, and Δ^0_1-comprehension schema (cf. [11, 12]).

Roughly speaking, the system RCA_0 corresponds to computable mathematics, which has enough power to show the approximate fixed point theorem (see [11, Section IV.7]). On the other hand, Orevkov [8] showed that the Brouwer fixed point theorem is invalid in computable mathematics in a rigorous sense; hence RCA_0 is not enough to prove the actual fixed point theorem.

In Bishop-style constructive mathematics, it is claimed that a uniform continuous version of the invariance of dimension theorem has a constructive proof (cf. [1, Section I.19]). Similarly, in the same constructive setting, Julian et al. [4] studied the Alexander duality theorem and the Jordan-Brouwer separation theorem (which are basic tools to show the invariance of domain theorem in modern algebraic topology; cf. [3]). However, these constructive versions are significantly different from the original ones (from constructive and computable viewpoints).

Concerning the original theorems, Shioji and Tanaka [10] (see also [11, Section IV.7]) utilized Orevkov's idea to show that over RCA_0, the Brouwer fixed point theorem is equivalent to the *weak König's lemma* (WKL): Every infinite binary tree has an infinite path. Other examples equivalent to WKL include the Jordan curve theorem and the Schönflies theorem [9].

In [12], Stillwell wrote, 'finding the exact strength of the Brouwer invariance theorems seems to me one of the most interesting open problems in reverse mathematics.' In this article, we solve this problem by showing that some forms of the Brouwer invariance theorems are equivalent to WKL over the base system RCA_0.

Theorem 1. *The following are equivalent over* RCA_0*:*

1. *The weak König's lemma.*
2. *(Invariance of domain) Let $U \subseteq \mathbb{R}^m$ be an open set and $f: U \to \mathbb{R}^m$ be a continuous injection. Then the image $f[U]$ is also open.*
3. *(Invariance of dimension I) If $m > n$, then there is no continuous injection from $\mathbb{R}^m$ into $\mathbb{R}^n$.*
4. *(Invariance of dimension II) If $m > n$, then there is no topological embedding of $\mathbb{R}^m$ into $\mathbb{R}^n$.*

Proof. For (1)⇒(2), as mentioned in [12], the usual algebraic topology machineries (cf. [3]) are available in WKL_0. A simpler proof of the invariance of domain theorem is presented in [13, Section 6.2], which can also be carried out in WKL_0.

For (2)⇒(3), suppose $m > n$ and that there is a continuous injection f from $\mathbb{R}^m$ into $\mathbb{R}^n$. Define $g: \mathbb{R}^m \to \mathbb{R}^m$ by $g(x) = (f(x), 0, 0, \dots, 0)$. Then g is also a continuous injection. Hence, by invariance of domain, the image of g is open. However, if $m > n$, then $\{(z, 0, 0, \dots, 0) \in \mathbb{R}^m : z \in \mathbb{R}^n\}$ does not contain a nonempty open set. Thus, we get $m \leq n$.

The implication (3)⇒(4) is obvious. We devote the rest of the paper to proving the implication (4)⇒(1). □

We first describe the outline of our strategy for (the contrapositive of) (4)⇒(1).

First, we will show that several basic results in topological dimension theory are provable in RCA_0. More explicitly, RCA_0 proves that whenever the n-sphere $\mathbb{S}^n$ is an absolute extensor for X, the covering dimension of X is at most n. We will also show that the Nöbeling embedding theorem (stating that every n-dimensional Polish space is topologically embedded into a 'universal' n-dimensional subspace of $\mathbb{R}^{2n+1}$) is provable in RCA_0.

Then, under $\mathsf{RCA}_0 + \neg\mathsf{WKL}$, we will show that the 1-sphere $\mathbb{S}^1$ is an absolute extensor (for all Polish spaces). This means that under $\neg\mathsf{WKL}$, *every Polish space is at most one-dimensional*, and therefore, by the Nöbeling embedding theorem, every Polish space is topologically embedded into $\mathbb{R}^3$. In particular, we will show that assuming $\neg\mathsf{WKL}$, a topological embedding of $\mathbb{R}^4$ into $\mathbb{R}^3$ *does* exist. However, the following two questions remain open.

Question 1. Does RCA_0 prove that there is no topological embedding of $\mathbb{R}^3$ into $\mathbb{R}^2$?

Question 2. Does RCA_0 prove that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ whenever $m \neq n$?

1.1. Preliminaries

We assume that the reader is familiar with reverse mathematics (see [11, 12]). In particular, we use standard formulations of mathematical concepts in second-order arithmetic: A real number is coded as a Cauchy sequence of rational numbers with modulus of convergence ([11, Definition II.4.4]). A Polish space X is coded as a pair of a countable set $A \subseteq \mathbb{N}$ (which represents a countable dense subset of a space X) and a function $d\colon A^2 \to \mathbb{R}$ ([11, Definition II.5.2]). A code of an open set $U \subseteq X$ is any sequence of rational open balls B_n whose union is U ([11, Definition II.5.6]). A code of a closed set $U \subseteq X$ is a code of its complement (as an open set). A code of an open set and a code of a closed set in this sense are sometimes called a *positive* code and a *negative* code, respectively. A code of a partial continuous function $f\colon\subseteq X \to Y$ is any data Φ specifying a modulus of point-wise continuity for f; that is, if (a, r, b, s) is enumerated into Φ at some round, then $x \in \mathrm{dom}(f)$ and $d_X(x, a) < r$ imply $d_Y(f(x), b) \leq s$ ([11, Definition II.6.1]). A topological embedding f of X into Y is coded as a pair of (codes of) continuous functions (f, g) such that $g \circ f(x) = x$ for any $x \in X$.

In particular, we note that a 'code' of some mathematical object can always be considered as an element of $\mathbb{N}^{\mathbb{N}}$. In reverse mathematics, we often use sentences like 'for a given x one can *effectively* find a y such that ...' when there is a partial continuous function $f\colon\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that if $\dot{x}$ is a code of x, then $f(\dot{x})$ is defined and returns a code of such a y.

2. Proof of (4)⇒(1)

2.1. Coincidence of dimension

In this section, we discuss a few basic results in topological dimension theory within RCA_0. For basics on classical topological dimension theory, see [2, 7]. Throughout this section, a space always means a Polish space.

It is not hard to see that the results we will discuss in this section are provable within RCA (i.e., RCA_0 plus full induction); however, most basic results in topological dimension theory involve induction argument (see Lemma 1 and Lemma 4), so we will need a few tricks to make the proofs work with only Σ^0_1-induction.

