EQUICONTINUITY ON HARMONIC SPACES

CORNELIU CONSTANTINESCU

Dedicated to Professor KIYOSHI NOSHIRO on his 60th birthday

G. Mokobodzki proved [5] that on any harmonic space with countable basis satisfying the axioms 1, 2, T_+, K_D [2] [1] any equally bounded set of harmonic functions is equicontinuous. P. Loeb and B. Walsh showed [4] that the same property holds on a harmonic space without countable basis, if Brelot's axiom 3 is fulfilled. The aim of this paper is to generalize these results to a harmonic space X satisfying only the axioms 1, 2_0, K_1, [2] [1] where 2_0 is a weakened form of axiom 2. As a corollary we get: if any point of X possesses two open neighbourhoods U, V such that the set of harmonic functions on U separates the points of $U \cap V$, then X has locally a countable basis.

Throughout this paper Bourbaki's notations and terminology will be used.

1. Family of measures

Throughout this paragraph we shall denote by X, Y two compact spaces and by $(\omega_x)_{x\in X}$ a family of (nonnegative) measures on Y such that for any equally bounded upper directed family $(f_\iota)_{\iota\in I}$ of Borel functions on Y the function on X

$$x \to \sup_{\iota\in I} \int f_\iota d\omega_x$$

is continuous. We denote for any bounded Borel function f on Y by f' the function on X

$$x \to \int f d\omega_x.$$

It is a continuous function. We denote further for any measure μ on X by μ' the measure on Y

$$f \to \int f' d\mu \qquad (f \in \mathscr{K}(Y)).$$

Received March 22, 1966.

https://doi.org/10.1017/S0027763000024089 Published online by Cambridge University Press

For any bounded Borel function f on Y and any measure μ on X we have

$$\int f' d\mu = \int f d\mu'.$$

LEMMA 1. *There exists a countable set $A \subset X$ such that for any nonnegative bounded Borel function f on Y f' vanishes on X if it vanishes on A.*

Let us denote by $\mathfrak{M}$ the set of finite subsets of X and for any $M \in \mathfrak{M}$ by $\mathscr{F}_M$ the set of Borel functions f on Y, $0 \leq f \leq 1$, such that f' vanishes on M. For any $f, g \in \mathscr{F}_M$ and any $x \in M$ we have

$$\int \sup(f, g) d\omega_x \leq \int f d\omega_x + \int g d\omega_x = 0.$$

Hence $\sup(f, g) \in \mathscr{F}_M$ and $\mathscr{F}_M$ is upper directed. It follows that the least upper bound u_M of the family $(f')_{f \in \mathscr{F}_M}$ is continuous. Since it vanishes on M we deduce

$$\inf_{M \in \mathfrak{M}} u_M(x) = 0$$

for any $x \in X$. Hence there exists, by Dini's theorem, an increasing sequence (M_n) in $\mathfrak{M}$ such that

$$\lim_{n \to \infty} u_{M_n}(x) = 0$$

for any $x \in X$. We set

$$A = \bigcup_{n=1}^{\infty} M_n.$$

Let f be a Borel function on Y, $0 \leq f \leq 1$, such that f' vanishes on A. f belongs to $\mathscr{F}_M$ for any M. Hence for any $x \in X$ and any $M \in \mathfrak{M}$

$$f'(x) \leq u_M(x).$$

It follows that f' vanishes on X.

COROLLARY 1. *There exists an atomic measure μ on X such that for any measure ν on X ν' is absolutely continuous with respect to μ'.*

Let μ be an atomic measure on X such that

$$\mu(\{x\}) > 0$$

for any $x \in A$. Let ν be an arbitrary measure on X and let f be a nonnegative

https://doi.org/10.1017/S0027763000024089 Published online by Cambridge University Press

bounded Borel function on Y such that

$$\int f d\mu' = 0.$$

Then f' vanishes on A and therefore on X and we get

$$\int f d\nu' = \int f' d\nu = 0.$$

THEOREM 1. *Let $\mathscr{C}$ be the Banach space of real continuous functions on X with the norm of uniform convergence, let $\mathscr{F}$ be the set of Borel functions f on Y such that $|f| \leq 1$ and let $\mathscr{F}'$ be the set $\{f' | f \in \mathscr{F}\}$. $\mathscr{F}'$ is compact with respect to the weak topology of $\mathscr{C}$ and any sequence in $\mathscr{F}'$ contains a convergent subsequence (also with respect to the weak topology of $\mathscr{C}$).*

