and, using the notation introduced above, we have $d = 14$, $R^2 = 882$, $r = 7\sqrt{6}$ and $M = 28$. Now (4) shows that for any solution (x, y, z) of (5), we have $z = 14 - 14 \cos\theta$ so that $|z| \leqslant 28$. However, as $9d^2 > R^2$, all entries of all solutions will be non-negative, so that $0 \leqslant z \leqslant 28$. It follows that if (x, y, z) is an integer solution to (5) then $z \in \{0, \ldots, 28\}$,

$$x = \frac{42 - z \pm \sqrt{3z(28 - z)}}{2}, \qquad y = 42 - x - z.$$

If we now compute the values of (x, y, z) for $z = 0, 1, \ldots, 28$, we find that the only integer solutions of (5) are $(0, 21, 21)$, $(7, 7, 28)$, $(3, 12, 27)$, $(1, 16, 25)$ and all triples obtained by permuting the entries.

Example 2.2: The simultaneous equations

$$x + y + z = -1, \qquad x^2 + y^2 + z^2 = 21 \tag{6}$$

provide an example in which some of the entries are negative. The discussion above shows that if (x, y, z) is an integer solution of these equations, then $z \in \{-4, \ldots, 4\}$ and, in addition,

$$x = \frac{-(1 + z) + \varepsilon\sqrt{41 - 2z - 3z^2}}{2}.$$

A computation now shows that the only integer solutions to (6) are $(1, 2, -4)$ and all triples obtained by permuting the entries.

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107.29   A cautionary example relating to the interpretation of numerical computations

1. *An example case*

The availability of software tools [1, 2, 3, 4, 5, 6] and multi-function calculators has unquestionably had an impact on how students interpret numerical solutions to mathematical problems. However, these tools can also lead to an overly-casual attitude about how to interpret the effects of numerical precision. For example, students quickly learn to interpret a numerical result of 0.99999999 as being exactly 1, or 3.54e-16 as being zero. The problem, of course, is that they may become so habituated to disregarding low-order terms that they fail to recognise results that are *nearly* an integer – *but are not*. The following trig expression (in radians) provides a cautionary example:

$$\arcsin\left(1 + \sin\left(11\right)\right) - \sin\left(11\right). \tag{1}$$

Figure 1 shows the result for (1) using *WolframAlpha*, which is probably the most widely used online mathematical tool. As can be seen, it seems to indicate that the expression evaluates to 1. Figures 2 and 3 show results obtained from *Octave/Matlab* and *Mathematica*, respectively, that are similarly 1 to within the precision of the display. In fact, a student could be forgiven for concluding from the *Mathematica* result in Figure 3 that the answer given is *exactly* 1, i.e., not just *nearly* 1 up to some level of precison.

If a student were to increase the displayed precision using *WolframAlpha* or *Mathematica*, the result would be what is shown in Figure 4 with zeros out to 16 decimal places. A student might very well conclude that digits beyond that long sequence of zeros are just due to numerical inaccuracy. This is a natural assumption to make based on practical experience with calculators and other common mathematical tools. However, students are likely to be very surprised when they discover that the number shown in Figure 4 is in fact accurate to the precision shown, i.e., that the expression of (1) does *not* equal 1.

2. Considerations on Numerology

There are multiple factors that may contribute to students drawing a mistaken inference from the example of the previous section. First is the fact that some calculators display spurious degrees of precision. This can occur when equations are solved to a precision less than the default display precision. One might expect a few spurious low-order digits to be of little consequence – let alone be noticed. However, student homework problems are typically designed to be amenable to solving by hand and often have integer solutions. Thus, students become habituated to seeing slight deviations from known integer results due to numerical imprecision from their calculators. It is no wonder, then, that such experience could lead to false presumptions when real-world problems *happen* to produce near-integer solutions.

WolframAlpha

Input: N $[\sin^{-1}(1 + \sin(11)) - \sin(11)]$

Result: 1.000000000000000...

FIGURE 1: Result of the evaluation of (1) using WolframAlpha [1] with default precision (and defined in radians, not degrees). Note that the ellipsis (...) can reasonably be taken to imply that the sequence of decimal zeros continues indefinitely, or it can be taken to imply there are additional significant digits (possibly nonzero) that are not shown.

```
Octave
octave: output precision(17)
error: output precision: arg must be less than or equal to 16
octave: output precision(16)
octave: asin(1+sin(11)) - sin(11)
ans = 1.000000000000000
```

FIGURE 2: Result of the evaluation of (1) using Octave [3] with maximum output/
display precision.

```
Mathematica 12.1
In[1]:= N [sin⁻¹ (1 + sin(11)) − sin(11)]
Out[1]= 1.
```

FIGURE 3: Result of the evaluation of (1) using Mathematica [2] with default
precision. It may not be clear from the form of the result (1.) whether the answer is
precisely the integer 1 or has been rounded.

```
WolframAlpha
Input
sin⁻¹ (1 + sin(11)) − sin(11)
Decimal approximation
1.0000000000000001565509787806395074264893435166601462092
(result in radians)
```

FIGURE 4: Result of the evaluation of (1) using WolframAlpha with increased
displayed precision. The shown digits are entirely accurate, i.e. the expression does
not equal 1, but at first glance the natural inclination might be to assume that the
nonzero decimal digits are simply *numerical inaccuracy*.

A natural response to this concern might be to point out that such events
are relatively rare. After all, the chances of a solution to a 'random' problem
being within 6 or 7 digits of an integer is on the order of one in a million.
Yes, that's roughly correct, but what if a solution *happens* to be near π, 2π,
$\frac{1}{2}\pi$, $\sqrt{2}$, or some other commonly-encountered special number? Is it likely
that a student might assume a result beginning with '3.14159...' is
identically π even though discrepancies appear at the 6th decimal place?
Probably so. Thus, opportunities for numerological misadventure are
somewhat greater than might be estimated at first glance.

The risk of numerological seduction is further complicated by the fact
that proximity to an integer, or some other special number, could potentially
hint that 'something deeper' may be at play. For example, in 1859 Charles
Hermite recognized that $\exp\left(\pi\sqrt{163}\right)$ (now commonly referred to as

Ramanujan's constant [7]) is a near-integer to 12 decimal places, and this has inspired subsequent efforts to explain *why* this is the case. Yes, there could be something deeper, but if its near-integer status is purely coincidental then search for an 'explanation' can only yield another numerological chimera that is no more revealing than the existence of a line that connects two given points.

3. *Discussion*

One way to reduce susceptibility to mistakes of the kind discussed in this Note is to avoid using calculators that display spurious digits of precision. Educators can also emphasise to students that many modern tools (e.g., *WolframAlpha / Mathematica*) never display spurious digits, so if a displayed result deviates from an integer or other special number, then it truly is *not* equal to that special number. More generally, a full discussion of this topic with students is likely to be stimulating and provoke greater sensitivity to the possiblity that tantalizing features of a given solution may be purely accidental.

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107.30 Remark on Cauchy–Schwarz inequality

In the mathematical sciences, inequalities play an important role. There are many mathematical inequalities, some of which are the basis for constructing other inequalities. One of the fundamental inequalities is the well-known *Cauchy–Schwarz inequality*. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be real numbers. The Cauchy–Schwarz inequality states that (see, e.g., [1])