2.1.1. Normality

A space X is *normal* if for any (negative codes of) disjoint closed sets $P_0, P_1 \subseteq X$, one can find (positive codes of) disjoint open sets $S_0, S_1 \subseteq X$ such that $P_0 \subseteq S_0$ and $P_1 \subseteq S_1$. A space X is *perfectly normal* if for any disjoint closed sets $P_0, P_1 \subseteq X$, one can effectively find a (code of) continuous function $g\colon X \to [0, 1]$ such that for all $x \in X$ and $i < 2$, $x \in C_i$ if and only if $g(x) = i$. Note that effectivity is required for all notions to reduce the complexity of induction involved in our proofs. It is known that the effective version of Urysohn's lemma is provable within RCA_0 as follows:

Fact 1 (cf. [11, Lemma II.7.3])**.** Over RCA_0, every Polish space is perfectly normal. □

Let $\mathcal{U}$ be a cover of a space X. A cover $\mathcal{V}$ of X is a *refinement of* $\mathcal{U}$ if for any $B \in \mathcal{V}$ there is $A \in \mathcal{U}$ such that $B \subseteq A$. A *shrinking of a cover* $\mathcal{U} = (U_i)_{i<s}$ of X is a cover $\mathcal{V} = (V_i)_{i<s}$ of X such that $V_i \subseteq U_i$ for any $i < s$.

Lemma 1 (RCA_0)**.** *Let X be a perfectly normal space. Then for every finite open cover $\mathcal{U}$ of X, one can effectively find a closed shrinking of $\mathcal{U}$.*

Proof. Let $\mathcal{U} = \{U_i\}_{i<k}$ be a finite open cover. By perfect normality of X, for each $i < k$ one can effectively find a continuous function $g_i : X \to [0, 1]$ such that $g_i(x) > 0$ if and only if $x \in U_i$ for any $x \in X$. One can effectively construct (a code of) the following sequence $\langle g_i', \tilde{g}_i\rangle_{i<k}$ of (possibly partial) continuous functions:

$$\tilde{g}_i(x) = \frac{g_i(x)}{g_i(x) + \max\{g_s'(x), g_t(x) : s < i < t < k\}},$$
$$g_i'(x) = \max\left\{0, \tilde{g}_i(x) - \frac{1}{2}\right\}.$$

Fix $x \in X$. By Σ^0_1-induction, we show that the denominator in the definition of $\tilde{g}_i(x)$ is nonzero. Note that $g_i(x) > 0$ for some $i < k$, since $(U_i)_{i<k}$ covers X. This verifies the base case. We inductively assume that the denominator of $\tilde{g}_i(x)$ is nonzero, that is, $g_s'(x) > 0$ for some $s < i$ or $g_t(x) > 0$ for some $t \geq i$. Suppose that the denominator of $\tilde{g}_{i+1}(x)$ is zero, that is, $g_s'(x) = 0$ for any $s \leq i$ or $g_t(x) = 0$ for any $t > i$. Note that $g_i'(x) = 0$ implies $\tilde{g}_i(x) \leq 1/2$, and therefore, by the definition of $\tilde{g}_i$, we have

$$g_i(x) \leq \max\{g_s'(x), g_t(x) : s < i < t < k\} = 0.$$

However, this contradicts the induction hypothesis. Hence, $\langle g_i', \tilde{g}_i\rangle_{i<k}$ defines a sequence of total continuous functions, and for any $x \in X$, we have $g_i'(x) > 0$ for some $i < k$, as seen earlier. This means that $W_i = \{x \in X : g_i'(x) > 0\} = \{x \in X : \tilde{g}_i(x) > 1/2\}$ covers X. Therefore, $F_i = \{x \in X : \tilde{g}_i(x) \geq 1/2\}$ also covers X. Now if $g_i(x) = 0$, then clearly $\tilde{g}_i(x) = 0 < 1/2$; hence we have $W_i \subseteq F_i \subseteq U_i$. This concludes that $(F_i)_{i<k}$ is a closed shrinking of $(U_i)_{i<k}$. □

2.1.2. Star refinement

Let $S \subseteq X$ and $\mathcal{U}$ be a cover of a space X. A *star of S w.r.t.* $\mathcal{U}$ is defined as

$$\mathrm{st}(S, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : S \cap U \neq \emptyset\}.$$

Define $\mathcal{U}^\star$ by $\{\mathrm{st}(U, \mathcal{U}) : U \in \mathcal{U}\}$. A *star refinement* of a cover $\mathcal{U}$ of X is a cover $\mathcal{V}$ of X such that $\mathcal{V}^\star$ is a refinement of $\mathcal{U}$. It is known that a space is normal if and only if every finite open cover has a finite open star refinement.

Lemma 2 (RCA_0)**.** *Let X be a normal space. Then for every finite open cover $\mathcal{U}$ of X, one can effectively find a finite open star refinement of $\mathcal{U}$.*

Proof. Given a finite open cover $\mathcal{U} = \{U_i\}_{i<k}$ of X, as in the proof of Lemma 1, one can effectively find a closed shrinking $\{F_i\}_{i<k}$ and an open shrinking $\mathcal{W} = \{W_i\}_{i<k}$ such that $W_i \subseteq F_i \subseteq U_i$ for each $i < k$. Then $\mathcal{V}_i = \{X \setminus F_i, U_i\}$ is an open cover of X. Define $\mathcal{V}$ as the following open cover of X:

$$\mathcal{V} = \mathcal{W} \wedge \bigwedge_{i<k} \mathcal{V}_i := \left\{W \cap \bigcap_{i<k} V_i : W \in \mathcal{W},\ V_i \in \mathcal{V}_i\right\}.$$

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We claim that if $V \in \mathcal{V}$ is of the form $W_\ell \cap \bigcap_{i<k} V_i$, then $\mathrm{st}(V, \mathcal{V}) \subseteq U_\ell$. For any $V^* \in \mathcal{V}$ of the form $W_m \cap \bigcap_{i<k} V_i^*$, if $V \cap V^* \neq \emptyset$, then $V_\ell^* \neq X \setminus F_\ell$, since $V \subseteq W_\ell \subseteq F_\ell$. Therefore, $V^* \subseteq V_\ell^* = U_\ell$. Consequently, $\mathcal{V}$ is an open star refinement of $\mathcal{U}$ as desired. □

Also define $\mathcal{U}^\triangle$ by $\{\mathrm{st}(\{x\}, \mathcal{U}) : x \in X\}$. A *point-star refinement* (or a *barycentric refinement*) of a cover $\mathcal{U}$ of X is a cover $\mathcal{V}$ of X such that $\mathcal{V}^\triangle$ is a refinement of $\mathcal{U}$. Clearly, every star refinement is a point-star refinement.