Let μ be a measure on X such that for any measure ν on X ν' is absolutely continuous with respect to μ'. Let for any measure ν on X g_ν be a function of $\mathscr{L}^1(\mu')$ such that $\nu' = g_\nu \cdot \mu'$. The map φ of $\mathscr{L}^\infty(\mu')$ into $\mathscr{C}$ defined by

$$\varphi(f)(x) = \int f g_{\omega_x} d\mu' \qquad (f \in \mathscr{L}^\infty(\mu'))$$

is continuous with respect to the weak topology of $\mathscr{C}$ and the topology $\sigma(\mathscr{L}^\infty(\mu'), \mathscr{L}^1(\mu'))$ of $\mathscr{L}^\infty(\mu')$ (i.e. the least fine topology on $\mathscr{L}^\infty(\mu')$ for which all linear forms $g \to \int g h d\mu'$ $(h \in L^1(\mu'))$ are continuous). Since $L^\infty(\mu')$ is the dual of $\mathscr{L}^1(\mu')$ it follows that $\mathscr{F}$ is quasi-compact with respect to the topology $\sigma(\mathscr{L}^\infty(\mu'), \mathscr{L}^1(\mu'))$ of $\mathscr{L}^\infty(\mu')$ (we used here an idea of Mokobodzki [4]). Hence $\mathscr{F}' = \varphi(\mathscr{F})$ is compact with respect to the weak topology of $\mathscr{C}$. The last assertion follows from Šmulian Dieudonné-Schwartz theorem ([3] page 314) (we followed in this point P. Loeb and B. Walsh [4]).

THEOREM 2. *Let X, Y, Z be compact spaces and $(\omega_x)_{x \in X}$ (resp. $(\rho_y)_{y \in Y}$) be a family of measures on Y (resp. on Z) such that for any equally bounded upper directed family $(f_\iota)_{\iota \in I}$ of Borel functions on Y (resp. on Z) the function on X (resp. on Y)*

$$x \to \sup_{\iota \in I} \int f_\iota d\omega_x \qquad (\text{resp. } y \to \sup_{\iota \in I} \int f_\iota d\rho_y)$$

is continuous. Let $\mathscr{G}$ be the set of Borel functions g on Z such that $|g| \leq 1$. The set of functions on X

$$\{x \to \int (\int g d\rho_y) d\omega_x(y) \,|\, g \in \mathscr{G}\}$$

is compact with respect to the topology of uniform convergence on X.

https://doi.org/10.1017/S0027763000024089 Published online by Cambridge University Press

Let (g_n) be a sequence of functions of $\mathscr{G}$. By the preceding theorem there exists a subsequence (g_{n_k}) and a function $g \in \mathscr{G}$ such that

$$\lim_{k\to\infty}\int g_{n_k} d\rho_y = \int g d\rho_y$$

for any $y \in Y$. Let us denote for any natural number m by $f_{1,m}$ (resp. $f_{2,m}$) the function on Y

$$y \to \inf_{k \geq m}\int g_{n_k} d\rho_y \qquad (\text{resp. } y \to \sup_{k \geq m}\int g_{n_k} d\rho_y).$$

The sequence $(f_{1,m})$ (resp. $f_{2,m}$) is nondecreasing (resp. nonincreasing) and converges to the function

$$y \to \int g d\rho_y.$$

Hence $(f'_{1,m})_m$ (resp. $f'_{2,m})_m$ is a nondecreasing (resp. nonincreasing) sequence of continuous functions on X converging to the continuous function

$$x \to \int (\int g d\rho_y) d\omega_x(y).$$

By Dini's theorem the convergence is uniform on X. Hence the sequence

$$(x \to \int (\int g_{n_k} d\rho_y) d\omega_x(y))_k$$

converges uniformly on X to the function

$$x \to \int (\int g d\rho_y) d\omega_x(y).$$

2. Harmonic spaces

Let X be a locally compact space and $\mathscr{H}$ be a sheaf of real vector spaces of real continuous functions on X. More exactly this means that for any open set U of X $\mathscr{H}(U)$ is a set of real continuous functions on U, **called harmonic functions (on U)** such that: *a*) if $u, v \in \mathscr{H}(U)$ and if α, β are real numbers then $\alpha u + \beta v \in \mathscr{H}(U)$; *b*) if V is an open nonempty subset of the open set U, then the restriction to V of any harmonic function on U is a harmonic function on V; *c*) if $(U_\iota)_{\iota \in I}$ is a family of open nonempty sets of X and if u is a real function on $\bigcup_{\iota \in I} U_\iota$ such that its restriction to any U_ι is a harmonic function on U_ι then u is a harmonic function on $\bigcup_{\iota \in I} U_\iota$. Let V be an open relatively compact

https://doi.org/10.1017/S0027763000024089 Published online by Cambridge University Press

set of X. A family $(\omega_x)_{x \in V}$ of measures on the boundary ∂V of V is called a **family of harmonic measures on** V if:

$a)$ for any bounded Borel function f on ∂V the function on V

$$x \to \int f d\omega_x$$

is harmonic;

$b)$ if u is a harmonic function on U, $U \supset \bar{V}$, then for any $x \in V$

$$u(x) = \int u d\omega_x.$$

An open relatively compact set V of X for which there exists a family of harmonic measures on V will be called **pseudoregular**.

We suppose the following axioms.

Axiom 2_0. *The set of pseudoregular sets is a basis of X.*

Axiom K_1. *For any open set U of X the least upper bound of any equally bounded upper directed family of harmonic functions on U is harmonic on U.*

Lemma 2. *Let V be an open relatively compact set of X and let $(\omega_x)_{x \in V}$ be a family of harmonic measures on V. If $(f_\iota)_{\iota \in I}$ is an equally bounded upper directed family of Borel functions on ∂V then the function on V*

$$x \to \sup_{\iota \in I} \int f_\iota d\omega_x$$

is continuous.

The lemma follows immediately from axiom K_1.

Theorem 3. *The set $\mathcal{U}$ of harmonic functions u on X such that $|u| \leq 1$ is compact with respect to the uniform convergence on compact sets of X.*

Let U, V be pseudoregular sets, $\bar{U} \subset V$, and K be a compact subset of U. Let $(\omega_x)_{x \in U}$ (resp. $(\rho_y)_{y \in V}$) be a family of harmonic measures on U (resp. V). For any $y \in \partial U$, any $x \in K$ and any $u \in \mathcal{U}$ we have

$$u(y) = \int u d\rho_y, \qquad u(x) = \int \left(\int u d\rho_y \right) d\omega_x(y).$$

Let $\mathfrak{U}$ be an ultrafilter on $\mathcal{U}$. By lemma 2 and theorem 2 there exists a bounded Borel function f on ∂V such that u converges along $\mathfrak{U}$ uniformly on

https://doi.org/10.1017/S0027763000024089 Published online by Cambridge University Press

K to the function

$$x \to \int \left(\int f d\rho_y \right) d\omega_x(y).$$

This function being harmonic on U it follows that the function

$$x \to \lim_{u, \mathfrak{U}} u(x)$$

belongs to $\mathscr{U}$. Further it follows from axiom 2_0 that $\mathfrak{U}$ converges to it uniformly on any compact set of X.

Corollary 2. *Any equally bounded set of harmonic functions on X is equicontinuous.*

This is a consequence of Ascoli's theorem.

Corollary 3. *If any point of X possesses two open neighbourhoods U, V such that the set of harmonic functions on U separates the points of $U \cap V$ then X has locally a countable basis.*

Let $\mathscr{U}$ be the set of harmonic functions u on U such that $|u| \leq 1$. We may suppose that $\mathscr{U}$ separates the points of $U \cap V$. Let K be a compact subset of $U \cap V$. For any $x, y \in K$ we set

$$d(x, y) = \sup_{u \in \mathscr{U}} |u(x) - u(y)|.$$

Since $\mathscr{U}$ separates the points of K d is a distance on K. Since $\mathscr{U}$ is equicontinuous the topology of K is finer then the topology associated to d. It follows that these topologies coincide and K possesses a countable basis.

Bibliography

[1] H. Bauer, Axiomatische Behandlung des Dirichletschen Problems für elliptische und parabolische Differentialgleichungen, Math. Ann. **146** (1962), 1-59.

[2] M. Brelot, Lectures on Potential Theory (part. IV), Tata Institute of Fundamental Research, Bombay, (1960).

[3] G. Köthe, Topologische lineare Räume, (zweite Auflage) Springer Verlag, Berlin-Heidelberg-New York (1966).

[4] P. Loeb, B. Walsh, The Equivalence of Harnack's Principle and Harnack's Inequality in the Axiomatic System of Brelot, Ann. Inst. Fourier, **15**, **2** (1965), 597-600.

[5] G. Mokobodzki, Représentations intégrales des fonctions harmoniques et surharmoniques (1964), unpublished.

Institutul de Matematică
str. M. Eminescu 47
București 3, Rumania