2.1.3. Absolute extensor

A space K is called an *absolute extensor* for a space X if for any continuous map $f : P \to K$ on a closed set $P \subseteq X$, one can find a continuous map $g : X \to K$ extending f, that is, $g \upharpoonright P = f \upharpoonright P$. It is known that the topological dimension (and the cohomological dimension) of a normal space can be restated in the context of the absolute extensor. Classically, it is known that the covering dimension of X is at most n if and only if the n-sphere $\mathbb{S}^n$ is an absolute extensor for X (cf. [2, Theorem 1.9.3] or [7, Theorem III.2]). This equivalence is due to Eilenberg and Otto. To prove the equivalence, Eilenberg and Otto introduced the notion of an essential family.

We will need effectivity for inessentiality to reduce the complexity of induction. Therefore, instead of considering the essentiality of a family, consider the following notion: A space X is $(n+1)$*-inessential* if for any sequence $(A_i, B_i)_{i<n+1}$ of disjoint pairs of closed sets in X, one can effectively find a sequence $(U_i, V_i)_{i<n+1}$ of disjoint open sets in X such that $A_i \subseteq U_i$ and $B_i \subseteq V_i$ for each $i \leq n$, and $(U_i \cup V_i)_{i<n+1}$ covers X.

Lemma 3 (RCA_0)**.** *Let X be a Polish space. If the n-sphere $\mathbb{S}^n$ is an absolute extensor for X, then X is $(n+1)$-inessential.*

Proof. As the boundary $\partial\mathbb{I}^{n+1}$ of the $(n+1)$-hypercube $\mathbb{I}^{n+1}$ is homeomorphic to $\mathbb{S}^n$, we can assume that $\partial\mathbb{I}^{n+1}$ is an absolute extensor for X. Given a sequence $(A_i, B_i)_{i<n+1}$ of disjoint pairs of closed sets, one can define $f : \bigcup_{i<n+1}(A_i \cup B_i) \to \partial\mathbb{I}^{n+1}$ such that $(\pi_i \circ f)^{-1}\{0\} = A_i$ and $(\pi_i \circ f)^{-1}\{1\} = B_i$ by perfect normality (Fact 1), where π_i is the projection into the ith coordinate. Then, by our assumption, we have $g : X \to \partial\mathbb{I}^{n+1}$, which agrees with f on $\bigcup_{i<n+1}(A_i \cup B_i)$. Define $U_i := (\pi_i \circ g)^{-1}[0, 1/2)$ and $V_i := (\pi_i \circ g)^{-1}(1/2, 1]$. Then $(U_i, V_i)_{i<n+1}$ covers X, since the range of g is contained in $\partial\mathbb{I}^{n+1}$. Hence, the sequence (U_i, V_i) observes the condition of $(n+1)$-inessentiality. □

2.1.4. Covering dimension

Let $\mathcal{U}$ be a cover of a space X. The *order of* $\mathcal{U}$ is at most n if for any $U_0, U_1, \ldots, U_{n+1} \in \mathcal{U}$ we have $\bigcap_{i<n+2} U_i = \emptyset$. A space X has the *covering dimension at most n* if for any finite open cover of X, one can effectively find a finite open refinement of order at most n.

Lemma 4 (RCA_0)**.** *Let X be a Polish space. If X is $(n+1)$-inessential, then the covering dimension of X is at most n.*

Proof. We first show the following claim.

Claim 1 (RCA_0)**.** *If X is $(n+1)$-inessential, then for any open cover $\mathcal{U} = (U_i)_{i<n+2}$ of X, one can effectively find an open shrinking $\mathcal{W} = (W_i)_{i<n+2}$ of $\mathcal{U}$ such that $\bigcap \mathcal{W} = \emptyset$.*

Proof. We follow the argument in [2, Theorem 1.7.9]. Given an open cover $\mathcal{U} = (U_i)_{i<n+2}$ of X, pick a closed shrinking $(F_i)_{i<n+2}$ by Lemma 1. Then consider the sequence $(U_i, X \setminus F_i)_{i<n+1}$ of open covers. By $(n+1)$-inessentiality, one can find a sequence of disjoint open sets $(W_i, V_i)_{i<n+1}$ in X such that $W_i \subseteq U_i$, $V_i \subseteq (X \setminus F_i)$ and $\bigcup_{i<n+1} W_i \cup V_i$ covers X. Define $W_{n+1} := U_{n+1} \cap \bigcup_{i<n+1} V_i$. As $F_{n+1} \subseteq U_{n+1}$, we have

$$\bigcup \mathcal{W} = \left[\bigcup_{i<n+1} W_i \cup U_{n+1}\right] \cap \left[\bigcup_{i<n+1} W_i \cup \bigcup_{i<n+1} V_i\right] \supseteq \bigcup_{i<n+2} F_i = X.$$

Thus, $\mathcal{W} = (W_i)_{i<n+2}$ is an open cover of X. Moreover, as V_i and W_i are disjoint, we have

$$\bigcap_{i<n+2} W_i = \bigcap_{i<n+1} W_i \cap \left[U_{n+1} \cap \bigcup_{i<n+1} V_i\right] \subseteq \bigcap_{i<n+1} W_i \cap \bigcup_{i<n+1} V_i = \emptyset.$$

This concludes that $\mathcal{W}$ is an open refinement of $\mathcal{U}$ of order at most n, as desired. □

We then follow the argument in [2, Theorem 1.6.10]. Suppose that $\mathcal{U} = \{U_i\}_{i<s}$ is a finite open cover of X. Let $[s]^{n+2}$ be the collection of all set $D \subseteq s$ such that $|D| = n+2$, and D_e be the eth element in $[s]^{n+2}$. Put $b := |[s]^{n+2}| = \binom{s}{n+2}$ and set $U_i^{-1} = U_i$. We will construct a sequence $(F_i^e, U_i^e)_{e<b}$ of pairs of a closed set F_i^e and an open set U_i^e such that $(U_i^e)_{i<s}$ is an open shrinking of $\mathcal{U}$ and, moreover,

$$(\forall i < s)\ U_i^e \subseteq F_i^e \subseteq U_i^{e-1}, \text{ and } \bigcap_{i \in D_e} U_i^e = \emptyset.$$

Given a sequence $\mathcal{U} = (U_i)_{i<s}$ of open sets which is given as cozero sets of $(u_i)_{i<s}$, by Claim 1 one can effectively find a code of a sequence $(w_i)_{i \in D_e}$ of partial continuous functions such that whenever $\mathcal{U}$ is a cover of X, w_i is total, the cozero sets $\mathcal{W} = (W_i)_{i\in D_e}$ of $(w_i)_{i\in D_e}$ are an open shrinking of $(U_i)_{i\in D_e}$ and $\mathcal{U}' := (U_i, W_j : i \in D_e,\ j \notin D_e)$ covers X. Set $u_i' = u_i$ for $i \notin D_e$ and $u_i' = w_i$ for $i \in D_e$. Then $\mathcal{U}'$ is given as a collection of cozero sets of u_i''s.

Then, by Lemma 1, one can effectively find a code of a sequence $(\tilde{v}_i)_{i<s}$ of partial continuous functions such that whenever $\mathcal{U}$ is a cover of X, v_i is total, $u_i'(x) = 0$ implies $\tilde{v}_i(x) = 0$ and $V_i = \{x : \tilde{v}_i(x) > 1/2\}$ covers X. Set $F_i = \{x : \tilde{v}_i(x) \geq 1/2\}$ and $v_i(x) = \max\{0, \tilde{v}_i(x) - 1/2\}$.

It is clear that if $\mathcal{U}$ is an open cover of X, then $(V_i)_{i<s}$ is an open shrinking of $\mathcal{U}$, and moreover,

$$V_i \subseteq F_i \subseteq U_i, \text{ and } \bigcap_{i\in D_e} V_i = \emptyset.$$

To reduce the complexity of induction, we now note that the construction $(u_i)_{i<s} \mapsto (v_i, \tilde{v}_i)_{i<s}$ is effective, that is, has an explicit Σ^0_1-description Φ. Hence, one can effectively obtain (a code of) a sequence $(\tilde{g}_i^e, g_i^e)_{e,i}$ such that $(u_i)_{i<s} = (g_i^{e-1})$ and $(\tilde{v}_i, v_i)_{i<s} = (\tilde{g}_i^e, g_i^e)_{i<s}$ satisfies the Σ^0_1-condition Φ describing the construction. Then define $U_i^e = \{x : \tilde{g}_i^e(x) > 1/2\}$ and $F_i^e = \{x : \tilde{g}_i^e(x) \geq 1/2\}$.

We first check that $(U_i^e)_{i<s}$ forms an open cover for any $e < b$. Fix $x \in X$. By Σ^0_1-induction, one can easily show that for any e, $x \in U_i^e$ for some $i < s$. Next we see that $U_i^d \subseteq U_i^e$ for any $e \leq d < b$. Fix $x \in X$. Note that $g_i^{e-1}(x) = 0$ implies $\tilde{g}_i^e(x) < 1/2$, and this condition is Σ^0_1. For $d > e$, inductively assume that $g_i^{e-1}(x) = 0$ implies $\tilde{g}_i^d(x) < 1/2$. Then $\tilde{g}_i^d(x) < 1/2$ clearly implies $g_i^d(x) = 0$, and therefore $\tilde{g}_i^{d+1}(x) < 1/2$. By Σ^0_1-induction, $g_i^{e-1}(x) = 0$ implies $\tilde{g}_i^d(x) < 1/2$ for any $d > e$. Hence, $g_i^{e-1}(x) = 0$ implies $g_i^d(x) = 0$ for $d > e$, which implies that $U_i^d \subseteq U_i^e$ for any $e \leq d < b$.

Finally, set $V_i = U_i^{b-1}$. We have shown that $(V_i)_{i<s}$ is an open shrinking of $\mathcal{U}$. It remains to show that the order of $(V_i)_{i<s}$ is at most n. To see this, it suffices to show that for any e, $\bigcap_{i\in D_e} V_i = \emptyset$. As shown earlier, $\mathcal{U}^{e-1} = (U_i^{e-1})_{i<s}$ forms an open cover. Therefore, $(U_i^e)_{i<s}$ is an open shrinking of $\mathcal{U}^{e-1}$ such that $\bigcap_{i\in D_e} U_i^e = \emptyset$. Then, as seen before, we have $V_i = U_i^{b-1} \subseteq U_i^e$ for any $i < s$. Therefore, $\bigcap_{i\in D_e} V_i = \emptyset$ as desired. □

2.2. Nöbeling's embedding theorem

The *n-dimensional Nöbeling space* N^n is a subspace of $\mathbb{I}^{2n+1}$ consisting of points with at most n rational coordinates. The Nöbeling embedding theorem says that an n-dimensional separable metrizable space is topologically embedded into the n-dimensional Nöbeling space. We will see that the Nöbeling imbedding theorem is provable in RCA_0 in the following sense.

Theorem 2 (RCA_0). *If the covering dimension of a Polish space X is at most n, then X can be topologically embedded into the n-dimensional Nöbeling space.*

More precisely, there is a topological embedding f of X into $\mathbb{I}^{2n+1}$ such that for any $x \in X$, at most n coordinates of $f(x)$ are rational.

2.2.1. The modified Kuratowski mapping

We say that points $\{p_i\}_{i<\ell}$ in $\mathbb{I}^{d+1}$ are in a *general position*, that is, if $0 \leq m \leq d$, then any $m+2$ points from $\{p_i\}_{i<\ell}$ do not lie in an m-dimensional hyperplane of $\mathbb{I}^{d+1}$. The following is an easy effectivization of a very basic observation (cf. [2, Theorem 1.10.2]).

Observation 1 (RCA_0). Given $\varepsilon > 0$ and points $q_1, \dots, q_k \in \mathbb{R}^m$, one can effectively find $p_1, \dots p_k \in \mathbb{R}^m$ in general position such that $d(p_i, q_i) < \varepsilon$ for any $i \leq k$. □

A *polyhedron* is a geometric realization $|\mathcal{K}|$ of a simplicial complex $\mathcal{K}$ in a Euclidean space. We approximate a given space by a polyhedron as follows: Let $\mathcal{U} = (U_i)_{i<k}$ be a finite open cover of X. The *nerve of* $\mathcal{U}$ is an abstract simplicial complex $\mathcal{N}(\mathcal{U})$ with k vertices $\{p_i\}_{i<k}$ such that an m-simplex $\{p_{j_0}, \dots, p_{j_{m+1}}\}$ belongs to $\mathcal{N}(\mathcal{U})$ if and only if $U_{j_0} \cap \dots \cap U_{j_{m+1}}$ is nonempty. We define the function $\kappa : X \to |\mathcal{N}(\mathcal{U})|$ as

$$\kappa(x) = \frac{\sum_{i=0}^{k-1} d(x, X \setminus U_i) p_i}{\sum_{j=0}^{k-1} d(x, X \setminus U_j)}.$$

The function κ is called *the κ-mapping (or Kuratowski mapping) determined by $\mathcal{U}$ and $(p_i)_{i<k}$*. For basics on κ-mapping, see also [2, Definition 1.10.15] and [7, Section IV.5].

However, we cannot ensure the existence of $(x, i) \mapsto d(x, X \setminus U_i)$ within RCA_0. Therefore, we introduce a replacement for the κ-mapping. Recall that within RCA_0, given an open set U_i, one can effectively find a continuous function $u_i : X \to [0, 1]$ whose cozero set is exactly U_i. The *modified κ-mapping* $\kappa : X \to \mathbb{I}^{2n+1}$ determined by $(u_i)_{i<s}$ and $(z_i)_{i<s}$ is defined as

$$\kappa(x) = \frac{\sum_{i<s} u_i(x) z_i}{\sum_{j<s} u_j(x)}.$$

The denominator of this formula is nonzero whenever $\mathcal{U}$ is a cover of X. Given $x \in X$, let $\Lambda(x)$ be the list of all indices $e < s$ such that $x \in U_e$. Such sets exist by bounded Σ_1^0 comprehension within RCA_0. Let $Z(x)$ be the hyperplane spanned by $(z_e : e \in \Lambda(x))$.

Claim 2 (RCA_0). *For any $x \in X$, $\kappa(x)$ is contained in the convex hull of $(z_e : e \in \Lambda(x))$, and in particular, $\kappa(x) \in Z(x)$.* □

Proof. Fix $x \in X$. By definition of u_i, $x \notin U_i$ (that is, $i \in \Lambda(x)$) implies $u_i(x) = 0$. Set $\lambda_i = u_i(x)/(\sum_{j \in \Lambda(x)} u_j(x))$. Clearly, $\sum_{i \in \Lambda(x)} \lambda_i = 1$ and $\kappa(x) = \sum_{i \in \Lambda(x)} \lambda_i z_i$. Hence, $\kappa(x)$ is contained in the convex hull of $(z_e : e \in \Lambda(x))$. □

2.2.2. Proof of Theorem 2

First, note that to work within RCA_0, we need to avoid any use of compactness. Therefore, we cannot use the standard proof of Nöbeling's embedding theorem. However, we will show that one can remove compactness arguments from some proof of Nöbeling's embedding theorem, for example, given in [7, Theorem IV.8], by performing very careful work.

Proof. For $n+1$ coordinates $(c_i)_{i<n+1} \in (2n+1)^{n+1}$ and $n+1$ rationals $(r_i)_{i<n+1}$, consider the following hyperplane:

$$L = \{(x_j)_{j<2n+1} \in \mathbb{I}^{2n+1} : (\forall i < n+1)\ x_{c_i} = r_i\}.$$

Let $(L_t)_{t\in\mathbb{N}}$ be the list of all such hyperplanes. For a list $(V_e)_{e\in\mathbb{N}}$ of all basic open balls in X, let $\langle i, j \rangle$

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be the tth pair such that $\overline{V_i} \subseteq V_j$. Then consider the open cover $\mathcal{V}_t = \{V_j, X \setminus \overline{V_i}\}$, where $\overline{V_i}$ is the formal closure of V_i, that is, the closed ball whose center and radius are the same as V_i.

We first give an explicit construction of (a code of) a sequence $(f_t)_{t\in\mathbb{N}}$ of (possibly partial) continuous functions. We describe the construction at stage t. Suppose that a continuous function $f_t\colon X \to \mathbb{I}^{2n+1}$ and a positive rational $\delta_t > 0$ have already been constructed. Consider L_t and $\mathcal{V}_t$. We construct a $\mathcal{V}_t$-mapping f_{t+1} which avoids L_t.

By the total boundedness of $\mathbb{I}^{2n+1}$, one can easily find a collection $(x_j)_{j\leq m}$ of points in $\mathbb{I}^{2n+1}$ such that $(B(x_j;\delta_t))_{j\leq m}$ covers $\mathbb{I}^{2n+1}$, where $B(x;\delta)$ is the open ball centered at x of radius δ. Consider $\mathcal{W}_t = \{f_t^{-1}[B(x_j;\delta_t)] : j \leq m\}$. Since the covering dimension of X is at most n, one can effectively find an open refinement of $\mathcal{V}_t \wedge \mathcal{W}_t$ of order at most n. Apply Lemma 2 to this new open cover of X to get an open star refinement $\mathcal{U}_t = (U^t_i)_{i<s}$ of $\mathcal{V}_t \wedge \mathcal{W}_t$ of order at most n. Then one can effectively find a sequence of continuous functions $(u^t_i)_{i<s}$ such that U^t_i is the cozero set of u^t_i.

For each $i < s$, one can effectively choose $x_i \in U^t_i$ and then get the value $f_t(x_i)$. Then, by Observation 1, we can effectively choose $z^t_i \in X$ and $p^t_j \in L_t$ such that

$$d(f_t(x_i), z^t_i) < \delta, \text{ and } (z^t_i, p^t_j)_{i<s,j<n+1} \text{ are in a general position,}$$

that is, if $0 \leq m \leq 2n$, then any $m+2$ vertices do not lie in an m-dimensional hyperplane of $\mathbb{I}^{2n+1}$. Let $\kappa\colon X \to \mathbb{I}^{2n+1}$ be the modified κ-mapping determined by $(u_i)_{i<s}$ and $(z_i)_{i<s}$.

Claim 3 (RCA_0). *For any $x \in X$, $d(f_t(x), \kappa(x)) < 3\delta_t$.*

Proof. Let $x \in X$ be given. If $x \notin U^t_i$, then $u^t_i(x) = 0$. If $x \in U^t_i$, since $\mathcal{U}_t$ is a refinement of $\mathcal{W}_t$, we have $d(f_t(x), f_t(y)) < 2\delta_t$ for any $y \in U^t_i$. Therefore, $d(f_t(x), z^t_i) < 3\delta_t$, since $d(f_t(x_i), z^t_i) < \delta_t$, where $x_i \in U^t_i$. Hence, by the definition of the modified κ-mapping, we get $d(f_t(x), \kappa(x)) < 3\delta_t$ for any $x \in X$, since

$$d(f_t(x), \kappa(x)) = d\left(\sum_{i<s} \lambda_i(x) f_t(x), \sum_{i<s} \lambda_i(x) z^t_i\right) \leq \sum_{i<s} \lambda_i(x) d(f_t(x), z^t_i) < 3\delta_t,$$

where $\lambda_i(x)$ is defined as in Claim 2. The first equality follows from $\sum_{i<s} \lambda_i = 1$, and the middle inequality follows from the triangle inequality. □

Let $[s]^{\leq n}$ denote the set of all finite subsets $D \subseteq s$ with $|D| \leq n$, and Z^t_D be the hyperplane spanned by $(z^t_e : e \in D)$. Now one can calculate the following value:

$$\eta_t := \min\{d(Z^t_D, Z^t_E) : D, E \in [s]^{\leq n},\ Z^t_D \cap Z^t_E = \emptyset)\} > 0.$$

Recall that $(z^t_i)_{i\in\Lambda(x)}$ and $(p^t_j)_{j<n+1}$ are in a general position, and L_t is spanned by $(p^t_j)_{j<n+1}$, which implies that $d(Z^t_D, L_t) > 0$ for any $D \in [s]^{\leq n}$. One can also calculate the following value:

$$\eta'_t := \min\{d(Z^t_D, L_t) : D \in [s]^{\leq n}\} > 0.$$

Now define $f_{t+1} = \kappa$ (where κ is the modified κ-mapping defined before Claim 3) and $\delta_{t+1} = \min\{\delta_t, \eta_t/8, \eta'_t/4\}/3$. To reduce the complexity of induction, note that the construction $(f_t, \delta_t) \mapsto (f_{t+1}, \delta_{t+1}, \eta_t, \eta'_t)$ is effective, that is, has an explicit Σ^0_1-description. We then have a sequence $(f_t, \delta_t, \eta_t, \eta'_t)_{t\in\mathbb{N}}$ with auxiliary parameters $(z^t_i)_{t\in\mathbb{N},i<s}$ and $(p^t_j)_{t\in\mathbb{N},j<n+1}$. A simple induction shows $\delta_t < 2^{-t}$. By Σ^0_1-induction with Claim 3, for any $t \leq s$ one can also show that $d(f_t(x), f_s(x)) < \sum_{s\geq t} \delta_{s+1} < 2^{-t}$; hence this is classically a uniform convergent sequence. Note that the uniform limit theorem is provable within RCA_0, since a modulus of point-wise continuity of the uniform limit $f = \lim_{t\to\infty} f_t$ is effectively calculated from a sequence of moduli of point-wise continuity of $(f_t)_{t\in\mathbb{N}}$ and the modulus of uniform convergence 2^{-t}. Hence, the uniform limit $f = \lim_{t\to\infty} f_t$ exists. By the definition of δ_t, we also get $d(f, f_{t+1}) < \eta_t/4, \eta'_t/2$.

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Claim 4 (RCA$_0$)**.** *For any $t \in \mathbb{N}$ and $y \in \mathbb{I}^{2n+1}$, there is $V \in \mathcal{V}_t$ such that $f^{-1}[B(y;\eta_t/4)] \subseteq V$.*

Proof. Let $y \in \mathbb{I}^{2n+1}$ be given. For $x, x' \in f^{-1}[B(y;\eta_t/4)]$, we have $d(f(x), f(x')) < \eta_t/2$. As $d(f, f_{t+1}) < \eta_t/4$, we have $d(f_{t+1}(x), f_{t+1}(x')) < \eta_t$. By Claim 2, we have $f_{t+1}(x) = \kappa(x) \in Z^t(x)$ and $f_{t+1}(x') = \kappa(x') \in Z^t(x')$, where $Z^t(x)$ is defined in a similar manner as before. By the choice of η_t, we have $Z^t(x) \cap Z^t(x') \neq \emptyset$.

Assume that $Z^t(x)$ is spanned by $(z^t_{i_\ell})_{\ell<t}$ and $Z^t(x')$ is spanned by $(z^t_{j_\ell})_{\ell<u}$. Since $Z^t(x) \cap Z^t(x') \neq \emptyset$, $(z^t_{i_\ell}, z^t_{j_m})_{\ell<t,m<u}$ lie on a $((t-1)+(u-1))$-dimensional hyperplane. By our choice, the open cover $\mathcal{U}_t$ has order at most n, and therefore $t, u \leq n+1$; hence $t + u \leq 2n+2$. Since $\{z_{i_\ell}, z_{j_m}\}_{\ell<t,m<u}$ are in a general position, $t+u$ vertices do not lie in a $(t+u-2)$-dimensional hyperplane. Hence, there must be ℓ and m such that $z_{i_\ell} = z_{j_m}$. This implies that $x, x' \in U_{i_\ell}$.

Consequently, if $x, x' \in f_t^{-1}[B(y;\eta_t/4)]$, then x' belongs to the star of $\{x\}$ w.r.t. $\mathcal{U}_t$, that is, $x' \in \mathrm{st}(\{x\}, \mathcal{U}_t)$. As $\mathcal{U}_t$ is a star refinement of $\mathcal{V}_t$, we obtain $V \in \mathcal{V}_t$ such that $f^{-1}[B(y;\eta_t/4)] \subseteq \mathrm{st}(\{x\}, \mathcal{U}_t) \subseteq V$. □

Claim 5. *For any $x \in X$ and $p \in L_t$, $d(f(x), p) > \eta'_t/2$.*

Proof. By the definition of η'_t, we have $d(Z_t(x), L_t) \geq \eta'_t$ for any $x \in X$. By Claim 2, we also have $f_{t+1}(x) \in Z_t(x)$, and therefore $d(f_{t+1}(x), L_t) \geq \eta'_t$. Hence $d(f(x), L_t) \geq \eta'_t/2$. □

Claim 5 ensures that the range of f avoids L_t; hence, f is a continuous map from X into the n-dimensional Nöbeling space $N^n \subseteq \mathbb{R}^{2n+1}$.

Claim 6. *f is injective.*

Proof. Let W be any open neighborhood of $x \in X$. Then, by the perfect normality of X (Fact 1), there are basic open balls V_i and V_j such that $x \in V_i \subseteq \overline{V_i} \subseteq V_j \subseteq W$. By applying Claim 4 to the code t of pairs $\langle i, j\rangle$, that is, $\mathcal{V}_t = \{V_j, X \setminus \overline{V_i}\}$, we get an open neighborhood B of $f(x)$ such that either $f^{-1}[B] \subseteq V_j$ or $f^{-1}[B] \subseteq X \setminus \overline{V_i}$. However, as $x \in V_i$, we have $x \in f^{-1}[B] \cap V_i \neq \emptyset$; hence $f^{-1}[B] \subseteq V_j$. Therefore, if $x' \notin W$, then, as $W \supseteq V_j$, we get $f(x') \notin B$. This implies that f is injective. □

It remains to show that f^{-1} is continuous in RCA$_0$. In the usual proof, by using the property that f is an ε-mapping for all $\varepsilon > 0$, we conclude that f is a closed map. However, it is unclear, from the property being an ε-mapping, how one can effectively obtain a code of the closed image $f[A]$ of a closed set $A \subseteq X$ (without using any compactness arguments). Fortunately, Claim 4 has more information than just saying that f is an ε-mapping, which can be used to show that f is an effective open map.

Claim 7. *f is an open map.*

Proof. Say that an open ball $B_X(x;q)$ in X is formally (resp. strictly) included in $B_X(y;p)$ if $d(x,y) \leq p - q$ (resp. $d(x,y) < p - q$). Note that if $B_X(x;q)$ is strictly included in $B_X(y;p)$, then $\overline{B_X(x;q)} \subseteq B_X(y;p)$. Let $U = \bigcup_e V_{u(e)} \subseteq X$ be an open set given as a union of open balls. Then make a new list $(V_{v(e,j)})_{e,j\in\mathbb{N}}$ of all open balls V_j such that V_j is strictly included in $V_{u(e)}$.

Let $t(e,j)$ be the code of the pair $\langle v(e,j), u(e)\rangle$, that is, $\mathcal{V}_{t(e,j)} = \{V_{u(e)}, X \setminus \overline{V}_{v(e,j)}\}$. We now consider a list $(B^{e,j}_k)_{k\in\mathbb{N}}$ of all open balls of radius $\leq \eta_{t(e,j)}/4$ in $\mathbb{I}^{2n+1}$. By Claim 4, either $f^{-1}[B^{e,j}_k] \subseteq V_{u(e)}$ or $f^{-1}[B^{e,j}_k] \subseteq X \setminus \overline{V}_{v(e,j)}$ holds. As we have already shown that f is continuous, we get a code of the open set $f^{-1}[B^{e,j}_k] = \bigcup_m V_{s(e,j,k,m)}$. If $V_{s(e,j,k,m)}$ is formally included in $V_{v(e,j)}$ for some m, then we must have $f^{-1}[B^{e,j}_k] \subseteq V_{u(e)}$. Let $(J_i)_{i\in\mathbb{N}}$ be a list of all such open balls $B^{e,j}_k$, that is,

$$\{J_i\}_{i\in\mathbb{N}} = \{B^{e,j}_k : V_{s(e,j,k,m)} \text{ is formally included in } V_{v(e,j)} \text{ for some } m\}.$$

We claim that $f[U] = \bigcup_{i\in\mathbb{N}} J_i$. If $x \in U$, then $x \in V_{u(e)}$ for some e, and so $x \in V_{v(e,j)}$ for some j. By Claim 4, if B is a sufficiently small basic open ball containing $f(x)$, then $f^{-1}[B] \subseteq V_{v(e,j)}$. Hence, $f^{-1}[B]$ contains an open ball which is formally included in $V_{v(e,j)}$. Therefore, $f(x) \in B = J_i$ for some

$i \in \mathbb{N}$. For the converse, if $J_i = B$, then $f^{-1}[B] \subseteq V_{u(e)}$ for corresponding e, as already mentioned, and therefore $f^{-1}[B] \subseteq V_{u(e)} \subseteq U$. Consequently, $B \subseteq f[U]$. □

By Claim 7, one can effectively obtain a code of f^{-1} as a continuous function. This concludes the proof of Theorem 2. □

2.3. Every Polish space is at most one-dimensional

We say that K is an *absolute extensor* if it is an absolute extensor for any Polish space. In other words, if X is a Polish space, for any continuous map $f\colon P \to K$ on a closed set $P \subseteq X$ one can find a continuous map $g\colon X \to K$ extending f. The Tietze extension theorem states that the unit interval $\mathbb{I}$ is an absolute extensor. This clearly implies that $\mathbb{I}^n$ is also an absolute extensor by coordinate-wise extending $f = (f_i)_{i<n}\colon P \to \mathbb{I}^n$ to $g = (g_i)_{i<n}\colon X \to \mathbb{I}^n$. It is known that the effective version of the Tietze extension theorem is provable within RCA_0 as follows:

Fact 2 (see [11, Theorem II.7.5])**.** The Tietze extension theorem is provable in RCA_0, that is, $\mathbb{I}^n$ is an absolute extensor. □

It is intuitively obvious that the topological dimension of the n-hypercube $\mathbb{I}^n$ is n (but the proof is not so easy even in the classical world). Surprisingly, however, under $\neg\mathsf{WKL}$, *every Polish space is at most one-dimensional* in the following sense.

Lemma 5 ($\mathsf{RCA}_0 + \neg\mathsf{WKL}$)**.** *If X is a Polish space, then the* 1*-sphere $\mathbb{S}^1$ is an absolute extensor for X.*

Proof. By Orevkov's construction [8] (cf. [10]), if the weak König's lemma fails, then there is a continuous retraction $r\colon \mathbb{I}^2 \to \partial\mathbb{I}^2$. Note that the one-dimensional sphere $\mathbb{S}^1$ is homeomorphic to $\partial\mathbb{I}^2$. Let $f\colon P \to \partial\mathbb{I}^2$ be a continuous map on a closed set $P \subseteq X$. Then, since $\mathbb{I}^2$ is an absolute extensor by Fact 2, one can effectively find a continuous extension $f^*\colon X \to \mathbb{I}^2$ of f such that $f^* \restriction P = f \restriction P$. Then $g = r \circ f^*\colon X \to \partial\mathbb{I}^2$ is continuous and extends f, since r is a continuous retraction. This concludes that $\mathbb{S}^1$ is an absolute extensor for X as $\mathbb{S}^1 \simeq \partial\mathbb{I}^2$. □

Proof (Proof of Theorem 1 (4)⇒(1)) Suppose $\neg\mathsf{WKL}$. Then by Lemma 5, $\mathbb{S}^1$ is an absolute extensor for $\mathbb{R}^m$. By Lemmata 3 and 4, the covering dimension of $\mathbb{R}^m$ is at most one. By Theorem 2, there is a topological embedding f of $\mathbb{R}^m$ into the one-dimensional Nöbeling space; that is, for any $x \in \mathbb{R}^m$, at most one coordinate of $f(x) \in \mathbb{R}^3$ is rational. Consequently, there is a topological embedding of $\mathbb{R}^m$ into $\mathbb{R}^3$. □

3. Continuous degrees

In this section, we mention some relationships between reverse mathematics of topological dimension theory and J. Miller's work on continuous degrees [6].

Classically, a space is countable-dimensional if it is a countable union of zero-dimensional subspaces. However, within RCA_0 it is difficult to handle with the notion of a subspace. Instead, we use the following definition. A *copy of a subspace of Y in X* is a pair $S = (f, g)$ of (codes of) partial continuous functions $f\colon\subseteq X \to Y$ and $g\colon\subseteq Y \to X$. Then we say that $x \in X$ *is a point in* $S = (f, g)$ if $f(x)$ is defined, and $g \circ f(x)$ is defined and equal to x. A separable metric space X is *countable-dimensional* if X is a union of countably many copies of subspaces of $\mathbb{N}^{\mathbb{N}}$; that is, there is a sequence $(S_e)_{e\in\mathbb{N}}$ of copies of subspaces of $\mathbb{N}^{\mathbb{N}}$ such that every $x \in X$ is a point in S_e for some $e \in \mathbb{N}$.

Theorem 3. *The following are equivalent over* RCA_0*:*

1. *The weak König's lemma.*
2. *The Hilbert cube $\mathbb{I}^{\mathbb{N}}$ is not countable-dimensional.*

Proof. (1)⇒(2): The usual argument (cf. [2, Theorem 1.8.20]) only uses the Brouwer fixed point theorem, which can be carried out in WKL_0 [10].

(2)⇒(1): As $\mathbb{I}^{\mathbb{N}}$ is Polish, if we assume $\neg\mathsf{WKL}$ then, by Lemma 5, $\mathbb{S}^1$ is an absolute extensor for $\mathbb{I}^{\mathbb{N}}$. Therefore, by Lemmata 3 and 4 and Theorem 2, $\mathbb{I}^{\mathbb{N}}$ can be embedded into the one-dimensional Nöbeling space N^1. It is clear that N^1 is a finite union of zero-dimensional subspaces. □

Indeed, the instance-wise version of Theorem 3 holds. We now consider the instance-wise version in an ω-model $(\omega, \mathcal{S})$ of RCA_0: For (1)⇒(2), if $(S_e)_{e\in\omega} \in \mathcal{S}$ is a sequence of copies of subspaces of ω^ω, then there is an infinite binary tree $T \in \mathcal{S}$ such that every infinite path through T computes a point $x \in \mathbb{I}^\omega$ which is not a point of S_e for any $e \in \omega$. For (2)⇒(1), if $T \in \mathcal{S}$ is an infinite binary tree, then there is a sequence $(S_e)_{e\in\omega} \in \mathcal{S}$ of copies of subspaces of ω^ω such that if $x \in \mathbb{I}^\omega$ is not a point in S_e for any $e \in \omega$, then x computes an infinite path through T.

We now interpret this instance-wise ω-model version of Theorem 3 in the context of continuous degrees. We say that $\mathbf{b}$ *is PA-above* $\mathbf{a}$ (written $\mathbf{a} \ll \mathbf{b}$) if for any $\mathbf{a}$-computable infinite binary tree there is a $\mathbf{b}$-computable infinite path. Miller [6] reduced the first-order definability of PA-aboveness to that of continuous degrees: Whenever $\mathbf{a}$ and $\mathbf{b}$ are total degrees, $\mathbf{a} \ll \mathbf{b}$ if and only if there is a non-total continuous degree $\mathbf{v}$ such that $\mathbf{a} < \mathbf{v} < \mathbf{b}$. For continuous and total degrees, see [6].

(1)⇒(2) implies [6, Theorem 8.2]: If $\mathbf{a}$ and $\mathbf{b}$ are total degrees and $\mathbf{b} \ll \mathbf{a}$, then there is a non-total continuous degree $\mathbf{v}$ with $\mathbf{b} < \mathbf{v} < \mathbf{a}$. To see this, consider the topped ω-model of RCA_0 consisting of all sets of Turing degree $\leq \mathbf{b}$. Then, as in [5], take the list (f_e, g_e) of all pairs of Turing reductions (more precisely, all reductions in the sense of representation reducibility), considered copies in $\mathbb{I}^\omega$ of subspaces of ω^ω. By (1)⇒(2), there is an infinite binary tree T of Turing degree $\mathbf{b}$ such that any path computes $x \in \mathbb{I}^\omega$, which is not a point in (f_e, g_e). Such an x is non-total, since there is no $\alpha \in \omega^\omega$ such that $f_e(x) = \alpha$ and $g_e(\alpha) = x$. As $\mathbf{b} \ll \mathbf{a}$, such an x is computable in $\mathbf{a}$. If necessary, by adding a new coordinate x to code $\mathbf{b}$, we can conclude that there is a non-total degree $\mathbf{v}$ with $\mathbf{b} < \mathbf{v} < \mathbf{a}$.

(2)⇒(1) implies [6, Theorem 8.4]: If $\mathbf{v}$ is a non-total continuous degree and $\mathbf{b} < \mathbf{v}$ is total, then there is a total degree $\mathbf{c}$ with $\mathbf{b} \ll \mathbf{c} < \mathbf{v}$. To see this, consider the same ω-model $\mathcal{S}$ as before. As in [5], we consider a copy $S \in \mathcal{S}$ of a subspace of ω^ω in $\mathbb{I}^\omega$ as a pair of $\mathbf{b}$-relative Turing reductions. As $\mathbf{v}$ is non-total and $\mathbf{b} \leq \mathbf{v}$, a point $x \in \mathbb{I}^\omega$ of degree $\mathbf{v}$ avoids any sequence of copies $(S_e)_{e\in\omega} \in \mathcal{S}$ of subspaces of ω^ω in $\mathbb{I}^\omega$. Hence, by (2)⇒(1), for any infinite binary tree $T \in \mathcal{S}$, x computes an infinite path c through T. Consequently, we have $\mathbf{b} \ll \mathbf{c} < \mathbf{v}$ for some $\mathbf{c}$.

This argument indicates that (some of) Miller's work [6] (on definability of PA degrees via continuous degrees) can be considered as the computable instance-wise version of Theorem 3.

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